Moduli Spaces and their Birational Geometry

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Part I

Introduction to moduli spaces

Classification problems

Problem

Find all possible mathematical objects with given conditions or axioms.

- Finite dimensional vector spaces up to isomorphism
- Cyclic groups
- Finite simple groups
- Poincaré conjecture: a consequence of the classification of three dimensional compact manifolds

Moduli spaces

In many natural geometric classification problems,

- there are infinitely many objects,
- but it depends on several parameters,
- parameters form a space (with good structures).

A moduli space is a space of parameters we need to classify certain geometric objects.

In concrete terms, a moduli space is a dictionary of geometric objects.

Toy example: moduli space of circles

To describe a circle on $\mathbb{R}^2$, we need two pieces of information: the center (x_0, y_0) and the radius r.

Also for any choice of (x_0, y_0) and $r > 0$, we can construct a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$.

Therefore, the moduli space M_C of circles on $\mathbb{R}^2$ is

$$M_C = \{(x_0, y_0, r) \in \mathbb{R}^3 \mid r > 0\},$$

which is an open subset of $\mathbb{R}^3$.

Universal family

Define

$$U_C = \{(x, y, x_0, y_0, r) \in \mathbb{R}^5 \mid (x - x_0)^2 + (y - y_0)^2 = r^2, r > 0\}.$$

There is a map

$$\begin{aligned} \pi : U_C &\to M_C \\ (x, y, x_0, y_0, r) &\mapsto (x_0, y_0, r). \end{aligned}$$

For a point $(1, 2, 3)$,

$$\pi^{-1}(1, 2, 3) = \{(x, y) \in \mathbb{R}^2 \mid (x - 1)^2 + (y - 2)^2 = 3^2\}.$$

- Moduli space M_C has all information about circles on $\mathbb{R}^2$.
- U_C contains all circles on $\mathbb{R}^2$.

U_C is called the universal family of M_C.

Moduli space of triangles and two lessons

A triangle on $\mathbb{R}^2$ can be described by three vertices, $v_1 = (x_1, y_1), v_2 = (x_2, y_2), v_3 = (x_3, y_3)$.

To get a triangle, three vertices v_1, v_2, v_3 must not be collinear. Set

$$C = \{(v_1, v_2, v_3) \in \mathbb{R}^6 \mid v_1, v_2, v_3 \text{ are collinear}\}$$

Then the moduli space M_T of triangles seems to be

$$M_T = \mathbb{R}^6 - C.$$

But...

Need quotient spaces

But three points v_2, v_3, v_1 define the same triangle.

More generally, a permutation of v_1, v_2, v_3 defines the same triangle.

In algebraic terms, there is a S_3 group action on $\mathbb{R}^6 - C$ and

$$M_T = (\mathbb{R}^6 - C)/S_3,$$

the quotient space (or orbit space).

Many moduli spaces are constructed in this way.

Lesson: Group action is a very important tool in moduli theory.

Moduli of degenerated objects

M_T is not compact.

There are many tools to study the geometry of compact spaces.

$\Rightarrow$ Want to compactify it.

M_T is not compact (closed) because

- Sometimes $\lim_{t\to 0}(v_1(t), v_2(t), v_3(t))$ is a triple of collinear points.
- Sometimes $\lim_{t\to 0} v_1(t)$ diverges.

We can remedy this problem by

- allow degenerated triangles,
- consider triangles in a projective plane instead of $\mathbb{R}^2$.

Lesson: A compactification of a moduli space may a moduli space of given objects and their degenerations.

Why do we study moduli spaces?

Many mathematical problems can be solved by studying geometry of moduli spaces.

Classical enumeration problems:

- How many lines in a plane pass through given 2 points?
- How many lines in a three dimensional vector space intersect given 4 general lines?
    1. 0
    2. 1
    3. ∞
    4. none of them

Approach using moduli space

M_L: moduli space of lines in a three dimensional space.

$\ell_1, \ell_2, \ell_3, \ell_4$: four given lines.

$S_i = \{\ell \in M_L \mid \ell \cap \ell_i \neq \emptyset\}$.

We want to find

$$|S_1 \cap S_2 \cap S_3 \cap S_4|.$$

By studying geometry of M_L (using cohomology), one can obtain the answer 2.

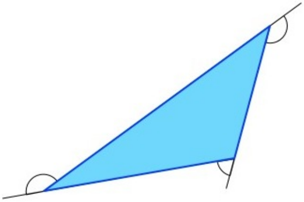
- (Steiner's problem) Find the number of conics which are tangent to given 5 general lines. Answer: 3264.

