DERIVED CATEGORY AND ACM BUNDLES OF
MODULI SPACE OF VECTOR BUNDLES ON A CURVE

KYOUNG-SEOG LEE AND HAN-BOM MOON

Dedicated to the memory of M. S. Narasimhan

ABSTRACT. We show that the derived category of a curve is embedded into the derived category of the moduli space of vector bundles on the curve of coprime rank and degree. We also generalize the semiorthogonal decomposition constructed by Narasimhan and Belmans-Mukhopadhyay. Finally, we produce a one-dimensional family of ACM bundles over the moduli space.

1. INTRODUCTION

The purpose of this paper is to give a complete affirmative answer to two problems on the moduli space of vector bundles on a curve. One is the embedding problem between derived categories, and the other is the construction of nontrivial arithmetically Cohen-Macaulay (ACM) bundles on the moduli space.

Let X be a smooth projective curve of genus $g \geq 2$. Fix two positive integers r, d such that $(r, d) = 1$ and $0 < d < r$, and fix $L \in \mathrm{Pic}^d(X)$. The moduli space $\mathrm{M}(r, L)$ of rank r, determinant L stable vector bundles on X is an $(r^2 - 1)(g - 1)$-dimensional smooth Fano variety of index two. Let $\mathcal{E}$ be the normalized Poincaré bundle on $X \times \mathrm{M}(r, L)$.

1.1. **Embedding of derived category.** We study the Fourier-Mukai transform $\Phi_{\mathcal{E}} : \mathrm{D}^b(X) \to \mathrm{D}^b(\mathrm{M}(r, L))$ with the kernel $\mathcal{E}$. Narasimhan proved that $\Phi_{\mathcal{E}}$ is an embedding when $r = 2$ in [Nar17, Nar18], by studying the Hecke correspondence. Fonarev and Kuznetsov proved the same result for a general X using different techniques [FK18]. Belmans and Mukhopadhyay extended Narasimhan's method and proved the embedding result for $r \geq 2$, $d = 1$ and $g \geq r + 3$ in [BM19]. In this paper, we lift all the assumptions on the rank, degree, and genus by employing birational geometry of moduli spaces of parabolic bundles, and the theory of derived categories of variation of GIT quotients, developed by Halpern-Leistner in [HL15] and Ballard, Favero, and Katzarkov in [BFK19].

Theorem A. *The functor* $\Phi_{\mathcal{E}} : \mathrm{D}^b(X) \to \mathrm{D}^b(\mathrm{M}(r, L))$ *is fully faithful.*

1.2. **Semiorthogonal decomposition.** To put our results in context, we explain a brief history of the question. After establishing his embedding theorem, Narasimhan made the following conjecture, which was announced in [Lee18]. In the paper, as numerical evidence, Lee proved that the motive of $\mathrm{M}(r, L)$ admits a compatible motivic decomposition. Further evidence was also provided by Gómez and Lee in [GL20]. Belmans-Galkin-Mukhopadhyay also independently proposed the same conjecture in [BGM18], with another numerical evidence in [BGM20].

Conjecture 1.1. *The category* $\mathrm{D}^b(\mathrm{M}(2, L))$ *has a semiorthogonal decomposition*

$$\mathrm{D}^b(\mathrm{M}(2, L)) = \langle \{\mathrm{D}^b(X_k), \mathrm{D}^b(X_k)\}_{0 \leq k \leq g-2}, \mathrm{D}^b(X_{g-1})\rangle,$$

where $X_k = X^k/S_k$ *is the* k*-th symmetric product of* X.

Toward the proof of Conjecture 1.1, Lee and Narasimhan showed that by analyzing the Hecke correspondence, $\mathrm{D}^b(X_2)$ is embedded [LN21] when X is non-hyperelliptic and $g \geq 16$. After this, Tevelev-Torres and Xu-Yau showed that the above building blocks are embedded in $\mathrm{D}^b(\mathrm{M}(2, L))$ with entirely different approaches [TT21, XY21]. After an early draft of this paper was circulated, very recently, the remaining generation part was proved by Tevelev [Tev23].

It is natural to guess the existence of a similar decomposition for $\mathrm{D}^b(\mathrm{M}(r, L))$ for general r and d. Based on [GL20], where the motivic decomposition of $\mathrm{M}(r, L)$ is studied, we expect the following statement. A more explicit version of the conjecture for $r = 3$ and its evidence can be found in [GL20, Conjecture 1.9].

Conjecture 1.2. *The category* $\mathrm{D}^b(\mathrm{M}(r, L))$ *has a semiorthogonal decomposition where each indecomposable component is the derived category of products of* X_k *and* $\mathrm{Jac}(X)$.

As we can see from Brill-Noether theory, the geometry of curves is very complicated in general. However, in the study of $\mathrm{D}^b(\mathrm{M}(r, L))$, we expect a uniform decomposition, which does not depend on each curve.

Except for $g = r = 2$, it seems that $\mathrm{D}^b(\mathrm{M}(r, L))$ contains at least two copies of $\mathrm{D}^b(\mathrm{pt})$ and two copies of $\mathrm{D}^b(X)$. In [BM19, Theorem B], the authors proved this is the case for $d = 1$ and (roughly) $g \geq 3r + 4$. We extend this result for arbitrary coprime degree and give a constant genus bound.

Theorem B. *If* $g \geq 6$, *there is a semiorthogonal decomposition*

$$\mathrm{D}^b(\mathrm{M}(r, L)) = \langle \mathcal{A}, {}^\perp\mathcal{A}\rangle$$

where $\mathcal{A} = \langle \mathcal{O}, \Phi_{\mathcal{E}}(\mathrm{D}^b(X)), \Theta, \Phi_{\mathcal{E}}(\mathrm{D}^b(X)) \otimes \Theta\rangle$, *where* Θ *is the ample generator of* $\mathrm{Pic}(\mathrm{M}(r, L))$.

We expect that Theorem B is true without the genus restriction (except $g = r = 2$ case), but we have only partial result for $g \leq 5$ (Theorem 9.5 and Remark 9.6).

1.3. **ACM bundles.** We also discover a family of ACM bundles on $\mathrm{M}(r, L)$. For an n-dimensional projective variety V with an ample line bundle A, a vector bundle F is called *ACM* if $\mathrm{H}^i(V, F \otimes A^j) = 0$ for all $j \in \mathbb{Z}$ and $0 < i < n$. An ACM bundle F is *Ulrich* if $\mathrm{H}^0(V, F \otimes A^{-1}) = 0$ and $\mathrm{H}^0(V, F) = \mathrm{rank}\ F \cdot \deg V$. ACM bundles naturally appear in matrix factorization [Eis80] and correspond to maximal Cohen-Macaulay modules in commutative algebra [Yos90]. Ulrich bundles enable us to compute their associated Chow forms, and Eisenbud and Schreyer conjectured that every projective variety admits an Ulrich sheaf [ES03]. However, since the above strong cohomology vanishing is difficult to expect and hard to verify, despite many works (see [Bea18, Cetal21, Fae13] and references therein), very few general results are known for higher dimensional varieties, even for the existence of ACM bundles except some trivial examples.

Theorem C. *The restricted Poincaré bundle $\mathcal{E}_x$ is ACM with respect to Θ. Thus, there is a one-dimensional family of ACM bundles on $\mathrm{M}(r, L)$, parametrized by X.*

1.4. **Structure of the paper.** Sections 2 and 3 review several basic results about the moduli space of parabolic bundles on a curve. In Section 4, we investigate the positivity of the restricted Poincaré bundle. In Section 5, we explicitly compute the wall-crossings in the case of two parabolic points. The result is essential in the following sections. Section 6 is devoted to calculating cohomology groups of certain line bundles via derived categories of the variation of GIT. Three main theorems are proved in the remaining sections.

Conventions. We work over $\mathbb{C}$. In this paper, X denotes a smooth connected projective curve of genus $g \geq 2$. The moduli space of rank r, determinant L (resp. degree d) semistable vector bundles is denoted by $\mathrm{M}(r, L)$ (resp. $\mathrm{M}(r, d)$). Unless stated explicitly, we assume $(r, d) = 1$, so $\mathrm{M}(r, L)$ is a smooth projective variety. Let ℓ be the unique integer such that $\ell d \equiv 1 \bmod r$ and $0 < \ell < r$. Let Θ be the ample generator on $\mathrm{Pic}(\mathrm{M}(r, L))$. Let $\mathcal{E}$ be the normalized Poincaré bundle on $X \times \mathrm{M}(r, L)$ such that for each $x \in X$, its restriction $\mathcal{E}_x$ to $x \times \mathrm{M}(r, L) \cong \mathrm{M}(r, L)$ has the determinant Θ^ℓ. For a vector space W, $\mathbb{P}(W)$ is the projective space of one-dimensional quotients of W. Every algebraic stack is defined over the fppf topology.

2. Moduli spaces of parabolic bundles and their birational geometry

This section explains the notion of parabolic vector bundles and their moduli space. This paper only considers the parabolic structure with at most one flag for each parabolic point. Fix a smooth connected projective curve X and a finite ordered set $\mathbf{x} := (x_1, x_2, \cdots, x_k)$ of distinct closed points of X, so $(X, \mathbf{x}) \in \mathcal{M}_{g,k}$.

Definition 2.1. A rank r *parabolic bundle* over $(X, \mathbf{x})$ is a collection of data $(E, V_\bullet)$ where

(1) E is a rank r vector bundle over X;
(2) $V_\bullet = (V_1, V_2, \cdots, V_k)$ where V_i is a subspace of $E|_{x_i}$. The dimension of V_i is called the *multiplicity* of V_i and denoted by m_i.

The sequence $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ is called the *multiplicity* of $(E, V_\bullet)$.

Definition 2.2. Let $\mathcal{M}_{(X,\mathbf{x})}(r, L, \mathbf{m})$ (resp. $\mathcal{M}_{(X,\mathbf{x})}(r, d, \mathbf{m})$) be the moduli stack of parabolic bundles $(E, V_\bullet)$ over $(X, \mathbf{x})$ of rank r, determinant L (resp. degree d), and multiplicity $\mathbf{m}$. If there is no confusion, we use $\mathcal{M}(r, L, \mathbf{m})$ (resp. $\mathcal{M}(r, d, \mathbf{m})$).

The moduli stack of all bundles are non-separated Artin stack (see [Mum66, Item 4 in Lecture 7] for an example that a locally trivial family of bundles having a nontrivial limit), hence $\mathcal{M}_{(X,\mathbf{x})}(r, L, \mathbf{m})$ is also highly non-separated. To obtain a projective coarse moduli space that enables us to do projective birational geometry, we need to introduce a stability condition.

For a parabolic bundle $(E, V_\bullet)$, a *parabolic subbundle* $(F, W_\bullet)$ is a pair such that $F \subset E$ is a subbundle and $W_i = F|_{x_i} \cap V_i$. A *parabolic quotient bundle* is defined as a parabolic bundle $(E/F, Y_\bullet)$ such that $Y_i = \mathrm{im}\,(V_i \to E/F|_{x_i})$. A *parabolic weight* $\mathbf{a} = (a_1, a_2, \cdots, a_k)$ is a sequence of rational numbers such that $0 < a_i < 1$. Intuitively, we may regard $\mathbf{a}$ as extra weight for the parabolic flags. For a parabolic bundle $(E, V_\bullet)$, its *parabolic degree* is $\mathrm{pardeg}(E, V_\bullet) := \deg E + \sum_{1 \le i \le k} m_i a_i$. The same parabolic weight can induce the parabolic degree for parabolic subbundles and parabolic quotient bundles of $(E, V_\bullet)$. The *parabolic slope* is $\mu(E, V_\bullet) := \mathrm{pardeg}(E, V_\bullet)/\mathrm{rank}\, E$.

Definition 2.3. Fix a parabolic weight $\mathbf{a}$. A parabolic bundle $(E, V_\bullet)$ is $\mathbf{a}$-*(semi)stable* if for every parabolic subbundle $(F, W_\bullet)$, $\mu(F, W_\bullet) (\le) < \mu(E, V_\bullet)$. A parabolic weight $\mathbf{a}$ is *general* if the $\mathbf{a}$-semistability coincides with the $\mathbf{a}$-stability.

Definition 2.4. Let $(X, \mathbf{x}) \in \mathcal{M}_{g,k}$ with $g \ge 2$. Let $\mathcal{M}(r, L, \mathbf{m}, \mathbf{a})$ (resp. $\mathcal{M}(r, d, \mathbf{m}, \mathbf{a})$) be the moduli stack of rank r, determinant L (resp. degree d), $\mathbf{a}$-semistable parabolic bundles over $(X, \mathbf{x})$. Let $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ (resp. $\mathrm{M}(r, d, \mathbf{m}, \mathbf{a})$) be its good moduli space, which is a normal projective variety of dimension $(r^2 - 1)(g - 1) + \sum m_i(r - m_i)$ (resp. $r^2(g - 1) + 1 + \sum m_i(r - m_i)$) [MS80, Theorem 4.1]. When $\mathbf{a}$ is general, both $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ and $\mathrm{M}(r, d, \mathbf{m}, \mathbf{a})$ are nonsingular.

Remark 2.5. When $g \leq 1$, the moduli space behaves differently. For instance, if $g = 0$, depending on $\mathbf{a}$, $\mathcal{M}(r, L, \mathbf{m}, \mathbf{a})$ may be empty. Consult [MY21].

Example 2.6. The inequality $\mu(F, W_\bullet) \leq \mu(E, V_\bullet)$ defining the a-semistability can be understood as a perturbation of the inequality $\mu(F) \leq \mu(E)$ for the semistability of the underlying bundle. If $(r, d = \deg L) = 1$, the inequality is strict for all $F \subset E$, and if each coefficient of $\mathbf{a}$ is sufficiently small and general, then $\mathbf{a}$ does not affect the stability. Therefore, a parabolic bundle $(E, V_\bullet)$ is a-stable if and only if the underlying bundle E is stable. Thus, the forgetful morphism $\mathcal{M}(r, L, \mathbf{m}, \mathbf{a}) \to \mathcal{M}(r, L)$ induces a map between coarse moduli spaces

$$\pi : \mathrm{M}(r, L, \mathbf{m}, \mathbf{a}) \to \mathrm{M}(r, L)$$

and π is a $\times \mathrm{Gr}(m_i, r)$-fibration. Indeed, for a fixed Poincaré bundle $\mathcal{E}$ over $X \times \mathrm{M}(r, L)$,

$$\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}) \cong \times_{\mathrm{M}(r,L)} \mathrm{Gr}(m_i, \mathcal{E}_{x_i}).$$

Example 2.7. More generally, if $\mathbf{a} = (a_i)$ is general and one a_i is sufficiently small, then forgetting one flag does not affect on the stability calculation. Thus, there is a forgetful morphism

$$\pi : \mathrm{M}_{(X,\mathbf{x})}(r, L, \mathbf{m}, \mathbf{a}) \to \mathrm{M}_{(X,\mathbf{x}')}(r, L, \mathbf{m}', \mathbf{a}')$$

where $\mathbf{x}' = \mathbf{x} \setminus \{x_i\}$, $\mathbf{m}' = \mathbf{m} \setminus \{m_i\}$, and $\mathbf{a}' = \mathbf{a} \setminus \{a_i\}$. This is a $\mathrm{Gr}(m_i, r)$-fibration.

