Birational Geometry of Moduli Spaces

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Part I

Moduli Spaces

Classification Problem

Problem

Find all possible mathematical objects with given conditions or axioms.

- Smooth plane conics: circle, ellipse, parabola, and hyperbola
- Finite dimensional vector spaces
- Finite cyclic groups
- Finite simple groups
- Poincaré conjecture (Perelman's theorem, $1,000,000!): a consequence of the classification of three dimensional compact manifolds

Classification Problem

Question

Describe all circles on the plane.

We need to know the center (x_0, y_0) and the radius r.

$$(X - x_0)^2 + (Y - y_0)^2 = r^2$$

The space of circles:

$$\mathrm{M}_C = \{(x_0, y_0, r) \in \mathbb{R}^3 \mid r > 0\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \subset \mathbb{R}^3$$

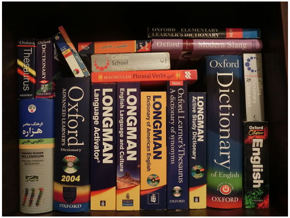
M_C: the moduli space of circles.

- Infinitely many objects, finite parameters.
- M_C is a topological space.

Definition

A moduli space is a space parametrizing a certain kind of geometric object.

We may think a moduli space as a dictionary of geometric objects.



Moduli Space of Triangles

A triangle on the plane can be described by three vertices:

$$v_1 = (x_1, y_1), \;\; v_2 = (x_2, y_2), \;\; v_3 = (x_3, y_3).$$

To get a triangle, three vertices v_1, v_2, v_3 must not be collinear.

$$C = \{(v_1, v_2, v_3) \in \mathbb{R}^6 \mid v_1, v_2, v_3 \text{ are collinear}\}$$

Then the moduli space M_T of triangles seems to be

$$\mathrm{M}_T = \mathbb{R}^6 - C,$$

but...

Moduli Space of Triangles

But three points v_2, v_3, v_1 define the same triangle.

More generally, a permutation of v_1, v_2, v_3 defines the same triangle.

There is an S_3-group action on $\mathbb{R}^6 - C$ and

$$\mathrm{M}_T = (\mathbb{R}^6 - C)/S_3,$$

the quotient space (or orbit space).

Many moduli spaces are constructed in this way.

Lesson

Group action is a very important tool in the moduli theory.

Moduli Space of Triangles

M_T is not compact, because

1. Sometimes $\lim_{t \to 0}(v_1(t), v_2(t), v_3(t))$ is a triple of collinear points.
2. Sometimes $\lim_{t \to 0}(v_1(t), v_2(t), v_3(t))$ diverges.

We can remedy these problems by

1. allowing degenerated triangles,
2. considering triangles in a projective space $\mathbb{P}^2$ instead of $\mathbb{R}^2$.

Lesson

A compactification of a moduli space may be a moduli space of given objects and their degenerations.

Moduli Spaces in Algebraic Geometry

Algebraic geometry is a study of geometric figures (algebraic varieties) which are zero sets of polynomials.

It is a nice field for moduli theory: many moduli spaces are finite dimensional algebraic varieties.

Three typical examples in algebraic geometry

- Abstract varieties: moduli space M_g of genus g Riemann surfaces (= complex curves)
- Subvarieties: moduli space $\mathrm{Hilb}(X)$ of subvarieties of a fixed variety X.
- Geometric structures: moduli space $\mathrm{M}_X(r)$ of rank r vector bundles on a fixed variety X.

Why do we study Moduli Spaces? - 1

Moduli spaces are explicit examples of higher dimensional varieties.

An algebraic variety in $\mathbb{C}^{1000}$ defined by $10{,}000$ equations $\cdots$ What is its dimension? Is it smooth? connected? nonempty?

For many moduli spaces,

1. local geometry (dimension, smoothness) $\leftrightarrow$ homological algebra
2. global geometry (topological invariants) can be studied without defining equations.

Laboratory mouse for algebraic geometers:

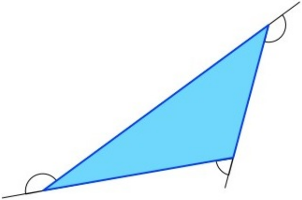
- Toric varieties (varieties which can be studied combinatorially)
- Moduli spaces

Why do we study Moduli Spaces? - 2

To study geometry of parametrized objects.