Using degeneration

Degeneration on a moduli space is very useful technique to prove many problems. Here is a toy example.

Question

Show that the sum of angle defects of a triangle is always 360°*.*



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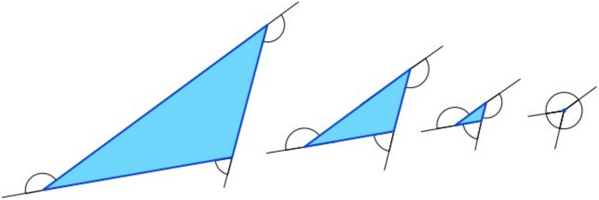


- When we deform our triangle to smaller similar triangles, the sum of angle defects is a constant.
- For the degenerated triangle (a point), the sum is 360°.

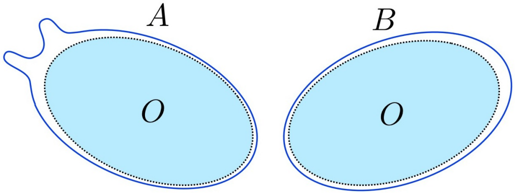
Part II

Birational geometry of moduli spaces

Birational geometry

One way to study a space: compare it with other similar spaces.

Two spaces A, B are birational if they share a common open dense subset O.



$f : A \to B$ is called a birational morphism if it preserves O.

If B is simpler than A,

Geometric data of $B \Rightarrow$ Understand the geometry of A.

Comparison of dictionaries

We can describe the birational geometry of moduli spaces as a comparison of dictionaries.

There is an one-to-one correspondence between many Korean words and English words, but...

Korean		English
설거지하다(dish) 세수하다(face) 씻다(hand)	$\leftrightarrow$	wash
이다	$\leftrightarrow$	be have been

Typical situations

Let X, Y be two birational moduli spaces.

- There is a surjective morphism $f : X \to Y$. X is finer than Y.
- There is no birational morphism between X and Y, but there is Z which has surjective birational morphisms from X and Y.

$$\begin{tikzcd} X \arrow[rr, leftrightarrow, dashed, "\times"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

- There is a moduli space W which has surjective birational morphisms to both X and Y.

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ X \arrow[rr, leftrightarrow, dashed, "\times"] & & Y \end{tikzcd}$$

Goals of the birational geometry of a moduli space M

- Find new moduli spaces which are birational to M.
- Understand the difference between them.
- Study geometric properties of M using birational moduli spaces.
- Interpret them using some theoretical frameworks.
- Classify birational models.

Part III

Moduli spaces of stable rational curves

Moduli spaces in algebraic geometry

Moduli theory is particularly useful in algebraic geometry because many moduli spaces are finite dimensional spaces (variety, scheme, stack, $\cdots$).

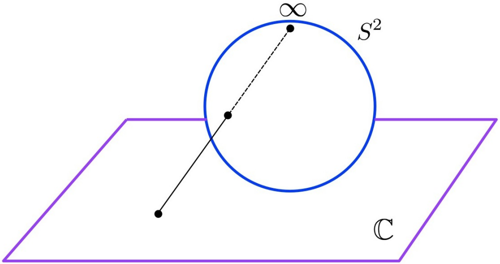
- Grassmannian $G(k,n)$: moduli space of k-dimensional sub vectorspaces of $\mathbb{C}^n$.
- Hilbert scheme $\mathrm{Hilb}(X)$: moduli space of subschemes of a fixed scheme X.
- M_g: moduli space of smooth algebraic curves of genus g.
- $M_C(r)$: moduli space of rank r vector bundles over a curve C.

My favorite is the moduli space $\overline{M}_{0,n}$ of stable rational curves.

Pointed rational curves

A smooth rational curve is a complex curve isomorphic to $\mathbb{CP}^1$.

- $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$
- Topologically, it is homeomorphic to S^2, a 2-dimensional sphere.



An n-pointed smooth rational curve is $(\mathbb{CP}^1, p_1, \cdots, p_n)$ where $p_1, \cdots, p_n$ are distinct points on $\mathbb{CP}^1$.

Pointed rational curves

Invertible linear fractions act on $\mathbb{CP}^1$ as

$$z \mapsto \frac{az+b}{cz+d}.$$

Two n-pointed smooth rational curves $(\mathbb{CP}^1, p_1, \cdots, p_n)$ and $(\mathbb{CP}^1, q_1, \cdots, q_n)$ are isomorphic if there exists an invertible linear fraction f such that $f(p_i) = q_i$.