Example 2.8. Fix $(X, \mathbf{x}) \in \mathcal{M}_{g,n}$. Let $\mathbf{x}' := \mathbf{x} \setminus \{x_k\}$. Let $\mathbf{m}' = (m_i)_{1 \leq i \leq k-1}$ and $\mathbf{a}' = (a_i)_{1 \leq i \leq k-1}$. Suppose that $m_k = 0$ or r. Then

$$\mathrm{M}_{(X,\mathbf{x})}(r, L, \mathbf{m}, \mathbf{a}) \cong \mathrm{M}_{(X,\mathbf{x}')}(r, L, \mathbf{m}', \mathbf{a}').$$

When one of $\mathbf{a}$ is sufficiently close to one, there is another contraction.

Proposition 2.9. *We use the notation in Example 2.8. For a general parabolic weight $\mathbf{a} = (a_i)$, assume that a_k is sufficiently close to one. Then there exists a morphism*

$$\pi_1 : \mathrm{M}_{(X,\mathbf{x})}(r, L, \mathbf{m}, \mathbf{a}) \to \mathrm{M}_{(X,\mathbf{x}')}(r, L(-(r - m_k)x_k), \mathbf{m}', \mathbf{a}').$$

Proof. It is sufficient to construct a morphism

$$\mathcal{M}_{(X,\mathbf{x})}(r, L, \mathbf{m}, \mathbf{a}) \to \mathcal{M}_{(X,\mathbf{x}')}(r, L(-(r - m_k)x_k), \mathbf{m}', \mathbf{a}')$$

between algebraic stacks.

Let $\widetilde{\mathbf{m}} = (\widetilde{m}_i)$ be a multiplicity such that $\widetilde{m}_i = m_i$ for $1 \leq i \leq k - 1$ and $\widetilde{m}_k = r$. By Example 2.8, there is a functorial isomorphism $\mathcal{M}_{(X,\mathbf{x})}(r, L(-(r - m_k)x_k), \widetilde{\mathbf{m}}, \mathbf{a}) \cong \mathcal{M}_{(X,\mathbf{x}')}(r, L(-(r - m_k)x_k), \mathbf{m}', \mathbf{a}')$. Thus, it is sufficient to show that there is a morphism

$$\mathcal{M}(r, L, \mathbf{m}, \mathbf{a}) \to \mathcal{M}(r, L(-(r - m_k)x_k), \widetilde{\mathbf{m}}, \mathbf{a}).$$

For a stable bundle $(E, V_\bullet) \in \mathcal{M}(r, L, \mathbf{m}, \mathbf{a})$, let E' be the kernel of the restriction map $E \to E|_{x_k} \to E|_{x_k}/V_{x_k}$. Then for each $i \neq k$, $E'|_{x_i}$ can be identified with $E|_{x_i}$. Set $V_i' = V_i$

under this identification. On the other hand, the restriction $f : E'|_{x_k} \to E|_{x_k}$ is a linear map with image V_k. We set $V'_k := f^{-1}(V_k) = E'|_{x_k}$. Then we obtain a parabolic bundle $(E', V'_\bullet) \in \mathcal{M}(r, L(-(r - m_k)x_k), \widetilde{\mathbf{m}}, \mathbf{a})$. Thus, we have a morphism

$$
\begin{aligned}
\mathcal{M}(r, L, \mathbf{m}, \mathbf{a}) &\to \mathcal{M}(r, L(-(r - m_k)x_k), \widetilde{\mathbf{m}}) \\
(E, V_\bullet) &\mapsto (E', V'_\bullet).
\end{aligned}
$$
(1)

We claim that $(E', V'_\bullet)$ is $\mathbf{a}$-semistable. Then the morphism in Equation (1) factors through $\mathcal{M}(r, L(-(r - m_k)x_k), \widetilde{\mathbf{m}}, \mathbf{a})$.

Suppose not. Then there is a parabolic subbundle $(F', W'_\bullet)$ of $(E', V'_\bullet)$ such that $\mu(F', W'_\bullet) > \mu(E', V'_\bullet)$. Let $\operatorname{rank} F' = s$, $\deg F' = e$, and $n_i = \dim W'_i$. Note that $n_k = s$.

Set $d = \deg L$. Then

$$
\begin{aligned}
\mu(E, V_\bullet) - \mu(E', V'_\bullet) &= \frac{d + \sum m_i a_i}{r} - \frac{d - (r - m_k) + \sum_{i \neq k} m_i a_i + r a_k}{r} \\
&= \frac{(r - m_k)(1 - a_k)}{r}.
\end{aligned}
$$
(2)

In general, F' is not a subbundle of E. But there is a subbundle F of E such that F/F' is a sheaf supported on x_k and $\dim(F/F')|_{x_k} = s - c$, where $c := \dim F|_{x_k} \cap V_{x_k}$. For the induced parabolic subbundle $(F, W_\bullet)$ of $(E, V_\bullet)$,

$$
\begin{aligned}
\mu(F, W_\bullet) - \mu(F', W'_\bullet) &= \frac{e + (s - c) + \sum_{i \neq k} a_i n_i + a_k c}{s} - \frac{e + \sum_{i \neq k} a_i n_i + a_k s}{s} \\
&= \frac{(s - c)(1 - a_k)}{s}.
\end{aligned}
$$
(3)

By combining (2) and (3), we have

$$
\mu(E, V_\bullet) - \mu(F, W_\bullet) = \mu(E', V'_\bullet) - \mu(F', W'_\bullet) + (1 - a_k)\left(\frac{r - m_k}{r} - \frac{s - c}{s}\right).
$$

Note that $\mu(E', V'_\bullet) - \mu(F', W'_\bullet)$ is independent from a_k, as the coefficient of a_k in each term is one. Thus, if a_k is sufficiently close to one, then the last term is negligible. By the assumption, $\mu(E', V'_\bullet) - \mu(F', W'_\bullet) < 0$ and hence the left hand side is also negative. It violates the stability of $(E, V_\bullet)$ and obtain a contradiction. $\qquad\square$

Remark 2.10. The morphism in Proposition 2.9 can be understood as a *generalized Hecke correspondence*. When $d = k = 1$ and $m = r - 1$, up to taking the dual bundle, we obtain the classical Hecke correspondence in the sense of [NR75, Section 4]. A difference in the $d > 1$ case is that $\mathrm{M}(r, L, m, a)$ does not admit morphisms to both $\mathrm{M}(r, L)$ and $\mathrm{M}(r, L(-x))$, so we need a birational modification on $\mathrm{M}(r, L, m, a)$. It can be explained in terms of parabolic wall-crossing as Section 3 below.

3. WALL-CROSSING

This section reviews how $M(r, L, \mathbf{m}, \mathbf{a})$ changes as $\mathbf{a}$ varies.

3.1. **General theory.** Let k be the number of parabolic points. Recall that a parabolic weight is, a length k sequence of rational number $\mathbf{a} = (a_i)$ with $0 < a_i < 1$. The closure of the set of parabolic weights is $[0, 1]^k \subset \mathbb{R}^k$.

There is a wall-chamber decomposition of $[0, 1]^k$. A parabolic bundle $(E, V_\bullet) \in M(r, L, \mathbf{m}, \mathbf{a})$ is strictly semi-stable if and only if there is a maximal destabilizing subbundle $(F, W_\bullet)$ such that $\mu(F, W_\bullet) = \mu(E, V_\bullet)$. More explicitly, this is true only if

$$(4) \qquad \frac{e + \sum n_i a_i}{s} = \frac{d + \sum m_i a_i}{r}$$

for some $0 < s < r$, $e \in \mathbb{Z}$, and $\mathbf{n} = (n_i)$. Here s is the rank, e is the degree, and $\mathbf{n}$ is the multiplicity of $(F, W_\bullet)$. So we require that $n_i \le \min\{s, m_i\}$. Let $\Delta(s, e, \mathbf{n})$ be the set of weights that satisfy (4). Note that this is an intersection of a hyperplane and $[0, 1]^k$. We call $\Delta(s, e, \mathbf{n})$ a *wall* if it is nonempty. We also obtain

$$(5) \qquad \Delta(s, e, \mathbf{n}) = \Delta(r - s, d - e, \mathbf{m} - \mathbf{n}).$$

Note that $\Delta(s, e, \mathbf{n}) = \Delta(ts, te, t\mathbf{n})$ if $ts < r$ for some $t > 1$. We call such a wall a *multiple wall*, and otherwise, it is a *simple wall*. A wall $\Delta(s, e, \mathbf{n})$ is simple if and only if $\{s, e, n_i\}$ are coprime and $\{r - s, d - e, m_i - n_i\}$ are coprime.

The stability changes only if a parabolic weight $\mathbf{a}$ lies on one of the walls. So for each open chamber $C \subset [0, 1]^k$, for any $\mathbf{a}, \mathbf{a}' \in C$, $M(r, L, \mathbf{m}, \mathbf{a}) \cong M(r, L, \mathbf{m}, \mathbf{a}')$. The stability coincides with the semistability if $\mathbf{a} \in (0, 1)^k \setminus \bigcup \Delta(s, e, \mathbf{n})$.

Let

$$(6) \qquad M(r, L, \mathbf{m}, \mathbf{a}^-) \dashleftarrow \dashrightarrow M(r, L, \mathbf{m}, \mathbf{a}^+)$$
$$\pi_- \searrow \qquad \swarrow \pi_+$$
$$M(r, L, \mathbf{m}, \mathbf{a})$$

be a wall-crossing. Suppose that $\mathbf{a}$ is a general point of $\Delta(s, e, \mathbf{n})$, and $\mathbf{a}^-$ and $\mathbf{a}^+$ are two very close weights on the opposite chambers. The contraction maps $\pi_\pm$ are birational surjections. Let $Y^\pm$ be the exceptional locus on $M(r, L, \mathbf{m}, \mathbf{a}^\pm)$ and let $Y := \pi_\pm(Y^\pm)$. The subvarieties $Y^\pm$ are called the *wall-crossing centers*. For our purpose, we need a lower bound of the codimension of $Y^\pm$. Observe that the parabolic bundles in Y^- are stable with respect to $\mathbf{a}^-$, but unstable with respect to $\mathbf{a}^+$. Thus, Y^- parametrizes unstable parabolic bundles with respect to some weight. The codimension of the unstable locus is estimated in [Sun00]. For an outline of the proof, see also [MY20, Section 3.2].

Theorem 3.1 ([Sun00, Proposition 5.1]). *In $\mathcal{M}(r, L, \mathbf{m})$, the codimension of the unstable locus with respect to a weight $\mathbf{a}$ is at least $(r - 1)(g - 1) + 1$.*

Corollary 3.2. *The codimension of the wall-crossing center is at least $(r-1)(g-1)+1$. In particular, if $g \geq 2$, every wall-crossing is a flip.*

We say a wall-crossing is *simple* if:

(1) The wall $\Delta(s, e, \mathbf{n})$ is a simple wall and;
(2) $\mathbf{a} \in \Delta(s, e, \mathbf{n})$ is on a unique wall.

A simple wall-crossing has an explicit description. The wall-crossing centers $Y^{\pm}$ are irreducible and their image $Y \cong \mathrm{M}(s, e, \mathbf{n}, \mathbf{a}) \times_{\mathrm{Pic}(X)} \mathrm{M}(r - s, d - e, \mathbf{m} - \mathbf{n}, \mathbf{a})$ is a smooth variety. For $(E, V_{\bullet}) \in Y^{-}$, there is a unique maximal $\mathbf{a}$-destabilizing subbundle $(E^{-}, V_{\bullet}^{-}) \in \mathrm{M}(s, e, \mathbf{n}, \mathbf{a})$, which fits into an exact sequence

$$0 \to (E^{-}, V_{\bullet}^{-}) \to (E, V_{\bullet}) \to (E^{+}, V_{\bullet}^{+}) \to 0$$

of parabolic bundles. The map π_{-} is restricted to the map $Y^{-} \to Y$, which sends $(E, V_{\bullet})$ to the S-equivalence class of $(E, V_{\bullet})$, which is the class of $(E^{-}, V_{\bullet}^{-}) \oplus (E^{+}, V_{\bullet}^{+})$. We denote this class by $((E^{-}, V_{\bullet}^{-}), (E^{+}, V_{\bullet}^{+}))$. Conversely, if $x := ((E^{-}, V_{\bullet}^{-}), (E^{+}, V_{\bullet}^{+}))$ is a general point in Y so that both $(E^{-}, V_{\bullet}^{-})$ and $(E^{+}, V_{\bullet}^{+})$ are stable, then the fiber $\pi_{-}^{-1}(x)$ is a projective space $\mathbb{P}(\mathrm{Ext}^{1}((E^{+}, V_{\bullet}^{+}), (E^{-}, V_{\bullet}^{-}))^{*})$ (see [Yok95, Section 1] for the derived functors on the category of parabolic sheaves). A functorial description is possible. Let $(\mathcal{E}^{-}, \mathcal{V}_{\bullet}^{-})$ (resp. $(\mathcal{E}^{+}, \mathcal{V}_{\bullet}^{+})$) be the Poincaré family over $\mathrm{M}(s, e, \mathbf{n}, \mathbf{a})$ (resp. $\mathrm{M}(r - s, d - e, \mathbf{m} - \mathbf{n}, \mathbf{a})$). The standard GIT construction and the descent method imply the existence of Poincaré bundle ([New78, Chapter 5], [HL10, Section 4.6]). Then $Y^{-} \cong \mathbb{P}(R^{1}\pi_{-*}\mathcal{P}ar\mathcal{H}om((\mathcal{E}^{+}, \mathcal{V}_{\bullet}^{+}), (\mathcal{E}^{-}, \mathcal{V}_{\bullet}^{-}))^{*})$ and $Y^{+} \cong \mathbb{P}(R^{1}\pi_{+*}\mathcal{P}ar\mathcal{H}om((\mathcal{E}^{-}, \mathcal{V}_{\bullet}^{-}), (\mathcal{E}^{+}, \mathcal{V}_{\bullet}^{+}))^{*})$. In particular, they are projective bundles over Y. Finally, it is well-known that the blow-up of $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}^{-})$ along Y^{-} is isomorphic to the blow-up of $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}^{+})$ along Y^{+}.