Question

Show that the sum of angle defects of a triangle is always 360°*.*



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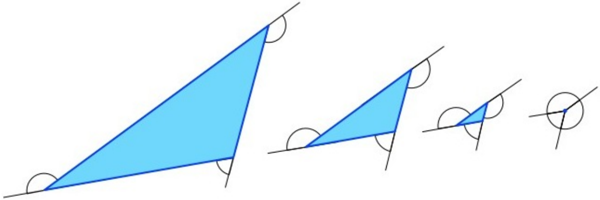

- When we deform our triangle to smaller similar triangles, the sum of angle defects is a constant.
- For the degenerated triangle (a point), the sum is 360°.

Why do we study Moduli Spaces? - 3

Many mathematical problems can be solved by studying geometry of moduli spaces.

Problem

How many lines in a three dimensional vector space intersect given 4 general lines?

1. 0
2. 1
3. ∞
4. *none of them*

Why do we study Moduli Spaces? - 3

Solution:

M_L: moduli space of lines in a three dimensional space

ℓ_1, ℓ_2, ℓ_3, ℓ_4: four given lines

$S_i := \{\ell \in \mathrm{M}_L \mid \ell \cap \ell_i \neq \emptyset\}$

Want: $|S_1 \cap S_2 \cap S_3 \cap S_4|$

By studying the geometry of M_L (via its cohomology ring), we obtain that the answer is 2.

Why do we study Moduli Spaces? - 3

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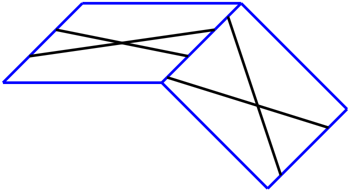

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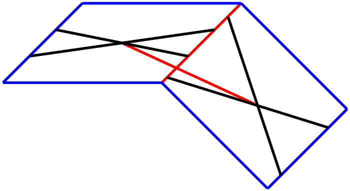


Why do we study Moduli Spaces? - 3

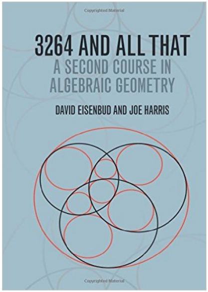
Problem (Steiner's problem 1848)

Find the number of conics which are tangent to given 5 general lines.

Incorrect answer of Steiner: $6^5 = 7776$

Correct answer (Chasles 1868) : 3264

Rigorous proof: (Fulton-MacPherson 1978)



Why do we study Moduli Spaces? - 4

They are essential in modern mathematical physics.

String theory:

- Our universe is 10 dimensional $\cdots$ 4 dimensional space-time + 6 dimensional hidden space (Calabi-Yau threefold)
- The fundamental objects consisting our universe are not particles, but strings.
- Important to understand the evolution of strings $\approx$ surfaces $\approx$ complex curves in CY threefolds.

Why do we study Moduli Spaces? - 4

A quintic threefold is a zero locus of a degree 5 homogeneous polynomial in $\mathbb{CP}^4$. It is a CY threefold.

Conjecture (Clemens (1984))

Let X be a general quintic threefold. For each $d \in \mathbb{N}$, there exist only finitely many genus 0 curves of degree d on X.

Proved for $d \leq 11$.

degree	number of curves
1	2,875
2	609,250
3	317,206,375
4	242,467,530,000
5	22,930,588,887,625
6	248,249,742,118,022,000
7	295,091,050,570,845,659,250

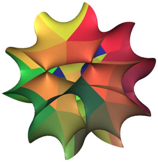
Main Question

Question

Study the shape of moduli spaces.

My research program

1. Study birational geometry of moduli spaces.
2. Apply birational geometry to the computation of topological invariants of various concrete moduli spaces.
3. Apply these results for the study of quantum invariants.



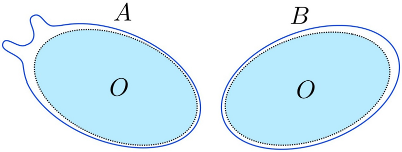
The art of doing mathematics consists in finding that special case which contains all the germs of generality. - D. Hilbert

Part II

Birational Geometry of Moduli Spaces

Definition

Two algebraic varieties A and B are birational if they share a common open dense subset O. Then B is called a birational model of A.