Lemma

For any three distinct points $a, b, c \in \mathbb{CP}^1$, there exists a unique linear fraction f such that $f(a) = 0, f(b) = 1, f(c) = \infty$.

So we may assume that $p_1 = 0, p_2 = 1, p_3 = \infty$.

Moduli of n-pointed smooth rational curves

Definition

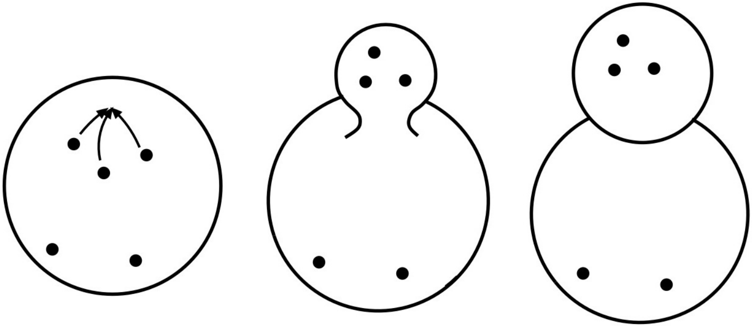
$M_{0,n}$ is the moduli space of isomorphism classes of n-pointed smooth rational curves.

$$
\begin{aligned}
M_{0,n} &= (\mathbb{CP}^1 - \{0,1,\infty\})^{n-3} - \{p_i = p_j \text{ for some } i \neq j\} \\
&= (\mathbb{C} - \{0,1\})^{n-3} - \{p_i = p_j \text{ for some } i \neq j\}
\end{aligned}
$$

- Open dense subset of $\mathbb{C}^{n-3} \Rightarrow$ smooth complex manifold
- $\dim M_{0,n} = n - 3$
- But it is NOT compact. (some points may collide at a point)
- Want to find a nice compactification.

Degeneration of n-pointed rational curves

When two or more points approaches, make a bubble at that point and distribute the points on the bubble.



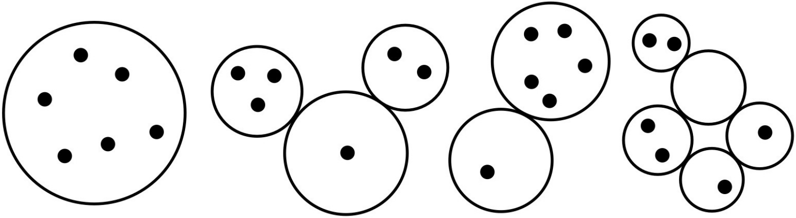
As a result, we can get a singular curve with distinct points at the smooth part.

Moduli space of stable rational curves

Definition

An n-pointed complex curve $(C, p_1, \cdots, p_n)$ is rational if all components are isomorphic to $\mathbb{CP}^1$ and there is no cycle of components. An n-pointed rational curve $(C, p_1, \cdots, p_n)$ is stable if

- At a singular point, locally it looks like $xy = 0$ in $\mathbb{C}^2$,
- p_i's are distinct smooth points,
- Each component of C has at least three special points.



Moduli space of stable rational curves

Definition

$\overline{M}_{0,n}$ is the moduli space of isomorphism classes of n-pointed stable rational curves.

- It is a compactification of $M_{0,n}$.
- It has dimension $n-3$.
- It is a projective complex manifold.
- Cohomology ring is known.
- Its birational geometry is hard and very complicated.

Why is it interesting?

1. By (Gibney-Keel-Morrison) and (Coskun-Harris-Starr), several questions about other moduli spaces of curves are reduced to the questions about $\overline{M}_{0,n}$.

2. It has many different directions of generalization.

- Moduli spaces of stable hyperplane arrangements
- Chow quotient $Gr(k,n)///(\mathbb{C}^*)^n$
- Cross ratio varieties for root systems
- Spaces of pointed trees of projective spaces

Why is it interesting?

3. It has very rich combinatorial structures.

- The universal family $U_{0,n}$ over $\overline{M}_{0,n}$ is isomorphic to $\overline{M}_{0,n+1}$.
- An irreducible component of $\overline{M}_{0,n} - M_{0,n}$ is isomorphic to $\overline{M}_{0,i} \times \overline{M}_{0,j}$.
- Consider a hypersimplex
$$\Delta(2,n) = \{(x_1, \cdots, x_n) \mid 0 \le x_i \le 1, \sum x_i = 2\}.$$
There is an one-to-one correspondence between the topological strata of $\overline{M}_{0,n}$ and decompositions of $\Delta(2,n)$ into matroid polytopes.
- Limit computation in $\overline{M}_{0,n} \Leftrightarrow$ Geometry of Bruhat-Tits building $PGL_2\mathbb{C}((z))/PGL_2\mathbb{C}[[z]]$

Part IV

Birational geometry of moduli space of stable rational curves

Three ways to approach - I. Algebraic stack

Define a moduli problem set theoretically, and show that there is an algebraic moduli space in a good algebraic category.