Over a multiple wall, the wall-crossing is more complicated. Consider the wall-crossing over a unique wall $\Delta(s, e, \mathbf{n})$, which is a multiple wall, such that there is $t \in \mathbb{Z}$ such that $\Delta(s, e, \mathbf{n}) = \Delta(ts, te, t\mathbf{n})$ with $ts < r$. Then the wall-crossing centers $Y^{\pm}$ has the image $Y \cong \bigcup \mathrm{M}(ts, te, t\mathbf{n}, \mathbf{a}) \times_{\mathrm{Pic}(X)} \mathrm{M}(r - ts, d - te, \mathbf{m} - t\mathbf{n}, \mathbf{a})$ and it is a reducible variety with nontrivial intersections and singularities. However, a general point of each irreducible component parametrizes a pair of stable parabolic bundles, thus over a general point, the wall-crossing is the same one with the case of a simple wall-crossing in the previous paragraph.

3.2. **GIT construction of moduli space.** The moduli spaces $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ can be constructed by GIT and each wall-crossing is indeed obtained by the variation of GIT. We review a GIT construction of $\mathrm{M}(r, L, \mathbf{a})$ after Bhosle [Bho89].

We fix a degree one line bundle $\mathcal{O}(1)$ on X. Fix an integer $m \gg 0$ such that $\mathrm{H}^{1}(E(m)) = 0$ and $E(m)$ is globally generated for every $(E, V_{\bullet}) \in \mathrm{M}(r, L, \mathbf{a})$. (Indeed, we may find such an m that works for all $\mathbf{a}$.) Let $\chi := h^{0}(E(m)) = d + r(m + 1 - g)$ and let $Q := \mathrm{Quot}(\mathcal{O}_{X}^{\chi})$ be

the quot scheme parametrizing quotients of $\mathcal{O}_X^\chi$ whose Hilbert polynomial is that of $E(m)$. Let $R \subset Q$ be a locally closed subscheme parametrizing the quotients $\mathcal{O}_X^\chi \xrightarrow{\varphi} F \to 0$ such that $\mathrm{H}^1(F) = 0$, $\mathrm{H}^0(\mathcal{O}_X^\chi) \xrightarrow[\cong]{\varphi} \mathrm{H}^0(F)$, F is locally free, and $\det F \cong L(rm)$. Let $\mathcal{O}_{R \times X}^\chi \to \mathcal{F} \to 0$ be the universal quotient over $R \times X$. For $x_1, x_2, \cdots, x_k \in X$, let

$$\widetilde{R} := \times_R \mathrm{Gr}(m_i, \mathcal{F}|_{x_i})$$

be the fiber product of Grassmannian bundles over R. There is a natural SL_χ-action on $\widetilde{R}$. Note that $\widetilde{R}$ parametrizes pairs $([\mathcal{O}_X^\chi \xrightarrow{\varphi} F \to 0], \{V_i \subset F|_{x_i}\})$.

We can make an explicit SL_χ-equivariant embedding of $\widetilde{R}$ into a product of elementary varieties as the following. Let $Z := \mathbb{P}\mathrm{Hom}(\wedge^r \mathbb{C}^\chi, \mathrm{H}^0(L(rm)))^*$. Then for any $[\mathcal{O}_X^\chi \xrightarrow{\varphi} F \to 0] \in R$,

$$\wedge^r \mathbb{C}^\chi \xrightarrow[\cong]{\wedge^r \varphi} \wedge^r \mathrm{H}^0(F) \to \mathrm{H}^0(\wedge^r F) \cong \mathrm{H}^0(L(rm))$$

gives a point in Z. Furthermore, for each x_i, by taking the inverse image $\psi_i^{-1}(V_i)$ for

$$\psi_i : \mathbb{C}^\chi \cong \mathrm{H}^0(F) \to F|_{x_i},$$

we obtain an element in $\mathrm{Gr}(\chi - r + m_i, \chi)$ for x_k. Therefore, we have an SL_χ-equivariant morphism

(7) $$\widetilde{R} \to \times_Z \mathrm{Gr}(\chi - r + m_i, \chi)$$

$$([\mathcal{O}_X^\chi \xrightarrow{\varphi} F \to 0], \{V_i \subset F|_{x_i}\}) \mapsto (\wedge^r \varphi, \{\psi_i^{-1}(V_i)\}).$$

In [Tha96, Section 7], it was shown that this morphism is indeed an embedding. In [Bho89], Bhosle calculated an explicit linearization $A(\mathbf{a})$, depending on $\mathbf{a}$, which gives $\widetilde{R}^{ss}(A(\mathbf{a}))/\mathrm{SL}_\chi \cong \mathrm{M}(r, L, \mathbf{a})$.

In summary, the wall-crossing of $\mathbf{a}$-stability is obtained by the variation of GIT.

3.3. **Mori's program.** The wall-crossing picture can be incorporated with projective birational geometry of $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ in the nicest way. Let $\mathbf{a}$ be a general parabolic weight. Then every rational contraction of $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ can be obtained in terms of wall-crossings, forgetful maps, and generalized Hecke correspondences. Proposition 3.3 can be recovered from [MY20, Section 5], but for the readers' convenience, we give the proof here.

Proposition 3.3. *Let $\mathbf{a} \in (0, 1)^k$ be a general parabolic weight. Then there is a linear isomorphism between a cone over $[0, 1]^k$ and the effective cone $\mathrm{Eff}(\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}))$ of divisors.*

Proof. By the GIT construction of $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ as an SL_χ-quotient in Section 3.2, all of them can be constructed as a GIT quotient of the same smooth variety $\widetilde{R}$ with various linearizations. Furthermore, the parabolic weights depend linearly on the choice of linearization. In particular, there is a linear embedding $(0, 1)^k \to \mathrm{N}^{1, \mathrm{SL}_\chi}(\widetilde{R})_\mathbb{R}$, where $\mathrm{N}^{1, \mathrm{SL}_\chi}(\widetilde{R})$ is the space of numerical classes of SL_χ-linearized line bundles on $\widetilde{R}$. Since the character

group of SL_χ is trivial and $\widetilde{R}$ is normal, $\mathrm{N}^{1,\mathrm{SL}_\chi}(\widetilde{R})_\mathbb{R} \to \mathrm{N}^1(\widetilde{R})_\mathbb{R}$ is bijective [MFK94, Corollary 1.6]. Applying Kempf's descent lemma [DN89, Theorem 2.3], we have a surjective linear map $\mathrm{N}^1(\widetilde{R})_\mathbb{R} \cong \mathrm{N}^{1,\mathrm{SL}_\chi}(\widetilde{R})_\mathbb{R} \to \mathrm{N}^1(\widetilde{R}/\!/_L\mathrm{SL}_\chi)_\mathbb{R} = \mathrm{N}^1(\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}))_\mathbb{R}$. This map is bijective because the unstable locus has codimension ≥ 2 (Theorem 3.1). In summary, there is a linear embedding $(0, 1)^k \to \mathrm{N}^{1,\mathrm{SL}_\chi}(\widetilde{R})_\mathbb{R} \to \mathrm{N}^1(\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}))_\mathbb{R}$, which induces a linear embedding of a cone over $[0, 1]^k$ to $\mathrm{N}^1(\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}))_\mathbb{R}$.

Now we show that the cone over the closure $[0, 1]^k$ of $(0, 1)^k$ can be identified with $\mathrm{Eff}(\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}))$. Recall that for any effective divisor D (or equivalently, a line bundle $\mathcal{O}(D)$) of a normal $\mathbb{Q}$-factorial projective variety V, we may associate a rational contraction $V \dashrightarrow V(D)$ where

$$V(D) := \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(V, \mathcal{O}(mD)).$$

Conversely, any rational contraction of V can be obtained in this way. If $D \in \mathrm{int}\ \mathrm{Eff}(V)$, then $V \dashrightarrow V(D)$ is a birational map and if $D \in \partial\mathrm{Eff}(V)$, $V \dashrightarrow V(D)$ is a contraction with positive dimensional general fibers.

Note that on the boundary $\partial[0, 1]^k$, one of the coordinates must be either zero or one. In the first case, we can obtain a rational contraction $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}) \to \mathrm{M}(r, L, \mathbf{m}', \mathbf{a}')$ in Example 2.7. In the latter case, we have a generalized Hecke modification in Proposition 2.9. All of them are contractions with positive dimensional fibers, so they must be associated with divisors on the boundary of the effective cone. Since the effective cone is convex, this is sufficient to obtain the result. $\qquad\square$

4. NEF VECTOR BUNDLES

Let $\mathcal{E}$ be the normalized Poincaré bundle over $X \times \mathrm{M}(r, L)$. Recall that for any $x \in X$, $\mathcal{E}_x$ is the vector bundle on $\mathrm{M}(r, L)$ obtained by restricting $\mathcal{E}$ to $x \times \mathrm{M}(r, L)$. We prove the nefness of $\mathcal{E}_x$ and some other positivity results. A key ingredient is the birational geometry of the moduli space of parabolic bundles with one parabolic point.

Theorem 4.1. *The restricted Poincaré bundle $\mathcal{E}_x$ is a nef vector bundle.*

Remark 4.2. The case of $d = 1$ of Theorem 4.1 is shown in [Nar17, Proposition 3.3] and [BM19, Lemma 13]. So we assume $d > 1$. When $d = 1$, the numerical computation in Lemma 4.4 is still valid. But we have $\ell = 1$ and thus, $a = 1$. Therefore, the first wall-crossing is precisely the fibration $\mathrm{M}(r, L, r - 1, \epsilon) \to \mathrm{M}(r, L(-x))$ in Proposition 2.9, that is, a contraction in the Hecke correspondence. In particular, there is no flip.

We obtain another nef bundle immediately.

Corollary 4.3. *The vector bundle $\mathcal{E}_x^* \otimes \Theta$ is nef.*

Proof. Fix a line bundle A of degree 1 on X. Consider the vector bundle $\mathcal{E}^* \otimes p^*A \otimes q^*\Theta$ on $X \times \mathrm{M}(r, L)$, where $p : X \times \mathrm{M}(r, L) \to X$ and $q : X \times \mathrm{M}(r, L) \to \mathrm{M}(r, L)$ are two projections. From the isomorphism $\mathrm{M}(r, L) \cong \mathrm{M}(r, L^*) \cong \mathrm{M}(r, A^r \otimes L^*)$, we see that $\mathcal{E}^* \otimes p^*A \otimes q^*\Theta$ is the normalized Poincaré bundle on $X \times \mathrm{M}(r, A^r \otimes L^*) \cong X \times \mathrm{M}(r, L)$. The restriction of $\mathcal{E}^* \otimes p^*A \otimes q^*\Theta$ to $x \times \mathrm{M}(r, L)$ is isomorphic to $\mathcal{E}_x^* \otimes \Theta$. From Theorem 4.1, we see that $\mathcal{E}_x^* \otimes \Theta$ is nef. $\qquad\square$

From now on, we prove the nefness of $\mathcal{E}_x$. By definition, we need to show that $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$ is nef. Observe that $\mathbb{P}(\mathcal{E}_x) \cong \mathrm{M}(r, L, r-1, \epsilon)$ for some very small $\epsilon > 0$ (Example 2.6). Here we use $\mathrm{M}(r, L, r-1, a)$ for the place $\mathrm{M}(r, L, (r-1), (a))$.

We explicitly analyze the first wall-crossing of the moduli space $\mathrm{M}(r, L, r-1, \epsilon)$ by increasing $\epsilon \to 1$. Recall that ℓ is a positive integer such that $\ell d \equiv 1 \bmod r$ and $0 < \ell < r$.

Lemma 4.4. *Let a be the smallest parabolic weight on a wall. Then $a = 1/\ell$. Furthermore, a maximal destabilizing subbundle has rank $t\ell$ and degree te for some $t \in \mathbb{Z}$ and an integer e satisfying $\ell d - re = 1$.*

Proof. Let $\Delta(s, e, n)$ be a wall. Note that n is either s or $s - 1$. By Equation (5), exchanging s by $r - s$ if necessary, we may assume that $n = s$. Then from $(e + sa)/s = (d + (r-1)a)/r$, we have $a = (sd - re)/s$. Since $(r, d) = 1$, we can find a unique positive $0 < s < r$ and $e \in \mathbb{Z}$ such that $sd - re = 1$, which is ℓ.

We claim that $a = (\ell d - re)/\ell = 1/\ell$ provides the first wall. Consider a wall $a' = (s'd - re')/s'$. Setting $t := s'd - re'$, $s'd \equiv t \bmod r$. On the other hand, $t\ell d \equiv t \bmod r$. So if $t\ell < r$, from the invertibility of d in $\mathbb{Z}/r\mathbb{Z}$, $s' = t\ell$ and $e' = te$. Then $a' = t/t\ell = 1/\ell = a$. If $t\ell \geq r$, there is a unique positive integer u such that $0 < s' = t\ell - ur < r$. Then $a' = t/s' = t/(t\ell - ur) > t/t\ell = 1/\ell$.

This computation tells us that $\Delta(\ell, e, \ell) = \Delta(s', e', s')$ only if $t\ell < r$ and $(s', e') = (t\ell, te)$. So we obtain the last assertion. $\qquad\square$

We have the following diagram:

(8)
$$\mathbb{P}(\mathcal{E}_x) = \mathrm{M}(r, L, r-1, \epsilon)$$

$\pi \swarrow \qquad\qquad \searrow \pi_-$

$$\mathrm{M}(r, L) \qquad\qquad\qquad \mathrm{M}(r, L, r-1, 1/\ell)$$

The first map π is a projective bundle and π_- is a small contraction by Corollary 3.2. And $\rho(\mathbb{P}(\mathcal{E}_x)) = \rho(\mathrm{M}(r, L)) + 1 = 2$. Since $\rho(\mathrm{M}(r, L, r-1, 1/\ell)) < \rho(\mathrm{M}(r, L, r-1, \epsilon)) = 2$, $\rho(\mathrm{M}(r, L, r-1, 1/\ell)) = 1$. Let A be an ample generator of $\mathrm{Pic}(\mathrm{M}(r, L, r-1, 1/\ell))$. Then $\pi^*\Theta$ and π_-^*A generates $\mathrm{N}^1(\mathbb{P}(\mathcal{E}_x))_{\mathbb{R}}$.

Definition 4.5. Fix a general point $((E^-, V^-), (E^+, V^+))$ in the component $\mathrm{M}(\ell, e, \ell, 1/\ell) \times_{\mathrm{Pic}(X)}$ $\mathrm{M}(r - \ell, d - e, r - \ell - 1, 1/\ell)$ of the wall-crossing center in $\mathrm{M}(r, L, r - 1, 1/\ell)$. Let C be a line class in the fiber $\pi_-^{-1}(((E^-, V^-), (E^+, V^+))) \cong \mathbb{P}(\mathrm{Ext}^1((E^+, V^+), (E^-, V^-))^*)$ (Section 3.1).