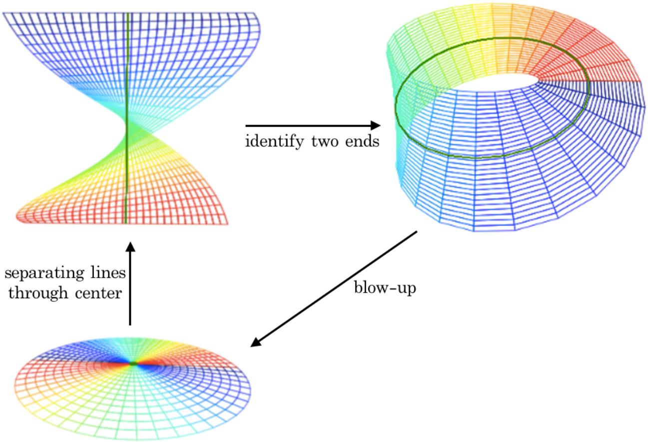


A surjective continuous map $f : A \to B$ is called a birational map if $f|_O = \mathrm{id}_O$.

- Geometrically, f is an algebro-geometric surgery of B.
- In this situation, B is simpler than A.

Birational Geometry - Example

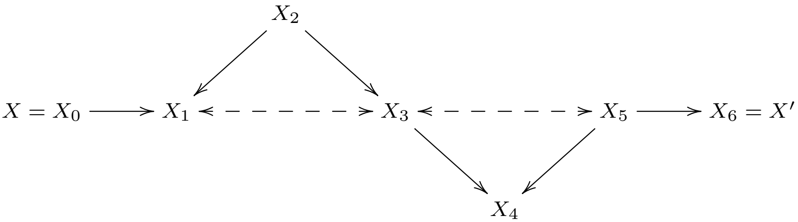
Below is an example of an explicit algebro-geometric surgery, the so-called blow-up.



Thus the circular disk is a birational model of the Möbius strip.

Application to Topological Invariants - Divide and Conquer

1. By applying algebro-geometric surgeries several times, we may construct a sequence of birational maps.



   X_i's are birational models of X.

2. It is relatively easy to measure the difference of topological invariants between X_i and X_{i+1}.
3. If it is easy to compute topological invariants of X', we can compute these of X.
4. In many cases, X' is a quotient space of some elementary variety.
5. Apply this strategy to moduli spaces!

In many cases, a birational model of a moduli space is again a moduli space of a different class of objects.

Several ways to understand a curve $C \subset \mathbb{P}^2$:

1. High school students: C is the solution set $\cdots$ $x^2 + y^2 - 1 = 0$
2. Calculus students: C is a parametrization $\cdots$ $X(t) = (\cos t, \sin t)$
3. Graduate students: C is an $\mathbb{R}[x, y]$-module $\cdots$ $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$
4. Algebraists: free resolution of the ideal of C

Each viewpoint provides a different moduli space.

1. $\mathrm{H}_{\mathbb{P}^2}(d)$: moduli space of subvarieties (Hilbert)
2. $\mathrm{K}_{\mathbb{P}^2}(d)$: moduli space of maps (Kontsevich)
3. $\mathrm{S}_{\mathbb{P}^2}(d)$: moduli space of modules (Simpson)
4. $\mathrm{B}_{\mathbb{P}^2}(d)$: moduli space of objects in a derived category (Bridgeland)

- Lines in $\mathbb{P}^r$:

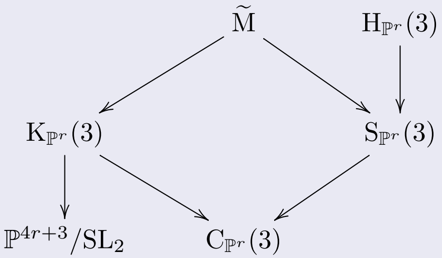
$$\mathrm{K}_{\mathbb{P}^r}(1) = \mathrm{H}_{\mathbb{P}^r}(1) = \mathrm{S}_{\mathbb{P}^r}(1) = \mathrm{Gr}(2, r+1)$$

- Conics in $\mathbb{P}^2$:

$$\mathrm{K}_{\mathbb{P}^2}(2) \to \mathrm{H}_{\mathbb{P}^2}(2) = \mathrm{S}_{\mathbb{P}^2}(2) = \mathbb{P}^5$$

- Cubics in $\mathbb{P}^r$:

Theorem (Kiem-M, Chung-Kiem)



Birational Geometry of Moduli Spaces - Example

$\mathrm{S}_{\mathbb{P}^2}(d)$: moduli space of stable sheaves ($\approx$ modules $\approx$ generalized vector spaces) on $\mathbb{P}^2$ with Hilbert polynomial $dm+1$.