Theorem (Smyth, 09)

As algebraic stacks, there are many moduli spaces $\overline{M}_{0,n}(Z)$ which are birational to $\overline{M}_{0,n}$. They obtained by allowing worse singularities and collisions of some points. They depend on certain combinatorial data Z.

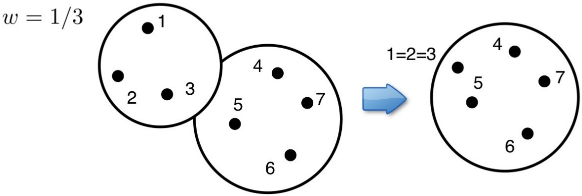
- By definition, it has a modular meaning.
- Hard to obtain good geometric properties, for example projectivity.

Three ways to approach - I. Algebraic stack

Example. (Hassett, 03) Define a new moduli problem:

Fix a weight $0 < w \leq 1$. $(C, p_1, \cdots, p_n)$ is w-stable if

- At a singular point, locally it looks like $xy = 0$ in $\mathbb{C}^2$,
- p_i's are smooth points, but $k \leq 1/w$ points can collide.
- For each component C', $w \cdot \#|\text{pts on } C'| + \#|\text{singular pts}| > 2$.



The moduli space of weighted stable curves $\overline{M}_{0,\mathcal{A}}$ of w-stable curves is an example of $\overline{M}_{0,n}(Z)$.

Three ways to approach - II. Construction using GIT

A standard technique of construction of moduli space is taking a quotient space of a larger moduli space.

In algebraic geometry, we use geometric invariant theory (GIT) quotient to obtain a projective quotient space.

Theorem (Swinarski, 08)

For any weight data $\mathcal{A}$, $\overline{M}_{g,\mathcal{A}}$ can be constructed as a GIT quotient of a closed subvariety of $\mathrm{Hilb}(\mathbb{P}^N) \times (\mathbb{P}^N)^n$ for a certain N.

Theorem (Giansiracusa, Jensen, M, 11)

Many of known birational models of $\overline{M}_{0,n}$ can be constructed by a GIT quotient $U_{d,n}//_L SL_{d+1}$ of a closed subvariety of $\mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n$. In particular, we can prove the projectivity of some of $\overline{M}_{0,n}(Z)$.

Three ways to approach - III. Mori's theory

For a pair (X, D) of a smooth projective variety X and a divisor (linear combination of codimension 1 subvarieties) D with some technical assumptions, one can construct a birational model as

$$X(D) := \mathrm{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)).$$

Fakhruddin described a way to study a new family of divisors on $\overline{M}_{0,n}$ so called conformal block divisors originated from the conformal field theory and the representation theory of affine Lie algebras.

Question

Is there any connection between these three approaches?

Interaction of three approaches - several results

For $(C, p_1, \cdots, p_n) \in \overline{M}_{0,n}$, take a cotangent space of C at p_i.

It forms a rank 1 bundle $\mathbb{L}_i$ on $\overline{M}_{0,n}$.

$\psi_i := c_1(L_i)$

Theorem (M, 11)

Let $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ be a weight datum.

$$\overline{M}_{0,n}(K + \sum_{i=1}^{n} a_i \psi_i) \cong \overline{M}_{0,\mathcal{A}}$$

where K is the canonical divisor of $\overline{M}_{0,n}$.

Interaction of three approaches - several results

Theorem (Jensen, Gibney, M, Swinarski, 12)

For any nontrivial symmetric $\mathfrak{sl}_2$ *weight* 1 *conformal block divisor* $\mathbb{D}(\mathfrak{sl}_2, \ell, (1, 1, \cdots, 1))$,

$$\overline{M}_{0,n}(\mathbb{D}(\mathfrak{sl}_2, \ell, (1, 1, \cdots, 1))) \cong U_{d,n} /\!/_{(\gamma, w)} SL_{d+1},$$

where $d = \lfloor \frac{n}{2} \rfloor - \ell$, $\gamma = \frac{\ell-1}{\ell+1}$, *and* $w = \frac{1}{\ell+1}$.