Lemma 4.6. *The intersection number $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1) \cdot C$ is zero.*

Proof. The image $\pi(\mathbb{P}(\mathrm{Ext}^1((E^+, V^+), (E^-, V^-)))^*) = \mathbb{P}(\mathrm{Ext}^1(E^+, E^-)^*) =: \mathbb{P}$ parametrizes isomorphism classes of extensions, and there is an exact sequence over $X \times \mathbb{P}$

$$0 \to p^* E^- \otimes q^* \mathcal{O}_{\mathbb{P}}(1) \to E \otimes q^* \mathcal{O}_{\mathbb{P}}(m) \to p^* E^+ \to 0$$

([Ram73, Lemma 2.3], [HL10, Example 2.1.12]). Here $p : X \times \mathbb{P} \to X$ and $q : X \times \mathbb{P} \to \mathbb{P}$ are two projections. If we restrict the exact sequence to $x \times C \cong x \times \mathbb{P}^1 \subset X \times \mathbb{P}$, we have

$$0 \to E_x^- \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to E_x \otimes \mathcal{O}_{\mathbb{P}^1}(m) \to E_x^+ \to 0.$$

Since $\mathcal{E}_x$ (and hence its restriction E_x) is normalized as $c_1(\mathcal{E}_x) = \Theta^\ell$ where $0 < \ell < r$, and E_x^- and E_x^+ are constant, $\ell = c_1(E_x^- \otimes \mathcal{O}_{\mathbb{P}^1}(1)) = c_1(E_x \otimes \mathcal{O}_{\mathbb{P}^1}(m)) = \ell + rm$. Thus, we have $m = 0$. Then $E_x|_{\pi(C)}$ fits in $0 \to \mathcal{O}_{\mathbb{P}^1}(1)^\ell \to E_x \to \mathcal{O}_{\mathbb{P}^1}^{r-\ell} \to 0$. A cohomology computation shows that this is a split extension. Therefore $\pi^{-1}(\pi(C)) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^\ell \oplus \mathcal{O}_{\mathbb{P}^1}^{r-\ell})$. The parabolic flag in E_x is determined by that of E_x^+ and it is fixed over C. This implies that $C \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^\ell \oplus \mathcal{O}_{\mathbb{P}^1}^{r-\ell})$. Therefore $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)|_C = \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1})}(1) = \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1) \cdot C = 0$. $\qquad\square$

Proof of Theorem 4.1. From $\rho(\mathbb{P}(\mathcal{E}_x)) = 2$, $\pi_-^* A \cdot C = 0$, and Lemma 4.6, we can conclude that $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$ and $\pi_-^* A$ are proportional. $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$ is a positive multiple of $\pi_-^* A$ because it intersects with the line class in a fiber of $\pi : \mathbb{P}(\mathcal{E}_x) \to \mathrm{M}(r, L)$ positively. Therefore $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$ is semi-ample, so $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$ and $\mathcal{E}_x$ are nef. $\qquad\square$

The following is essentially the same computation with [Nar17, Proposition 3.1].

Lemma 4.7. *Let $\mathcal{E}$ be the normalized Poincaré bundle on $X \times \mathrm{M}(r, L)$. Then*

$$\mathrm{Det}(\mathcal{E}^*) := \det(Rq_*(\mathcal{E}^*))^{-1} \cong \Theta^{\ell(1-g)-e}.$$

Proof. For the notational simplicity, let $\mathrm{M} := \mathrm{M}(r, L)$ and $\mathrm{M}' := \mathrm{M}(r, L^*)$. Then there is an isomorphism $\psi : \mathrm{M} \to \mathrm{M}'$. Since the isomorphism maps the unique ample generator $\Theta_{\mathrm{M}'}$ to Θ_{M}, by [Nar17, Proposition 2.1],

$$\Theta_{\mathrm{M}} = \psi^*(\Theta_{\mathrm{M}'}) = (\mathrm{Det}(\mathcal{E}^*))^r \otimes (\det(\mathcal{E}^*|_{\{x\} \times \mathrm{M}}))^{-d+r(1-g)} = \mathrm{Det}(\mathcal{E}^*)^r \otimes \Theta_{\mathrm{M}}^{-\ell(-d+r(1-g))}.$$

Thus, $\mathrm{Det}(\mathcal{E}^*) = \Theta_{\mathrm{M}}^{\frac{1+\ell(-d+r(1-g))}{r}} = \Theta_{\mathrm{M}}^{-e+\ell(1-g)}$. $\qquad\square$

Remark 4.8. Once we fix the parabolic points and the multiplicity, $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ are all birational, and for any general $\mathbf{a}$ and $\mathbf{a}'$, $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ and $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a}')$ are connected by finitely many flips (Section 3). In particular, their Picard groups are identified. For a notational simplicity, we will suppress all pull-backs (by flips and regular contractions) of

line bundles in our notation. For instance, when there is only one parabolic point x, there are two rational contractions $\pi : \mathrm{M}(r, L, r - 1, \epsilon) \to \mathrm{M}(r, L)$ and $\pi_1 : \mathrm{M}(r, L, r - 1, \epsilon) \dashrightarrow \mathrm{M}(r, L, r - 1, 1 - \epsilon) \to \mathrm{M}(r, L(-x))$. If there is no chance of confusion, we use $A \otimes B$ for $\pi^* A \otimes \pi_1^* B$. We denote $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(a)$ by $\mathcal{O}(a)$. Later, when there are two parabolic points, we will set $\mathcal{O}(a, b) := p_1^* \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(a) \otimes p_2^* \mathcal{O}_{\mathbb{P}(\mathcal{E}_y^*)}(b)$ where $p_1 : \mathbb{P}(\mathcal{E}_x) \times_{\mathrm{M}(r, L)} \mathbb{P}(\mathcal{E}_y^*) \to \mathbb{P}(\mathcal{E}_x)$ and $p_2 : \mathbb{P}(\mathcal{E}_x) \times_{\mathrm{M}(r, L)} \mathbb{P}(\mathcal{E}_y^*) \to \mathbb{P}(\mathcal{E}_y^*)$.

Lemma 4.9. *Let $k = (r, d - 1)$. On $\mathrm{M}(r, L, r - 1, a)$, $\Theta_{\mathrm{M}(r, L(-x))}^k = \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(r) \otimes \Theta_{\mathrm{M}(r, L)}^{1-\ell}$.*

Proof. The proof is a careful refinement of [Nar17, Proposition 3.3]. We may assume that a is sufficiently small, so $\mathrm{M}(r, L, r - 1, a) \cong \mathbb{P}(\mathcal{E}_x)$.

Let $p : X \times \mathbb{P}(\mathcal{E}_x) \to X$ and $q : X \times \mathbb{P}(\mathcal{E}_x) \to \mathbb{P}(\mathcal{E}_x)$ be two projections and $\pi : X \times \mathbb{P}(\mathcal{E}_x) \to X \times \mathrm{M}(r, L)$. Let $i_x : \mathbb{P}(\mathcal{E}_x) \cong x \times \mathbb{P}(\mathcal{E}_x) \hookrightarrow X \times \mathbb{P}(\mathcal{E}_x)$. Recall that there are two exact sequences that appear on the construction of the Hecke correspondence:

$$0 \to H(\mathcal{E}) \to \pi^\#(\mathcal{E}) \to p^* \mathcal{O}_x \otimes q^* \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1) \to 0$$

and

$$(9) \qquad 0 \to \pi^\#(\mathcal{E}^*) \to K(\mathcal{E}) \to i_{x*}(\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(-1) \otimes T_x) \to 0.$$

Here $\pi^\# \mathcal{E}$ is the pull-back of $\mathcal{E}$ to $X \times \mathbb{P}(\mathcal{E}_x)$ and T_x is the tangent space of X at x.

By [Nar17, Proposition 2.1],

$$\Theta_{\mathrm{M}(r, L(-x))}^k = \Theta_{\mathrm{M}(r, L^*(x))}^k = \mathrm{Det}(K(\mathcal{E}))^r \otimes (\det K(\mathcal{E})|_{z \times \mathbb{P}(\mathcal{E}_x)})^{1-d+r(1-g)}$$

for any $z \in X$. From (9), we have $\mathrm{Det}(\pi^\#(\mathcal{E}^*)) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1) = \mathrm{Det}(K(\mathcal{E}))$. Since $\mathrm{Det}(\pi^\#(\mathcal{E}^*)) = \pi^\# \mathrm{Det}(\mathcal{E})$ and $\pi^\#(\mathcal{E}^*)|_{z \times \mathbb{P}(\mathcal{E}_x)} \cong K(\mathcal{E})|_{z \times \mathbb{P}(\mathcal{E}_x)}$ for any $z \neq x$,

$$
\begin{aligned}
&\mathrm{Det}(K(\mathcal{E}))^r \otimes (\det K(\mathcal{E})|_{z \times \mathbb{P}(\mathcal{E}_x)})^{1-d+r(1-g))} \\
(10) \quad &= \mathrm{Det}(K(\mathcal{E}))^r \otimes (\det \pi^\#(\mathcal{E}^*)|_{z \times \mathbb{P}(\mathcal{E}_x)})^{1-d+r(1-g)} = \mathrm{Det}(K(\mathcal{E}))^r \otimes \Theta_{\mathrm{M}(r,L)}^{-\ell(1-d+r(1-g))} \\
&= \pi^\#(\mathrm{Det}(\mathcal{E}^*))^r \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(r) \otimes \Theta_{\mathrm{M}(r,L)}^{-\ell(1-d+r(1-g))} \\
&= \Theta_{\mathrm{M}(r,L)}^{r\ell(1-g)-re} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(r) \otimes \Theta_{\mathrm{M}(r,L)}^{-\ell(1-d+r(1-g))} = \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(r) \otimes \Theta_{\mathrm{M}(r,L)}^{1-\ell}.
\end{aligned}
$$

The second and the fourth equalities follow from the normalization of $\mathcal{E}$ and Lemma 4.7, respectively. $\qquad \square$

Corollary 4.10. *Let $k = (r, d - (r - 1))$. Then $\Theta_{\mathrm{M}(r, L(-(r-1)y))}^k = \mathcal{O}_{\mathbb{P}(\mathcal{E}_y^*)}(r) \otimes \Theta_{\mathrm{M}(r,L)}^{1+\ell}$.*

Proof. Within the identification $\mathrm{M}(r, L) \cong \mathrm{M}(r, L^*)$, the normalized Poincaré bundle over $\mathrm{M}(r, L^*)$ is $\mathcal{E}^* \otimes \Theta_{\mathrm{M}(r,L)}$, and $c_1(\mathcal{E}^* \otimes \Theta_{\mathrm{M}(r,L)}) = c_1(\Theta_{\mathrm{M}(r,L)}^{r-\ell})$. So $\mathrm{M}(r, L, 1, \epsilon) \cong \mathrm{M}(r, L^*, r - 1, \epsilon) \cong \mathbb{P}(\mathcal{E}_y^* \otimes \Theta_{\mathrm{M}(r,L)})$. When $a \to 1$, we obtain a contraction $\mathrm{M}(r, L^*, r - 1, a) \to \mathrm{M}(r, L^*(-y)) \cong \mathrm{M}(r, L(y)) \cong \mathrm{M}(r, L(-(r-1)y))$. By Lemma 4.9,

$$\Theta_{\mathrm{M}(r, L(-(r-1)y))}^k = \Theta_{\mathrm{M}(r,L^*)}^{1-(r-\ell)} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_y^* \otimes \Theta)}(r) = \Theta_{\mathrm{M}(r,L)}^{1-(r-\ell)} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_y^*)}(r) \otimes \Theta_{\mathrm{M}(r,L)}^r = \mathcal{O}_{\mathbb{P}(\mathcal{E}_y^*)}(r) \otimes \Theta_{\mathrm{M}(r,L)}^{1+\ell}.$$

$\square$

From (8), we obtain the nef cones of $\mathbb{P}(\mathcal{E}_x)$ and $\mathbb{P}(\mathcal{E}_x^*)$. The bigness in the statement follows from Lemma 4.9 and Corollary 4.10.

Corollary 4.11.   (1) *The nef cone of* $\mathbb{P}(\mathcal{E}_x) = \mathrm{M}(r, L, r-1, \epsilon)$ *is generated by* $\pi^*\Theta$ *and* $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$. *If* $d \neq 1$, $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$ *is big.*
  (2) *The nef cone of* $\mathbb{P}(\mathcal{E}_x^*) = \mathrm{M}(r, L, 1, \epsilon)$ *is generated by* $\pi^*\Theta$ *and* $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x^*)}(1) \otimes \pi^*\Theta$. *If* $d \neq r-1$, $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x^*)}(1) \otimes \pi^*\Theta$ *is big.*

5. MAIN EXAMPLE

From now on, we focus on the case that $k = 2$ and $\mathbf{m} = (r-1, 1)$. We set $\mathbf{x} = (x_1, x_2)$ and $\mathbf{a} = (a_1, a_2)$. We use $\mathrm{M}(r, L, \mathbf{a})$ for $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$.

5.1. Effective cone.
Let $\Delta(s, e, \mathbf{n})$ be a wall on $[0, 1]^2$ and let $\mathbf{a}$ be a general point on it. Let $(E, V_\bullet) \in Y \subset \mathrm{M}(r, L, \mathbf{a})$ be a general polystable parabolic bundle on the wall-crossing center. Then $(E, V_\bullet) \cong (F_1, W_{1\bullet}) \oplus (F_2, W_{2\bullet})$ and $\mu(E, V_\bullet) = \mu(F_1, W_{1\bullet}) = \mu(F_2, W_{2\bullet})$.

There are two possibilities. First of all, it is possible that one of F_i's (say F_1) has the largest possible intersection with the flags of E. That means, $\dim F_1|_{x_1} \cap V_1 = \dim F_1|_{x_1} = s$ and $\dim F_1|_{x_2} \cap V_2 = \dim V_2 = 1$. We have an equality

$$\frac{e + sa_1 + a_2}{s} = \frac{d + (r-1)a_1 + a_2}{r},$$

or equivalently, $sa_1 + (r-s)a_2 = sd - re$. The slope of the line on the (a_1, a_2)-plane is negative, so we will call the wall a *negative wall*. To intersect with the interior of $[0, 1]^2$, it is necessary that $0 < sd - re < r$. Since these walls are $\Delta(s, e, (s, 1)) = \Delta(r-s, d-e, (r-s-1, 0))$, they are simple walls (Section 3.1).

The other case is that $\dim F_1|_{x_1} \cap V_1 = \dim F_1|_{x_1} = s$ and $\dim F_1|_{x_2} \cap V_2 = 0$. Then

$$\frac{e + sa_1}{s} = \frac{d + (r-1)a_1 + a_2}{r},$$

so $sa_1 - sa_2 = sd - re$. The slope of the wall $\Delta(s, e, (s, 0))$ is one and we call it a *positive wall*. The nonempty intersection with $(0, 1)^2$ is equivalent to $-s < sd - re < s$. Since $(r, d) = 1$, $sd - re \neq 0$ and there is no wall passing through the origin. See Figure 1 for an example of the wall-chamber decomposition.

The line bundle Θ is the pull-back of Θ by $\mathrm{M}(r, L, \mathbf{a}) \dashrightarrow \mathrm{M}(r, L)$.