- $\mathrm{S}_{\mathbb{P}^2}(1) \cong \mathbb{P}^2$
- $\mathrm{S}_{\mathbb{P}^2}(2) \cong \mathbb{P}^5$
- $\mathrm{S}_{\mathbb{P}^2}(3) \cong$ a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^9$ (Le Potier, 1993)

Theorem (Chung-M)

There is a sequence of maps

$$\mathrm{S}_{\mathbb{P}^2}(4) \longleftarrow \mathrm{S}^{\epsilon}_{\mathbb{P}^2}(4) \longrightarrow \mathrm{B}_{\mathbb{P}^2}(4) \longrightarrow \mathbb{P}^{18}/\mathrm{SL}_2 \times \mathrm{SL}_3$$

where $\mathrm{S}^{\epsilon}_{\mathbb{P}^2}(4)$ *is the moduli space of stable pairs.*

The cohomology ring $\mathrm{H}^*(\mathrm{S}_{\mathbb{P}^2}(4), \mathbb{Q})$ is calculated.

$$\dim \mathrm{H}^{2i}(\mathrm{S}_{\mathbb{P}^2}(4), \mathbb{Q}) = 1, 2, 6, 10, 14, 15, 16, 16, \cdots$$

$$\chi(\mathrm{S}_{\mathbb{P}^2}(4)) = 192$$

Related to Gopakumar-Vafa invariants in mathematical physics.

Mori's Program for Moduli Spaces

Mori's program (generalized minimal model program): a theoretical framework for the classification of all birational models.

- Includes classification of codimension 1 subvarieties, curves, etc.
- Mori dream space $\Rightarrow$ the completion of the program is possible.
- Few completed examples.

Theorem (Chung-M, M-Yoo)

For the following moduli spaces, minimal model program is completed.

- $\mathrm{S}_{\mathbb{P}^2}(4)$
- $\mathrm{K}_{\mathrm{Gr}(2,n)}(2)$
- $\mathrm{M}_{\mathbf{p}}(r) \cdots$ *moduli space of rank r parabolic bundles over* $\mathbb{P}^1$

(Giansiracusa, Gibney, Jensen, M, Swinarski)

We have worked for the case of moduli spaces $\overline{\mathrm{M}}_{g,n}$ of curves.

Part III

Application to Quantum Invariants

Invariant Factor Problem

G : reductive group (GL_r, SL_r, SO_r, SP_r, $\cdots$)

$V_{\lambda^1}, V_{\lambda^2}, \cdots, V_{\lambda^n}$: finite dimensional irreducible G-representations

$V_{\vec{\lambda}}^G := (V_{\lambda^1} \otimes V_{\lambda^2} \otimes \cdots \otimes V_{\lambda^n})^G$: trivial subrepresentation

$V^G := \bigoplus_{\vec{\lambda}} V_{\vec{\lambda}}^G$: algebra of invariants

Question

1. $\dim V_{\vec{\lambda}}^G =$?
2. *When $V_{\vec{\lambda}}^G \neq 0$?*
3. *Is V^G finitely generated?*
4. *What is a good generating set?*

Example ($n = 3$, $G = \mathrm{GL}_r$)

1. $\Leftrightarrow$ Computation of Littlewood-Richardson coefficients
2. $\Leftrightarrow$ Horn's conjecture

Quantum Invariant

$\exists$ quantum generalizations of invariant factors, constructed via conformal field theory.

Input	CFT	Output
G: simple Lie group $p_1, p_2, \cdots, p_n \in \mathbb{P}^1$ $\ell \in \mathbb{Z}_{\geq 0}$ $\overrightarrow{\lambda} = (\lambda^1, \lambda^2, \cdots, \lambda^n)$	$\Rightarrow$	$\mathbb{V}_{\ell, \overrightarrow{\lambda}}$ $\cdots$ finite dimensional vector space

$\mathbb{V} := \bigoplus_{\ell, \overrightarrow{\lambda}} \mathbb{V}_{\ell, \overrightarrow{\lambda}} \cdots$ algebra of conformal blocks

Question

1. $\dim \mathbb{V}_{\ell, \overrightarrow{\lambda}} =$*?*
2. *When* $\mathbb{V}_{\ell, \overrightarrow{\lambda}} \neq 0$*?*
3. *Is* $\mathbb{V}$ *finitely generated?*
4. *What is a good generating set?*

Geometric Interpretation

$G = \mathrm{SL}_r$.