Application to geometry of moduli spaces

Theorem (M, 11)

There is an explicit inductive algorithm to compute Poincaré polynomial $P_t(\overline{M}_{0,\mathcal{A}}) = \sum_{k\geq 0} \dim H_k(\overline{M}_{0,\mathcal{A}}, \mathbb{Q}) t^k$ *of* $\overline{M}_{0,\mathcal{A}}$*.*

$\overline{M}_{0,5\cdot 1}$	$1 + 5t^2 + t^4$
$\overline{M}_{0,6\cdot 1}$	$1 + 16t^2 + 16t^4 + t^6$
$\overline{M}_{0,7\cdot 1}$	$1 + 42t^2 + 127t^4 + 42t^6 + t^8$
$\overline{M}_{0,7\cdot 1/3}$	$1 + 7t^2 + 22t^4 + 7t^6 + t^8$
$\overline{M}_{0,8\cdot 1}$	$1 + 99t^2 + 715t^4 + 715t^6 + 99t^8 + t^{10}$
$\overline{M}_{0,8\cdot 1/3}$	$1 + 43t^2 + 99t^6 + 99t^8 + 43t^8 + t^{10}$
$\overline{M}_{0,9\cdot 1}$	$1 + 219t^2 + 3292t^4 + 7723t^6 + 3292t^8 + 219t^{10} + t^{12}$
$\overline{M}_{0,9\cdot 1/3}$	$1 + 135t^2 + 604t^4 + 975t^6 + 604t^8 + 135t^{10} + t^{12}$
$\overline{M}_{0,9\cdot 1/4}$	$1 + 9t^2 + 37t^4 + 93t^6 + 37t^8 + 9t^{10} + t^{12}$
$\overline{M}_{0,10\cdot 1}$	$1 + 466t^2 + 13333t^4 + 63173t^6 + 63173t^8 + 13333t^{10} + 466t^{12} + t^{14}$
$\overline{M}_{0,10\cdot 1/3}$	$1 + 346t^2 + 3553t^4 + 8173t^6 + 8173t^8 + 3553t^{10} + 346t^{12} + t^{14}$
$\overline{M}_{0,10\cdot 1/4}$	$1 + 136t^2 + 298t^4 + 508t^6 + 508t^8 + 298t^{10} + 136t^{12} + t^{14}$

Application to geometry of moduli spaces

$\overline{M}_0(\mathbb{P}^r, d)$: moduli space of stable maps of genus 0 to $\mathbb{P}^r$ of degree d.
$\cdots$ a compactification of the moduli space of smooth rational curves in $\mathbb{P}^r$.

Theorem (Kiem, M, 10)

$$H^*(\overline{M}_0(\mathbb{P}^r, 2), \mathbb{Q}) = \mathbb{Q}[\xi, \alpha^2, \rho]/\langle(\rho+2\alpha+\xi)^{r+1} + (\rho-2\alpha+\xi)^{r+1}, \xi^{r+1}\rho \frac{(\rho+2\alpha+\xi)^{r+1}-\xi^{r+1}}{\rho+2\alpha} + \frac{(\rho-2\alpha+\xi)^{r+1}-\xi^{r+1}}{\rho-2\alpha}\rangle$$

$$H^*(\overline{M}_0(\mathbb{P}^\infty, 3), \mathbb{Q}) = \mathbb{Q}[\xi, \alpha^2, \rho_1^3, \rho_2^2, \rho_3, \sigma]/\langle\alpha^2\rho_1^3, \rho_1^3\sigma, \sigma^2 - \alpha^2\rho_3^2\rangle$$

$$P_t(\overline{M}_0(\mathbb{P}^r, 3)) = \left(\frac{1-t^{2r+10}}{1-t^6} + 2\frac{t^4-t^{2r+4}}{1-t^4}\right)\frac{(1-t^{2r+2})^2(1-t^{2r})}{(1-t^2)^2(1-t^4)}$$

Current projects

1. ALL known birational moduli spaces of $\overline{M}_{0,n}$ are contractions of $\overline{M}_{0,n}$.

Problem

Construct a modular flip of $\overline{M}_{0,n}$.

One way is to combine the idea of KKO compactification, Fulton-MacPherson space.

2. There are interesting combinatorial problems related to

- projectivity of Smyth's spaces ($\Leftrightarrow$ combinatorics of matroid decompositions),
- effective divisors on $\overline{M}_{0,n}$ ($\Leftrightarrow$ combinatorics of multinomials).

Thank you!