Lemma 5.1. *For a general weight* $\mathbf{a}$, *the dualizing bundle of* $\mathrm{M}(r, L, \mathbf{a})$ *is*

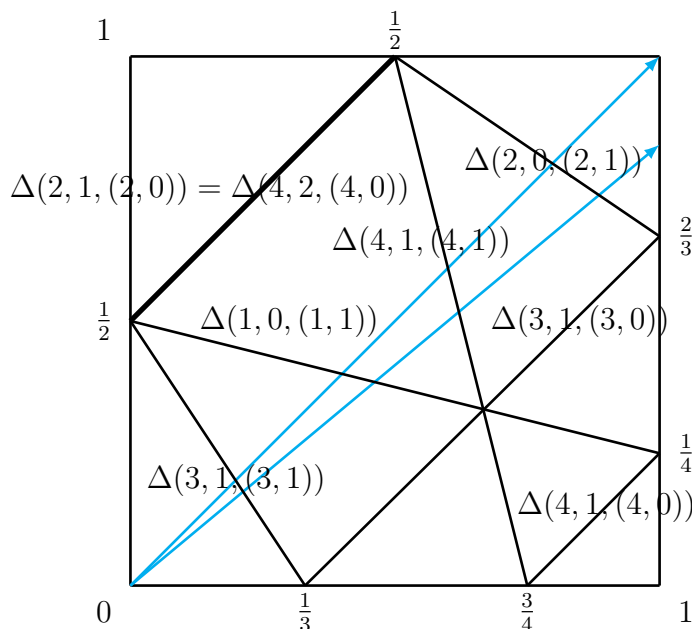$$\omega = \mathcal{O}(-r, -r) \otimes \Theta^{-2}.$$

FIGURE 1. The wall-chamber decomposition for $r = 5$ and $d = 2$. The thick line segment for $\Delta(2, 1, (2, 0))$ is a multiple wall. Two arrows denote diagonal and nearly diagonal wall crossing directions.

Proof. We may assume that $\mathbf{a}$ is sufficiently small and $\mathrm{M}(r, L, \mathbf{a}) \cong \mathbb{P}(\mathcal{E}_{x_1}) \times_{\mathrm{M}(r, L)} \mathbb{P}(\mathcal{E}_{x_2}^*)$. Apply the relative Euler sequence to $\mathbb{P}(\mathcal{E}_{x_1}) \to \mathrm{M}(r, L)$ and $\mathbb{P}(\mathcal{E}_{x_1}) \times_{\mathrm{M}(r, L)} \mathbb{P}(\mathcal{E}_{x_2}^*) \to \mathbb{P}(\mathcal{E}_{x_1})$. $\square$

Proposition 5.2. *Let* $\mathbf{a}$ *be a general weight. Then* $\mathrm{Eff}(\mathrm{M}(r, L, \mathbf{a}))$ *is generated by four extremal rays*

$$\Theta, \mathcal{O}(r, 0) \otimes \Theta^{1-\ell}, \mathcal{O}(0, r) \otimes \Theta^{1+\ell}, \mathcal{O}(r, r) \otimes \Theta.$$

Proof. By Proposition 3.3, it is sufficient to find four divisors associated to four extremal parabolic weights. For any big $\mathbb{Q}$-divisor $D \in \mathrm{int} \, \mathrm{Eff}(\mathrm{M}(r, L, \mathbf{a}))$, the associated birational model

$$\mathrm{M}(r, L, \mathbf{a})(D) := \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(\mathrm{M}(r, L, \mathbf{a}), \mathcal{O}(\lfloor mD \rfloor))$$

is $\mathrm{M}(r, L, \mathbf{a}')$. When $\mathbf{a}' = (0, 0)$, the associated rational contraction is $\mathrm{M}(r, L)$ and the associated divisor is a scalar multiple of Θ. When $\mathbf{a}' = (1/\ell, 0)$, by Section 4, the associated divisor is a multiple of $\mathcal{O}(1, 0)$. When $\mathbf{a}' = (1, 0)$, the associated rational contraction is $\mathrm{M}(r, L(-x))$ and the associated divisor is a scalar multiple of $\mathcal{O}(r, 0) \otimes \Theta^{1-\ell}$ by Lemma 4.9. For $\mathbf{a}' = (0, 1/(r - \ell))$, we have a multiple of $\mathcal{O}(0, 1) \otimes \Theta$. Finally, for $\mathbf{a}' = (0, 1)$, a multiple of $\mathcal{O}(0, r) \otimes \Theta^{1+\ell}$ is associated.

By an elementary computation, for each point $\mathbf{a}' = (a_1', a_2') \in [0,1]^2$, the associated divisor $\mathcal{O}(D) = \mathcal{O}(c_1, c_2) \otimes \Theta^d$ can be written as a positive multiple of $\Theta \otimes (\mathcal{O}(r,0) \otimes \Theta^{-\ell})^{a_1'} \otimes (\mathcal{O}(0,r) \otimes \Theta^{\ell})^{a_2'} = \mathcal{O}(ra_1', ra_2') \otimes \Theta^{1-\ell a_1' + \ell a_2'}$. A routine calculation shows that

$$(11) \qquad (a_1', a_2') = \left(\frac{c_1}{rd + \ell c_1 - \ell c_2}, \frac{c_2}{rd + \ell c_1 - \ell c_2} \right).$$

Thus, the last extremal ray, which is associated to $\mathbf{a}' = (1,1)$, is $\mathcal{O}(r,r) \otimes \Theta$. $\qquad\square$

5.2. Diagonal and nearly diagonal wall-crossings.

We say a wall crossing is a *diagonal* one if we cross a wall $\Delta(s, e, \mathbf{n})$ while the weight $\mathbf{a}$ is increasing along the line $a_1 = a_2$. A wall crossing is a *nearly diagonal* if we cross a wall $\Delta(s, e, \mathbf{n})$ while the weight $\mathbf{a}$ is increasing along $ra_1 = (r+1)a_2$. See Figure 1. We explicitly compute these wall-crossings.

Proposition 5.3. *All walls that appear in diagonal or nearly diagonal wall crossings are simple.*

Proof. For a negative wall $\Delta(s, e, (s,1)) = \Delta(r-s, d-e, (r-s-1,0))$, the greatest common divisor for both $\{s, e, s, 1\}$ and $\{r-s, d-e, r-s-1, 0\}$ are one. So every negative wall is simple. Then all multiple walls are positive walls, and hence parallel to the diagonal line $a_1 = a_2$. Such a wall is given by $a_1 - a_2 = (sd - re)/s$. Since $(r,d) = 1$, the right hand side is nonzero and it is disjoint from the diagonal line $a_1 = a_2$. Moreover, $|(sd - re)/s| \geq 1/s \geq 1/(r-1)$. It is a routine calculation to check that these walls do not intersect with $ra_1 = (r+1)a_2$ on $[0,1]^2$. $\qquad\square$

Remark 5.4. Several walls can meet at a weight during diagonal or nearly diagonal wall crossings. In this case, we may perturb the weight slightly, then the wall-crossing can be decomposed into a composition of several simple wall-crossings. Thus, we may assume that all wall-crossings are simple.

We can compute the dimension of all simple wall-crossing centers $Y_\pm$. For the theoretical background and details, see [MY21, Section 4]. Here we leave the computation for a negative wall $\Delta(s, e, (s,1))$. We keep the notation in the diagram (6).

For a point $((E^-, V_\bullet^-), (E^+, V_\bullet^+)) \in Y_0$, it is sufficient to evaluate $\dim \mathbb{P}(\mathrm{Ext}^1((E^\pm, V_\bullet^\pm), (E^\mp, V_\bullet^\mp))^*)$. By the Serre duality for parabolic bundles,

$$\mathrm{Ext}^1((E^+, V_\bullet^+), (E^-, V_\bullet^-)) \cong \mathrm{SParHom}((E^- \otimes \omega^*(-\mathbf{x}), V_\bullet^-), (E^+, V_\bullet^+))^*$$

[Yok95, Proposition 3.7]. There is an exact sequence of vector spaces [MY21, Section 4.2]

$$0 \to \mathrm{SParHom}((E^- \otimes \omega^*(-\mathbf{x}), V_\bullet^-), (E^+, V_\bullet^+)) \to \mathrm{Hom}(E^- \otimes \omega^*(-\mathbf{x}), E^+)$$

$$(12)$$
$$\to \bigoplus_{i=1}^{2} \mathrm{Hom}(E^- \otimes \omega^*(-\mathbf{x})|_{x_i}, E^+|_{x_i}) / \mathrm{N}_{x_i}((E^- \otimes \omega^*(-\mathbf{x}), V_\bullet^-), (E^+, V_\bullet^+)) \to 0,$$

where $N_x((E^-, V_\bullet^-), (E^+, V_\bullet^+))$ is the subspace of $\text{Hom}(E^-|_x, E^+|_x)$ which is strongly parabolic at x. Since the parabolic weight for V_i^- and V_i^+ are the same,

$$N_{x_i}((E^- \otimes \omega^*(-\mathbf{x}), V_\bullet^-), (E^+, V_\bullet^+))$$
$$= \{f \in \text{Hom}(E^- \otimes \omega^*(-\mathbf{x})|_{x_i}, E^+|_{x_i}) \mid f(E^- \otimes \omega^*(-\mathbf{x})|_{x_i}) \subset V_i^+, f(V_i^-) = 0\}.$$

From $\dim V_1^- = s = \text{rank}\, E^-$ and $\dim V_2^+ = 0$, $\dim N_{x_i}((E^- \otimes \omega^*(-\mathbf{x}), V_\bullet^-), (E^+, V_\bullet^+)) = 0$ for both $i = 1, 2$. Now

$$\dim \text{Ext}^1((E^+, V_\bullet^+), (E^-, V_\bullet^-))$$
(13)
$$= \dim \text{Hom}(E^- \otimes \omega^*(-\mathbf{x}), E^+) - 2s(r-s)$$
$$\geq \chi(E^{-*} \otimes E^+ \otimes \omega(\mathbf{x})) - 2s(r-s) = sd - re + s(r-s)(g-1).$$

By the same method, we obtain

(14)
$$\dim \text{Ext}^1((E^-, V_\bullet^-), (E^+, V_\bullet^+)) \geq re - sd + s(r-s)(g-1) + r,$$

so $\dim \text{Ext}^1((E^-, V_\bullet^-), (E^+, V_\bullet^+)) + \dim \text{Ext}^1((E^+, V_\bullet^+), (E^-, V_\bullet^-)) \geq 2s(r-s)(g-1)+r$. On the other hand,

$$\dim \mathbb{P}(\text{Ext}^1((E^-, V_\bullet^-), (E^+, V_\bullet^+))^*) + \dim \mathbb{P}(\text{Ext}^1((E^+, V_\bullet^+), (E^-, V_\bullet^-))^*)$$
$$= \dim \text{M}(r, L, \mathbf{a}) - \dim \text{M}(s, e, \mathbf{n}, \mathbf{a}) \times_{\text{Pic}^e(X)} \text{M}(r-s, d-e, \mathbf{m}-\mathbf{n}, \mathbf{a}) - 1$$
$$= 2s(r-s)(g-1) + r - 2.$$

Therefore, we obtain that (13) and (14) are indeed equalities. In summary:

Proposition 5.5. *Let $\Delta(s, e, (s, 1))$ be a negative wall. For the contraction map $\pi_\pm : \text{M}(r, L, \mathbf{a}^\pm) \to \text{M}(r, L, \mathbf{a})$ in (6), the dimension of the exceptional fiber of π_+ (resp. π_-) is $(re - sd) + s(r-s)(g-1) + r - 1$ (resp. $(sd - re) + s(r-s)(g-1) - 1$).*

6. COHOMOLOGY VIA WALL-CROSSING OF DERIVED CATEGORY

To prove the main theorems, a critical technical step is to identify the cohomology groups of the bundles on different birational models. Halpern-Leistner and Ballard-Favero-Katzarkov provided a systematic way to study the derived category of a variation of GIT [HL15, BFK19]. In this section, we review their works, in particular the quantization theorem. Technically, the results treat the derived category of a quotient stack. However, the following (well-known) lemma and its corollary show that it can be applied to the cohomology computation on the coarse moduli space. Let $\text{D}^{\text{per}}(\text{M})$ be the category of perfect complexes over M.

Lemma 6.1. *Let $\mathcal{M}$ be a smooth Artin stack and $\pi : \mathcal{M} \to \text{M}$ be its good moduli space. Then $L\pi^* : \text{D}^{\text{per}}(\text{M}) \to \text{D}^b(\mathcal{M})$ is fully faithful.*

Proof. We have an isomorphism $\mathcal{O}_{\mathrm{M}} \to R\pi_* \mathcal{O}_{\mathcal{M}}$ from the definition of a good moduli space. For any $F^\bullet, G^\bullet \in \mathrm{D}^{\mathrm{per}}(\mathrm{M})$ and $i \in \mathbb{Z}$, we have isomorphisms

$$\mathrm{Hom}(L\pi^* F^\bullet, L\pi^* G^\bullet[i]) \cong \mathrm{Hom}(F^\bullet, R\pi_* L\pi^* G^\bullet[i]) \cong \mathrm{Hom}(F^\bullet, G^\bullet[i])$$

by the adjunction formula and the projection formula ([HR17, Corollary 4.12], [Ols16, Proposition 9.3.6]). Therefore we see that $L\pi^*$ is fully faithful. □

Corollary 6.2. *We retain the same setup.*

  (1) If L is a vector bundle over M, then $\mathrm{H}^i(\mathrm{M}, L) \cong \mathrm{H}^i(\mathcal{M}, \pi^* L)$.
  (2) If M is smooth, $L\pi^* : \mathrm{D}^b(\mathrm{M}) \to \mathrm{D}^b(\mathcal{M})$ is fully faithful. In particular, for any $F^\bullet \in \mathrm{D}^b(\mathrm{M})$, $\mathrm{Hom}^i(\mathcal{O}_{\mathrm{M}}, F^\bullet) \cong \mathrm{Hom}^i(\mathcal{O}_{\mathcal{M}}, L\pi^* F^\bullet)$.

6.1. **Variation of GIT and derived category.** Let V be a smooth quasi projective variety equipped with a reductive group G-action and A be a linearization. The GIT quotient $V /\!/_A G$ is the good moduli space of the quotient stack $[V^{ss}(A)/G]$. Halpern-Leistner showed that, for a collection of integers $w = (w_i)$ for each Kempf-Ness stratum of the unstable locus, $\mathrm{D}^b([V/G])$ has a semiorthogonal decomposition

$$\mathrm{D}^b([V/G]) = \langle \mathrm{D}^b_{[V^{us}(A)/G]}([V/G])_{<w}, \mathbf{G}_w, \mathrm{D}^b_{[V^{us}(A)/G]}([V/G])_{\geq w} \rangle,$$

and moreover, the restriction functor $i^* : \mathbf{G}_w \to \mathrm{D}^b([V^{ss}(A)/G])$ is an equivalence of categories [HL15, Theorem 2.10].