$\mathrm{Fl}(V)$: the full flag variety of $V = \mathbb{C}^r$

By Borel-Weil theorem, $V_\lambda = \Gamma(\mathrm{Fl}(V), L_\lambda)$ for some L_λ.

$$V_{\vec{\lambda}} = \Gamma(\mathrm{Fl}(V)^n, L_{\vec{\lambda}})$$

$$V_{\vec{\lambda}}^G = \Gamma(\mathrm{Fl}(V)^n, L_{\vec{\lambda}})^G = \Gamma(\mathrm{Fl}(V)^n/G, \overline{L}_{\vec{\lambda}})$$

$$V^G = \bigoplus_{\vec{\lambda}} V_{\vec{\lambda}}^G = \bigoplus_L \Gamma(\mathrm{Fl}(V)^n/G, L) = \mathrm{Cox}(\mathrm{Fl}(V)^n/G)$$

$\cdots$ Cox ring, the ring of all 'functions' on $\mathrm{Fl}(V)^n/G$.

Theorem (Pauly)

$$\mathbb{V} = \bigoplus_{\ell, \vec{\lambda}} \mathbb{V}_{\ell, \vec{\lambda}} = \mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r)),$$

where $\mathrm{M}_{\mathbf{p}}(r)$ is the moduli space of rank r, degree 0 parabolic bundles (highly non-Hausdorff algebraic stack).

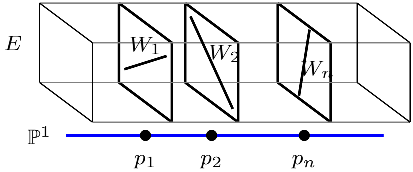
Parabolic Bundles

Fix n distinct points $\mathbf{p} = (p_1, \cdots, p_n)$ on $\mathbb{P}^1$.

Definition

A rank r parabolic bundle on $\mathbb{P}^1$ with parabolic points $\mathbf{p}$ is a collection of data $(E, \{W_i^\bullet\})$ where

1. E is a rank r vector bundle on $\mathbb{P}^1$;
2. $W_i^\bullet$ is a full flag of $E|_{p_i}$.



Moduli Space of Parabolic Bundles and Finite Generation

Theorem (M-Yoo)

The algebra $\mathbb{V}$ of conformal blocks is finitely generated.

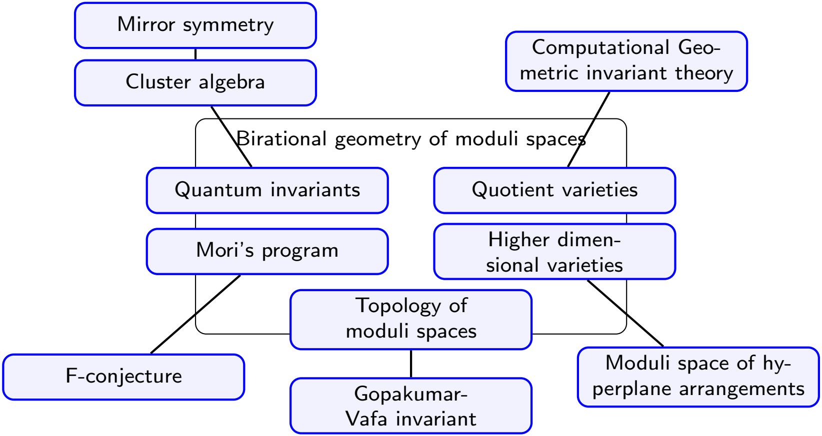
Ingredients of the proof:

- Stability condition
- Geometric invariant theory
- Minimal model program
- log Fano variety
- Deformation theory
- Wall-crossing computation
- $\cdots$

Part IV

Research Plan

Research Plan



Research Plan

Problem

Study the structure of $\mathbb{V}$.

- Theoretical aspects: Relation with cluster algebra and mirror symmetry
- Computational aspects: Mathematical experiments are possible (a great source of REU projects)

Problem (F-conjecture)

Classify all curves on the moduli space $\overline{\mathrm{M}}_{g,n}$ *of curves.*

Research Plan

Problem

Study geometry of moduli spaces of higher dimensional varieties with combinatorial structures (hyperplane arrangements, etc).

In many cases, the last stage of a sequence of birational maps is a quotient space.

The first step of the computation of algebro-geometric quotient space is a very heavy combinatorial calculation.

Problem

Develop an efficient algorithm to compute quotient spaces and implement it as an open source computer program.

Thank you!