From now on, we assume that there is only one unstable stratum S that is a smooth subvariety. It is determined by a one-parameter subgroup $\lambda(t)$ which minimizes the normalized weight $\mathrm{wt}_\lambda A/|\lambda|$ over the λ-fixed locus $Z \subset S$. Since a choice of w is arbitrary, we may set $w = 0$. Under this condition, $\mathbf{G}_w$ is characterized as the subcategory of complexes $F^\bullet$ such that the λ-weights of the hypercohomology $\mathcal{H}^*(F^\bullet|_Z)$ is supported on $[w, w + \eta)$ [HL15, Lemma 2.9]. Here η is the λ-weight of the top wedge product of $N^*_{S/V}|_Z$.

The following theorem is a key ingredient for our cohomology computation.

Theorem 6.3 (Quantization Theorem [HL15, Theorem 3.29]). *For $F^\bullet \in \mathrm{D}^b([V/G])$, suppose that the λ-weights of $\mathcal{H}^*(F^\bullet|_Z)$ are supported on $(-\infty, \eta)$. Then*

$$\mathrm{H}^i([V/G], F^\bullet) \cong \mathrm{H}^i([V^{ss}(A)/G], F^\bullet|_{[V^{ss}(A)/G]}).$$

We apply the above result to the variation of GIT setup. Let A_0 be a linearization such that $V^{ss}(A_0) \neq V^s(A_0)$. For a sufficiently small ϵ and a linearized ample line bundle A, let $A_\pm := A_0 \pm \epsilon A$. We assume that $V^{ss}(A_\pm) = V^s(A_\pm)$. Assume further that $V^s(A_\pm) = V^{ss}(A_0) \setminus S_\pm$ and $S_\pm$ are smooth irreducible varieties. If $\lambda_\pm$ are the one-parameter subgroups describing the Kempf-Ness strata $S_\pm$, then $\lambda_- = \lambda_+^{-1}$. Let $Z \subset S_+ \cap S_-$ be the $\lambda_\pm$-fixed locus, and $\eta_\pm$ be the $\lambda_\pm$-weight of the top wedge product of $N^*_{S_\pm/V}|_Z$.

Theorem 6.4 ([TT21, Theorem 3.15]). *If λ_--weights of $\mathcal{H}^*(F^\bullet|_Z)$ are supported on $(-\eta_+, \eta_-)$,*

$$\mathrm{H}^i([V^s(A_-)/G], F^\bullet|_{[V^s(A_-)/G]}) \cong \mathrm{H}^i([V^{ss}(A_0)/G], F^\bullet) \cong \mathrm{H}^i([V^s(A_+)/G], F^\bullet|_{[V^s(A_+)/G]}).$$

The first isomorphism follows from Theorem 6.3 and the second one is from the theorem and $\lambda_- = \lambda_+^{-1}$. In particular, for any line bundle E on $[V^{ss}(A_0)/G]$, if the magnitude of the λ_--weight is 'not too big,' then the cohomology of E on both sides of the wall can be identified.

6.2. **Weight computation.** All $\mathrm{M}(r, L, \mathbf{a})$ are constructed by GIT and they are connected by the variation of GIT (Section 3.2). For any simple wall-crossing, the technical assumptions we made in Section 6.1 hold. In this section, we compute the λ_--weight $\mathrm{wt}_{\lambda_-} F$ for every line bundle F and each simple wall that occurs during the diagonal and nearly diagonal wall-crossings.

Take a wall $\Delta(s, e, \mathbf{n})$ and pick a general weight $\mathbf{a} = (a_1, a_2) \in \Delta(s, e, \mathbf{n})$. Let A be an ample divisor associated to $\mathbf{a}$. Then $\widetilde{R}^{ss}(A)//\mathrm{SL}_\chi \cong \mathrm{M}(r, L, \mathbf{a})$. For two nearby weights $\mathbf{a}_\pm := (a_1 \pm \epsilon, a_2 \pm \epsilon)$, let $A_\pm$ be a line bundle such that $\mathrm{M}(r, L, \mathbf{a}_\pm) \cong \widetilde{R}^s(A_\pm)//\mathrm{SL}_\chi$.

Proposition 6.5. *Let $\Delta(s, e, \mathbf{n})$ be a simple wall, $\mathbf{a} \in \Delta(s, e, \mathbf{n})$, and A be an associated line bundle. Let λ_- be the one-parameter subgroup associated to the stratum $S_- := \widetilde{R}^{ss}(L_0) \setminus \widetilde{R}^s(L_-)$. Over the λ_--fixed locus $Z \subset S_-$,*

$$\mathrm{wt}_{\lambda_-} \Theta = -\chi(sd - re).$$

Proof. A general point in Z parametrizes a pair

$$([\mathcal{O}^{\chi^-} \oplus \mathcal{O}^{\chi^+} \xrightarrow{\varphi} E^-(m) \oplus E^+(m) \to 0], V_\bullet),$$

where E^- (resp. E^+) is a rank s (resp. $r - s$), degree e (resp. $d - e$) vector bundle, $\chi^\pm = \dim \mathrm{H}^0(E^\pm(m))$, and $\varphi = \varphi^- \oplus \varphi^+$ where $\varphi^\pm : \mathcal{O}^{\chi^\pm} \to E^\pm(m)$ and $\mathrm{H}^0(\mathcal{O}^{\chi^\pm}) \to \mathrm{H}^0(E^\pm(m))$ is a scalar multiple map. Because $\lambda_-(t)$ is a subgroup of SL_χ, $\lambda_-(t)$-weight on $E^+(m)$ is $u\chi^-$ and that on $E^-(m)$ is $-u\chi^+$ for some scalar u. Normalizing λ_-, we may assume that $u = 1$. By Riemann-Roch, it is straightforward to check that $\chi^- = e + sm + s(1 - g)$ and $\chi^+ = (d - e) + (r - s)m + (r - s)(1 - g)$.

For any vector bundle E over X, $\Theta|_{[E]} = \Theta|_{[E(m)]}$ is defined as

$$\mathrm{Det}(E(m))^r \otimes \det E(m)|_x^\chi = (\wedge^\chi \mathrm{H}^0(E(m))^*)^r \otimes \det E(m)|_x^\chi.$$

for some $x \in X$ [Nar17, Proposition 2.1]. Its λ_--weight is

$$-r(-\chi^+ \chi^- + \chi^- \chi^+) + \chi(-s\chi^+ + (r - s)\chi^-) = -\chi(sd - re).$$

$\square$

Lemma 6.6. *We identify the effective cone of $\mathrm{M}(r, L, \mathbf{m}, \mathbf{a})$ with the cone over $[0, 1]^2$, the closure of the space of parabolic weights (Proposition 3.3). Let $\tilde{A}$ be a line bundle on the same wall $\Delta(s, e, \mathbf{n})$, including A. Then $\mathrm{wt}_{\lambda_-} \tilde{A} = 0$.*

Proof. Recall that $\mathrm{M}(r, L, \mathbf{a}) = \widetilde{R}^{ss}(A)//\mathrm{SL}_\chi$. Since A descends to $\mathrm{M}(r, L, \mathbf{a})$, by Kempf's descent lemma [DN89, Theorem 2.3], for any closed SL_χ-orbit, the stabilizer group acts on the fiber of A trivially. In particular, at a point in Z, the stabilizer group λ_- acts trivially on the fiber, hence the λ_--weight is zero.

The same argument work for any small perturbation of A along $\Delta(s, e, \mathbf{n})$, as the stability and the stabilizer group λ_- does not change. By the linearity of weight, the same is true for arbitrary linear combination of A and its perturbation, which covers all $\Delta(s, e, \mathbf{n})$. $\square$

On the other hand, for a point $z := ([\mathcal{O}^{\chi^-} \oplus \mathcal{O}^{\chi^+} \to E^-(m) \oplus E^+(m) \to 0], V_\bullet) \in Z$, $N_{S_-/\widetilde{R}}|_z$ is identified with $\mathrm{Ext}^1((E^+, V_\bullet^+), (E^-, V_\bullet^-))$ and the action of λ_- on $N_{S_\pm/\widetilde{R}^{ss}(A)}|_z$ has weight $-\chi$ [Tha96, Section 7]. Thus, for a negative wall-crossing along $\Delta(s, e, (s, 1))$, we obtain

$$(15) \qquad \eta_- = \chi \dim \mathrm{Ext}^1((E^+, V_\bullet^+), (E^-, V_\bullet^-)) = \chi(sd - re + s(r - s)(g - 1)),$$

$$(16) \qquad \eta_+ = \chi \dim \mathrm{Ext}^1((E^-, V_\bullet^-), (E^+, V_\bullet^+)) = \chi(re - sd + s(r - s)(g - 1) + r).$$

by Proposition 5.5.

7. EMBEDDING OF DERIVED CATEGORY

In this section, we prove Theorem A.

7.1. **Bondal-Orlov criterion.**
Let $\mathcal{E}$ be the normalized Poincaré bundle over $X \times \mathrm{M}(r, L)$. Let $p : X \times \mathrm{M}(r, L) \to X$, $q : X \times \mathrm{M}(r, L) \to \mathrm{M}(r, L)$ be two projections. Consider the Fourier-Mukai transform

$$\begin{aligned} \Phi_\mathcal{E} : \mathrm{D}^b(X) &\to \mathrm{D}^b(\mathrm{M}(r, L)) \\ F^\bullet &\mapsto Rq_*(\mathcal{E} \otimes^L Lp^* F^\bullet). \end{aligned}$$

The Bondal-Orlov criterion [BO95, Theorem 1.1] provides the necessary and sufficient condition for the fully-faithfulness of a Fourier-Mukai transform between two smooth algebraic varieties. The next theorem is a version applied to $\Phi_\mathcal{E}$.

Theorem 7.1 (Bondal-Orlov criterion). *For each $x \in X$, let $\mathcal{E}_x$ be the restriction of the normalized Poincaré bundle on $\mathrm{M}(r, L)$. The functor $\Phi_\mathcal{E} : \mathrm{D}^b(X) \to \mathrm{D}^b(\mathrm{M}(r, L))$ is fully faithful if and only if the following conditions hold:*

(1) $\mathrm{H}^0(\mathrm{M}(r, L), \mathcal{E}_x \otimes \mathcal{E}_x^*) \cong \mathbb{C}$.
(2) $\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_x \otimes \mathcal{E}_x^*) = 0$ for $i \geq 2$.
(3) $\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^*) = 0$ for all $x_1 \neq x_2$ and all $i \in \mathbb{Z}$.

Proof of Theorem A. Items (1) and (2) of Theorem 7.1 are proved by [BM19, Section 3], by extending the work of Narasimhan and Ramanan in [NR75]. We show Item (3). Since

$$\mathrm{H}^i(\mathrm{M}(r, L, \mathbf{e}), \mathcal{O}(1,1)) \cong \mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^*)$$

for a small $\mathbf{e} = (\epsilon, \epsilon)$, it is sufficient to show that $\mathrm{H}^i(\mathrm{M}(r, L, \mathbf{e}), \mathcal{O}(1,1)) = 0$.

By Proposition 5.2 and the fact that there is no divisorial contraction on the wall-crossing (Proposition 5.5), there is a parabolic weight $\mathbf{a}$ such that $\mathcal{O}(r + 1, r + 1) \otimes \Theta^2$ is nef and big on $\mathrm{M}(r, L, \mathbf{a})$. Note that $\mathcal{O}(r + 1, r + 1) \otimes \Theta^2$ lies on a subspace generated by two extremal rays Θ and $\mathcal{O}(r, r) \otimes \Theta$ of $\mathrm{Eff}(\mathrm{M}(r, L, \mathbf{e}))$. To reach this line bundle, we may run a diagonal wall-crossing. By Proposition 5.3, we encounter only negative walls, which are all simple, to reach $\mathrm{M}(r, L, \mathbf{a})$ from $\mathrm{M}(r, L, \mathbf{e})$.

For each negative wall $\Delta(s, e, (s, 1))$, a parabolic weight $\mathbf{a}' = (a_1', a_2')$ lies on it if and only if it satisfies $sa_1' + (r - s)a_2' = sd - re$. Furthermore, if $\mathbf{a}'$ is on the diagonal, $a_1' = a_2' = (sd - re)/r$. Thus, by (11) in Section 5.1, the associated line bundle is a scalar multiple of

$$\mathcal{O}(sd - re, sd - re) \otimes \Theta.$$

The λ_--weight for this line bundle has to be zero by Lemma 6.6. By Proposition 6.5,

$$\mathrm{wt}_{\lambda_-} \mathcal{O}(1,1) = \chi.$$

On the other hand, since $g \geq 2$ and $0 < sd - re < r$, Equations (15) and (16) tell us $\eta_\pm > \chi$. Therefore, for any simple wall intersecting the diagonal, the λ_--weight of $\mathcal{O}(1,1)$ lies on $(-\eta_-, \eta_+)$. Theorem 6.4 implies that

$$\mathrm{H}^i(\mathrm{M}(r, L, \mathbf{a}'), \mathcal{O}(1,1)) \cong \mathrm{H}^i(\mathrm{M}(r, L, \mathbf{e}), \mathcal{O}(1,1))$$

for any $i \in \mathbb{Z}$ and any general diagonal weight $\mathbf{a}'$, including $\mathbf{a}$. For $i > 0$,

$$\mathrm{H}^i(\mathrm{M}(r, L, \mathbf{a}), \mathcal{O}(1,1)) = \mathrm{H}^i(\mathrm{M}(r, L, \mathbf{a}), \omega \otimes \mathcal{O}(r + 1, r + 1) \otimes \Theta^2) = 0$$

by Kawamata-Viehweg vanishing. And $\mathrm{H}^0(\mathrm{M}(r, L, \mathbf{a}), \mathcal{O}(1,1)) = 0$ since $\mathcal{O}(1,1) \notin \mathrm{Eff}(\mathrm{M}(r, L, \mathbf{a}))$. $\square$

8. Vanishing of cohomology

We prove the following vanishing result, which is used in both the computation of a semiorthogonal decomposition of $\mathrm{D}^b(\mathrm{M}(r, L))$ and the construction of ACM bundles.

Theorem 8.1. *For any $x \in X$ and $j \geq -1$, $\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_x \otimes \Theta^j) = 0$ for all $i > 0$.*

Proof. We divide the proof into several steps.

Step 1. Observe that

$$\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_x \otimes \Theta^j) \cong \mathrm{H}^i(\mathrm{M}(r, L, \mathbf{e}), \mathcal{O}(1, 0) \otimes \Theta^j) \cong \mathrm{H}^i(\mathrm{M}(r, L, \mathbf{e}), \mathcal{O}(r + 1, r) \otimes \Theta^{j+2} \otimes \omega).$$

Since
$$\mathcal{O}(r+1,r)\otimes\Theta^{j+2}=(\mathcal{O}(r,r)\otimes\Theta)\otimes\Theta^{\frac{\ell-1}{r}+(j+1)}\otimes(\mathcal{O}(r,0)\otimes\Theta^{1-\ell})^{\frac{1}{r}},$$
for $j\geq-1$, $\mathcal{O}(r+1,r)\otimes\Theta^{j+2}$ is on the effective cone of $\mathrm{M}(r,L,\mathbf{e})$ where $\mathbf{e}=((r+1)\epsilon,r\epsilon)$ for a small $0<\epsilon\ll1$. Moreover, unless $\ell=1$ and $j=-1$, it lies on the interior of the effective cone. (We will treat $\ell=1$, $j=-1$ case in **Step 4.**) Thus, if we take $\mathbf{a}$ as (possibly a slight perturbation of) the one associated to $\mathcal{O}(r+1,r)\otimes\Theta^{j+2}$, that is, $(\frac{r+1}{r(j+2)+\ell},\frac{r}{r(j+2)+\ell})$ by (11), $\mathcal{O}(r+1,r)\otimes\Theta^{j+2}$ is nef and big on $\mathrm{M}(r,L,\mathbf{a})$. By Kawamata-Viehweg vanishing, $\mathrm{H}^i(\mathrm{M}(r,L,\mathbf{a}),\mathcal{O}(r+1,r)\otimes\Theta^{j+2}\otimes\omega)=0$ for $i>0$. Thus, it is enough to show that $\mathrm{H}^i(\mathrm{M}(r,L,\mathbf{a}),\mathcal{O}(1,0)\otimes\Theta^j)\cong\mathrm{H}^i(\mathrm{M}(r,L,\mathbf{e}),\mathcal{O}(1,0)\otimes\Theta^j)$ for $j\geq-1$.

Step 2. We can move from $\mathbf{e}$ to $\mathbf{a}$ by a nearly diagonal wall-crossing (Section 5.2). All walls that we encounter are simple wall $\Delta(s,e,(s,1))$ (Proposition 5.3). The wall $\Delta(s,e,(s,1))$ is given by $sa_1+(r-s)a_2=sd-re$. So if the wall actually occurs while we move from $\mathbf{e}$ to $\mathbf{a}$,

$$(17)\qquad sd-re<s\frac{r+1}{r(j+2)+\ell}+(r-s)\frac{r}{r(j+2)+\ell}=\frac{r^2+s}{r(j+2)+\ell}.$$

Step 3. For each wall $\Delta(s,e,(s,1))$, let λ_- be the associated one-parameter subgroup. Combining Proposition 6.5 and Lemma 6.6, we have

$$\mathrm{wt}_{\lambda_-}\left(\mathcal{O}(1,0)\otimes\Theta^j\right)=\chi\left(\frac{s}{r}-\left(\frac{\ell}{r}+j\right)(sd-re)\right).$$

Since $0<sd-re<r$, for any wall, it is straightforward to check that $\mathrm{wt}_{\lambda_-}\left(\mathcal{O}(1,0)\otimes\Theta^j\right)\leq\mathrm{wt}_{\lambda_-}\left(\mathcal{O}(1,0)\otimes\Theta^{-1}\right)<\chi(s/r+(sd-re))<\eta_-$ for any $j\geq-1$ by comparing with (15).

Now we need to show that

$$(18)\qquad\qquad\qquad-\eta_+<\mathrm{wt}_{\lambda_-}\left(\mathcal{O}(1,0)\otimes\Theta^j\right)$$

for every wall $\Delta(s,e,(s,1))$ with the condition (17). Equation (18) is equivalent to

$$(\frac{\ell}{r}+j+1)(sd-re)<s(r-s)(g-1)+\frac{s}{r}+r.$$

Then

$$(\frac{\ell}{r}+j+1)(sd-re)<(\frac{\ell}{r}+j+1)\frac{r^2+s}{r(j+2)+\ell}=\frac{1}{r}(r(j+1)+\ell)\frac{r^2+s}{r(j+2)+\ell}$$
$$<\frac{r^2+s}{r}=r+\frac{s}{r}<s(r-s)(g-1)+\frac{s}{r}+r.$$

Thus, we have the inequality (18). Therefore, by Theorem 6.4,

$$\mathrm{H}^i(\mathrm{M}(r,L,\mathbf{a}),\mathcal{O}(1,0)\otimes\Theta^j)\cong\mathrm{H}^i(\mathrm{M}(r,L,\mathbf{e}),\mathcal{O}(1,0)\otimes\Theta^j).$$

Step 4. The only remaining case is that $\ell=1$ (hence $d=1$) and $j=-1$. We need to prove $\mathrm{H}^i(\mathrm{M}(r,L),\mathcal{E}_x\otimes\Theta^{-1})=0$ for $i>0$. Since $d=1$, there is a contraction map

$\pi_1 : \mathbb{P}(\mathcal{E}_x) = \mathrm{M}(r, L, r-1, \epsilon) \to \mathrm{M}(r, L(-x))$ (Remark 4.2). Then by [BM19, Lemma 13],

$$\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_x \otimes \Theta^{-1}) \cong \mathrm{H}^i(\mathbb{P}(\mathcal{E}_x), \mathcal{O}(1) \otimes \Theta^{-1}) = \mathrm{H}^i(\mathbb{P}(\mathcal{E}_x), \omega_{\mathbb{P}(\mathcal{E}_x)} \otimes \mathcal{O}(r+1))$$

$$= \mathrm{H}^i(\mathbb{P}(\mathcal{E}_x), \omega_{\mathbb{P}(\mathcal{E}_x)} \otimes \pi_1^* \Theta^{r+1}_{\mathrm{M}(r,L(-x))}).$$

By Kollár's vanishing [Kol86, Theorem 2.1], $R^i \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)}$ is torsion free for all i and

$$\mathrm{H}^k(\mathrm{M}(r, L(-x)), R^i \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)} \otimes \Theta^{r+1}_{\mathrm{M}(r,L(-x))}) = 0$$

for all $k > 0$. Since the Leray spectral sequence degenerates, $\mathrm{H}^0(\mathrm{M}(r, L(-x)), R^i \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)} \otimes \Theta^{r+1}_{\mathrm{M}(r,L(-x))}) \cong \mathrm{H}^i(\mathbb{P}(\mathcal{E}_x), \omega_{\mathbb{P}(\mathcal{E}_x)} \otimes \pi_1^* \Theta^{r+1}_{\mathrm{M}(r,L(-x))})$. On the other hand, over the stable locus $\mathrm{M}(r, L(-x))^s$, π_1 is a $\mathbb{P}^{r-1}$-fibration. Checking a general fiber, we can show that $R^i \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)} = 0$ for $i \neq r-1$. Thus, we obtain the desired vanishing for $1 \leq i \leq r-2$.

For $i = r-1$, since $R^{r-1} \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)}$ is a torsion free sheaf, we have an injective morphism $R^{r-1} \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)} \hookrightarrow (R^{r-1} \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)})^{\vee\vee}$. These two are isomorphic to $\omega_{\mathrm{M}(r,L(-x))}$ over an open subset of codimension ≥ 2 [Har77, Exercise III.8.4] and the latter is reflexive. Since $\mathrm{M}(r, L(-x))$ is locally factorial [DN89, Theorem A], $(R^{r-1} \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)})^{\vee\vee} \cong \omega_{\mathrm{M}(r,L(-x))} \cong \Theta^{-2r}_{\mathrm{M}(r,L(-x))}$ [DN89, Theorem F]. We have

$$\mathrm{H}^0(\mathrm{M}(r, L(-x)), R^{r-1} \pi_{1*} \omega_{\mathbb{P}(\mathcal{E}_x)} \otimes \Theta^{r+1}_{\mathrm{M}(r,L(-x))}) \hookrightarrow \mathrm{H}^0(\mathrm{M}(r, L(-x)), \omega_{\mathrm{M}(r,L(-x))} \otimes \Theta^{r+1}_{\mathrm{M}(r,L(-x))})$$

$$= \mathrm{H}^0(\mathrm{M}(r, L(-x)), \Theta^{-r+1}_{\mathrm{M}(r,L(-x))}) = 0.$$

$\square$

Remark 8.2. When $g = r = 2$, $\mathrm{M}(r, L)$ is an intersection of two quadrics in $\mathbb{P}^5$ and $\mathcal{E}_x$ is a spinor bundle [CKL19, FK18]. From this description, it was shown that $\mathcal{E}_x$ is ACM for all $x \in X$.

9. SEMIORTHOGONAL DECOMPOSITION

Since $\mathrm{M}(r, L)$ is an index two Fano variety of Picard number one [Ram73], $\mathcal{O}, \Theta$ form an exceptional collection. In this section, we prove Theorem B by showing that the exceptional collection and the image of $\mathrm{D}^b(X)$ form a part of a semiorthogonal decomposition of $\mathrm{D}^b(\mathrm{M}(r, L))$. It was proved for $r = 2$ in [Nar17, Nar18], and for $d = 1$ and $g \geq 3r+4$ in [BM19]. Since a stronger version of Theorem B is proved for $r = 2$ [TT21, Theorem 1.1], we assume that $r \geq 3$.

Proof of Theorem B. By Theorem A, we have four full subcategories

$$\mathcal{O}, \Phi_{\mathcal{E}}(\mathrm{D}^b(X)), \Theta, \Phi_{\mathcal{E}}(\mathrm{D}^b(X)) \otimes \Theta.$$

We will show that they are semiorthogonal in that order. We need to prove the orthogonality condition. Since $\{\mathbb{C}(x) \mid x \in X\}$ form a spanning class of $\mathrm{D}^b(X)$, $\{\mathcal{E}_x \mid x \in X\}$ (resp. $\{\mathcal{E}_x \otimes \Theta \mid x \in X\}$) form a spanning class of $\Phi_{\mathcal{E}}(\mathrm{D}^b(X))$ (resp. $\Phi_{\mathcal{E}}(\mathrm{D}^b(X)) \otimes \Theta$). Therefore, it is sufficient to prove the cohomology vanishing in Theorem 9.1 below. $\square$

Theorem 9.1. *Assume the $g(X) \geq 6$. For any $i \in \mathbb{Z}$ and not necessarily distinct two points $x_1, x_2 \in X$, the following cohomologies are trivial.*

(1) $\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1}^*)$;
(2) $\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \Theta^{-1})$;
(3) $\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1}^* \otimes \Theta^{-1})$;
(4) $\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^* \otimes \Theta^{-1})$.

We use Sommese's vanishing theorem for k-ample vector bundles. On a smooth variety V, a line bundle A on V is k-*ample* if it is semiample and the dimension of the fiber of the morphism $|mA| : V \to \mathbb{P}^N$ is less than or equal to k for $m \gg 0$. A vector bundle F is k-ample if $\mathcal{O}_{\mathbb{P}(F)}(1)$ is k-ample.

Theorem 9.2 ([Som78, Proposition 1.13], Sommese vanishing theorem)**.** *Let F be a rank r k-ample vector bundle on V. Then we have $\mathrm{H}^i(V, \omega_V \otimes F) = 0$ for $i \geq r + k$.*

We check the k-ampleness of bundles using the wall-crossing of moduli of parabolic bundles. Recall that $\mathbb{P}(\mathcal{E}_x) \cong \mathrm{M}(r, L, r-1, \epsilon)$. As we increase the parabolic weight ϵ, the first wall arises when the parabolic weight is $1/\ell$ and the wall is of the form $\Delta(\ell, e, \ell)$ (Lemma 4.4).

Lemma 9.3. *Suppose that $\ell \geq 2$ and the first wall crossing is a simple one. Over $\mathrm{M}(r, L)$, $\mathcal{E}_x$ is $(g-1)\ell(r-\ell)$-ample.*

Proof. The associated nef line bundle to the first small contraction $\pi_- : \mathrm{M}(r, L, r-1, \epsilon) \to \mathrm{M}(r, L, r-1, 1/\ell)$ is $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$, since $\mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1)$ is the unique (up to power) line bundle that is nef (Theorem 4.1) and trivially intersects with the fiber of π_- (Lemma 4.6). Thus, it is sufficient to compute the dimension of the exceptional fiber of π_-.

For a point $p := ((E^-, V^-) \oplus (E^+, V^+)) \in \mathrm{M}(r, L, r-1, 1/\ell)$, $\pi_-^{-1}(p) = \mathbb{P}(\mathrm{Ext}^1((E^+, V^+), (E^-, V^-))^*)$. We can compute its dimension, by modifying the exact sequence (12). After a standard computation, we obtain $\dim \mathbb{P}(\mathrm{Ext}^1((E^+, V^+), (E^-, V^-))^*) = (g-1)\ell(r-\ell)$. $\qquad\square$

Lemma 9.4. *For any two points $x_1, x_2 \in X$, $\mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^* \otimes \Theta$ is $(g-1)\ell(r-\ell)$-ample.*

Proof. First, suppose that $2 \leq \ell \leq r-2$. Note that $\mathcal{E}_{x_2}^* \otimes \Theta$ is a normalized Poincaré bundle over $\mathrm{M}(r, L^*(r))$, where $\deg L^*(r) = r - d$. For the wall crossing of $\mathbb{P}(\mathcal{E}_{x_1}) \cong \mathrm{M}(r, L, r-1, \epsilon)$, the first wall is $\Delta(\ell, e, \ell)$ and it is a multiple wall if and only if $2\ell < r$ (Lemma 4.4). On the other hand, $\mathbb{P}(\mathcal{E}_{x_2}^* \otimes \Theta) \cong \mathrm{M}(r, L^*(r), r-1, \epsilon)$ and its first wall is $\Delta(r-\ell, e', r-\ell)$ and it is a multiple wall if and only if $2(r-\ell) < r$. But since $2\ell + 2(r-\ell) = 2r$ and $\ell \nmid r$, one of these two walls is simple. Then we may apply Lemma 9.3 to compute the k-ampleness of one of them. By [LN20, Corollary 3.5], we can conclude that $\mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^* \otimes \Theta$ is (at least) $(g-1)\ell(r-\ell)$-ample.

Now suppose $\ell = 1$ ($\ell = r - 1$ case is the same). By [Nar17, Proposition 3.3], $\mathcal{E}_{x_1}$ is semi-ample. On the other hand, for $\mathbb{P}(\mathcal{E}_{x_2}^* \otimes \Theta) \cong \mathrm{M}(r, L^*(r), r - 1, \epsilon)$, the first wall is $\Delta(r - 1, e, r - 1)$. Since $2(r - 1) > r$ (because $r > 2$), this is a simple wall, so $\mathcal{E}_{x_2}^* \otimes \Theta$ is $(g-1)\ell(r-\ell) = (g-1)(r-1)$-ample by Lemma 9.3. By [LN20, Theorem 3.4], $\mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^* \otimes \Theta$ is $(g-1)\ell(r-\ell)$-ample, too. $\qquad\square$

Proof of Theorem 9.1. We first show Item (2). For $i \neq 0$, it follows from Theorem 8.1. From

$$\mathrm{H}^0(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \Theta^{-1}) \cong \mathrm{H}^0(\mathrm{M}(r, L, \mathbf{e}), \mathcal{O}(1, 0) \otimes \Theta^{-1})$$

and the fact that $\mathcal{O}(1, 0) \otimes \Theta^{-1} = \Theta^{\frac{\ell-1}{r} - 1} \otimes \left(\mathcal{O}(r, 0) \otimes \Theta^{1-\ell}\right)^{\frac{1}{r}}$ is not on $\mathrm{Eff}(\mathrm{M}(r, L, \mathbf{e}))$ (because $\ell < r$), it is trivial. Thus, we obtain Item (2). Since $\mathcal{E}_{x_1}^* = \mathcal{E}_{x_1}^* \otimes \Theta \otimes \Theta^{-1}$ and $\mathcal{E}_{x_1}^* \otimes \Theta$ is the normalized Poincaré bundle on $\mathrm{M}(r, L^*(r)) \cong \mathrm{M}(r, L)$, Item (1) follows from Item (2). Applying Serre duality,

$$\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1}^* \otimes \Theta^{-1}) \cong \mathrm{H}^{\dim \mathrm{M}(r,L)-i}(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \Theta \otimes \omega)^*$$
$$\cong \mathrm{H}^{\dim \mathrm{M}(r,L)-i}(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \Theta^{-1})^* = 0$$

and we obtain Item (3).

We move to Item (4). By Theorem 9.2 and Lemma 9.4,

$$\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^* \otimes \Theta^{-1}) \cong \mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^* \otimes \Theta \otimes \omega) = 0$$

if $i \geq r^2 + (g-1)\ell(r-\ell)$. Serre duality tells us that

$$\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_{x_1} \otimes \mathcal{E}_{x_2}^* \otimes \Theta^{-1}) \cong \mathrm{H}^{\dim \mathrm{M}(r,L)-i}(\mathrm{M}(r, L), \mathcal{E}_{x_1}^* \otimes \mathcal{E}_{x_2} \otimes \Theta^{-1})^* = 0$$

provided $i \leq (r^2 - 1 - \ell(r - \ell))(g - 1) - r^2$. Thus, if $r^2 + (g-1)\ell(r-\ell) \leq (r^2 - 1 - \ell(r - \ell))(g - 1) - r^2 + 1$, we obtain the desired vanishing. This is equivalent to

(19) $$\frac{2r^2 - 1}{r^2 - 1 - 2\ell(r - \ell)} \leq g - 1.$$

Since $0 < \ell < r$, if $r \geq 5$, we have

$$\frac{2r^2 - 1}{r^2 - 1 - 2\ell(r - \ell)} < \frac{2r^2 - 1}{r^2 - 1 - r^2/2} = \frac{4r^2 - 1}{r^2 - 2} \leq \frac{4 \cdot 5^2 - 1}{5^2 - 2} = \frac{99}{23}.$$

When $r = 3$ and 4, a direct computation gives $17/4$ and $31/9$, respectively. So if $g \geq 6$, the inequality (19) holds for all $r \geq 3$. $\qquad\square$

In the proof above, the genus bound is necessary only for Item (4), which is used to prove the orthogonality of $\Phi_\mathcal{E}(\mathrm{D}^b(X)) \otimes \Theta$. Thus, we obtain the following weaker version for all $g \geq 2$.

Theorem 9.5. *There is a semiorthogonal decomposition* $\mathrm{D}^b(\mathrm{M}(r, L)) = \langle \mathcal{A}', {}^\perp\mathcal{A}' \rangle$ *where* $\mathcal{A}' = \langle \mathcal{O}, \Phi_\mathcal{E}(\mathrm{D}^b(X)), \Theta \rangle$.

Remark 9.6. Whenever r and ℓ satisfy (19), we have the semiorthogonal decomposition of Theorem B. Therefore the genus bound can be improved, if we restrict $\deg L$. For instance, if $d = 1$ (so $\ell = 1$) and $r \geq 6$, the vanishing result holds for $g \geq 4$. Our method does not seem to work for $g = 2, 3$.

Remark 9.7. For $d = 1$, the semiorthogonal decomposition in Theorem B was obtained in [BM19, Theorem B]. Their genus bound is weaker than ours – for instance, for $r \geq 4$, they proved it for $g \geq 3r + 4$.

10. ACM BUNDLES ON $\mathrm{M}(r, L)$

Besides the structure of $\mathrm{D}^b(\mathrm{M}(r, L))$, another immediate application of the technique we developed in this paper is a construction of a one-dimensional family of ACM bundles.

Definition 10.1. Let V be an n-dimensional projective variety with an ample line bundle A. A vector bundle F on V is an *ACM bundle* with respect to A if $\mathrm{H}^i(V, F \otimes A^j) = 0$ for every $1 \leq i \leq n-1$ and $j \in \mathbb{Z}$. An ACM bundle F is *Ulrich* if $\mathrm{H}^0(V, F \otimes A^{-1}) = 0$ and $h^0(V, F) = \operatorname{rank} F \cdot \deg V = \operatorname{rank} F \cdot (A)^n$.

For a smooth Fano variety of Picard rank one, it is straightforward to verify that every line bundle is ACM. It is also clear that if F is ACM with respect to A, then $F \otimes A^k$ is ACM with respect to A for all $k \in \mathbb{Z}$. But finding a non-trivial example of an ACM bundle is not an easy task for higher dimensional varieties. In this section, we show that $\mathcal{E}_x$ is ACM.

Remark 10.2. Many authors assume that A to be very ample when they consider ACM bundles. Because the Picard number of $\mathrm{M}(r, L)$ is one, Theorem C implies that $\mathcal{E}_x$ is ACM for every very ample line bundle. On $\mathrm{M}(r, L)$, Θ^k is known to be very ample when $k \geq r^2 + r$ [EP04, Theorem A], but an optimal k for the very ampleness is unknown.

Proof of Theorem C. By Serre duality,

$$\mathrm{H}^i(\mathrm{M}(r, L), \mathcal{E}_x \otimes \Theta^j) \cong \mathrm{H}^{\dim \mathrm{M}(r,L)-i}(\mathrm{M}(r, L), \mathcal{E}_x^* \otimes \Theta \otimes \Theta^{-j-3})^*.$$

The vanishing for $\mathcal{E}_x \otimes \Theta^j$ for $j \leq -2$ follows from the vanishing for $\mathcal{E}_x^* \otimes \Theta \otimes \Theta^j$ for $j \geq -1$. Since $\mathcal{E}_x^* \otimes \Theta$ is the normalized Poincaré bundle over $\mathrm{M}(r, L^*(r)) \cong \mathrm{M}(r, L)$, it is sufficient to prove the vanishing for $j \geq -1$, which is Theorem 8.1.

For two different points $x_1, x_2 \in X$, $\mathcal{E}_{x_1} \neq \mathcal{E}_{x_2}$ [LN05, Theorem]. Thus, we obtain a one-dimensional family of ACM bundles. □

Remark 10.3. The bundle $\mathcal{E}_x$ is not Ulrich in general. If $g = r = 2$, $h^0(\mathrm{M}(r, L), \mathcal{E}_x) = 4 < 8 = 2 \deg(\mathrm{M}(r, L))$. It is an interesting problem to construct Ulrich bundles on $\mathrm{M}(r, L)$. See [CKL19] for an alternative construction of Ulrich bundles for $g = r = 2$ case.

References

[BFK19] M. Ballard, D. Favero, and L. Katzarkov, Variation of geometric invariant theory quotients and derived categories. *J. Reine Angew. Math.* 746 (2019), 235–303. 1, 17

[BGM18] P. Belmans, S. Galkin, and S. Mukhopadhyay, Semiorthogonal decompositions for moduli of sheaves on curves. *Oberwolfach Report* No. 24/2018, 9–11. 2

[BGM20] P. Belmans, S. Galkin, and S. Mukhopadhyay, Graph potentials and moduli spaces of rank two bundles on a curve. Preprint, arXiv:2009.05568. 2

[BM19] P. Belmans and S. Mukhopadhyay, Admissible subcategories in derived categories of moduli of vector bundles on curves. *Adv. Math.* 351 (2019), 653–675. 1, 2, 10, 21, 23, 26

[Bea18] A. Beauville, An introduction to Ulrich bundles. *Eur. J. Math.* 4, no. 1 (2018), 26-36. 3

[Bho89] U. Bhosle, Parabolic vector bundles on curves. *Ark. Mat.* 27 (1989), no. 1, 15–22. 8, 9

[BO95] A. Bondal and D. Orlov, Semiorthogonal decomposition for algebraic varieties. Preprint, arXiv:alg-geom/9506012, 1995. 20

[CKL19] Y. Cho, Y. Kim, and K.-S. Lee, Ulrich bundles on intersection of two 4-dimensional quadrics. International Mathematics Research Notices, 2019;, rnz320, https://doi.org/10.1093/imrn/rnz320 23, 26

[Cetal21] L. Costa, R. M. Miró-Roig and J. Pons-Llopis, *Ulrich Bundles: From Commutative Algebra to Algebraic Geometry*. Berlin, Boston: De Gruyter, 2021. 3

[DN89] J.-M. Drezet and M. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.* 97 (1989), no. 1, 53–94. 10, 20, 23

[Eis80] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.* 260, 35–64 (1980) 3

[ES03] D. Eisenbud and F.-O. Schreyer, Resultants and Chow forms via exterior syzygies. With an appendix by Jerzy Weyman. *J. Amer. Math. Soc.* 16 (2003), no. 3, 537–579. 3

[EP04] E. Esteves and M. Popa, Effective very ampleness for generalized theta divisors. *Duke Math. J.* 123 (2004), no. 3, 429–444. 26

[Fae13] D. Faenzi, *Some applications of vector bundles in algebraic geometry*. Habilitation á diriger des recherches, available at http://dfaenzi.perso.math.cnrs.fr/publi.html. 3

[FK18] A. Fonarev and A. Kuznetsov, Derived categories of curves as components of Fano manifolds. *J. London Math. Soc.* (2) 97 (2018) 24–46. 1, 23

[GL20] T. Gómez and K.-S. Lee, Motivic decompositions of moduli spaces of vector bundles on curves. Preprint, arXiv:2007.06067. 2

[HR17] J. Hall and D. Rydh, Perfect complexes on algebraic stacks. *Compos. Math.* 153 (2017), no. 11, 2318-2367. 18

[HL15] D. Halpern-Leistner, The derived category of a GIT quotient. *J. Amer. Math. Soc.* 28 (2015), no. 3, 871–912. 1, 17, 18

[Har77] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp. 23

[HL10] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves. Second edition.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010. xviii+325 pp. 8, 12

[Kol86] J. Kollár, Higher direct images of dualizing sheaves. I. *Ann. of Math.* (2) 123 (1986), no. 1, 11–42. 23

[LN05] H. Lange and P. E. Newstead, On Poincaré bundles of vector bundles on curves. *Manuscripta Math.* 117 (2005), no. 2, 173–181. 26

[LN20] F. Laytimi and W. Nahm, Semiample and k-ample vector bundles. *Comm. Algebra.* 48 (2020), 783-791. 24, 25

[Lee18]   K.-S. Lee, Remarks on motives of moduli spaces of rank 2 vector bundles on curves. Preprint, arXiv:1806.11101. 2

[LN21]   K.-S. Lee and M. S. Narasimhan, Symmetric products and moduli spaces of vector bundles of curves. Preprint, arXiv:2106.04872. 2

[MS80]   V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures. *Math. Ann.* 248, 205–239 (1980). 4

[MY20]   H-B. Moon and S-B. Yoo, Finite generation of the algebra of type A conformal blocks via birational geometry II: higher genus. *Proc. Lond. Math. Soc.* (3) 120 (2020), no. 2, 242–264. 7, 9

[MY21]   H-B. Moon and S-B. Yoo, Finite generation of the algebra of type A conformal blocks via birational geometry. *Int. Math. Res. Not. IMRN*, (2021), no. 7, 4941–4974. 5, 16

[Mum66] D. Mumford, *Lectures on curves on an algebraic surface.* With a section by G. M. Bergman Annals of Mathematics Studies, No. 59 Princeton University Press, Princeton, N.J., 1966. xi+200 pp. 4

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory. Third edition.*, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34. Springer-Verlag, Berlin, 1994. xiv+292 pp. 10

[Nar17]   M. S. Narasimhan, Derived categories of moduli spaces of vector bundles on curves. *J. Geom. Phys.* 122 (2017), 53–58. 1, 10, 12, 13, 19, 23, 25

[Nar18]   M. S. Narasimhan, Derived categories of moduli spaces of vector bundles on curves II. *Geometry, algebra, number theory, and their information technology applications*, 375-382, Springer Proc. Math. Stat., 251, Springer, Cham, 2018. 1, 23

[NR75]   M. S. Narasimhan and S. Ramanan, Deformations of the moduli space of vector bundles over an algebraic curve. *Ann. of Math.*, vol. 101, no. 3, (1975), 391–417. 6, 21

[New78]   P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, volume 51 of TIFR Lectures on Mathematics and Physics. Bombay: Tata Institute of Fundamental Research, 1978. 8

[Ols16]   M. Olsson, *Algebraic spaces and stacks.* American Mathematical Society Colloquium Publications, 62. American Mathematical Society, Providence, RI, 2016. xi+298 pp. 18

[Ram73]   S. Ramanan, The moduli spaces of vector bundles over an algebraic curve. *Math. Ann.* 200 (1973), 69–84. 12, 23

[Som78]   A. Sommese, Submanifolds of Abelian varieties. *Math. Ann.* 233 (1978), no. 3, 229–256. 24

[Sun00]   X. Sun, Degeneration of moduli spaces and generalized theta functions. *J. Algebraic Geom.* 9 (2000), no. 3, 459–527. 7

[Tev23]   J. Tevelev, Braid and Phantom, Preprint, arXiv:2304.01825. 2

[TT21]   J. Tevelev and S. Torres, The BGMN conjecture via stable pairs. Preprint, arXiv:2108.11951. 2, 19, 23

[Tha96]   M. Thaddeus, Geometric invariant theory and flips. *J. Amer. Math. Soc.* 9 (1996), no. 3, 691–723. 9, 20

[XY21]   K. Xu and S.-T. Yau, Semiorthogonal decomposition of $D^b(\mathrm{Bun}_2^L)$. Preprint, arXiv:2108.13353. 2

[Yok95]   K. Yokogawa, Infinitesimal deformation of parabolic Higgs sheaves. *Internat. J. Math.* 6 (1995), no. 1, 125–148. 8, 16

[Yos90]   Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings.* London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990. viii+177 pp. 3

KYOUNG-SEOG LEE, INSTITUTE OF THE MATHEMATICAL SCIENCES OF THE AMERICAS, UNIVERSITY OF MIAMI, 1365 MEMORIAL DRIVE, UNGAR 515, CORAL GABLES, FL 33146

Email address: kyoungseog02@gmail.com

HAN-BOM MOON, DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, NEW YORK, NY 10023

Email address: hmoon8@fordham.edu