A FAMILY OF DIVISORS ON $\overline{\mathcal{M}}_{g,n}$ AND THEIR LOG CANONICAL MODELS

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ABSTRACT. We prove a formula of log canonical models for moduli stack $\overline{\mathcal{M}}_{g,n}$ of pointed stable curves which describes all Hassett's moduli spaces of weighted pointed stable curves in a single equation. This is a generalization of the preceding result for genus zero to all genera.

1. INTRODUCTION

A central problem in algebraic geometry when studying a variety X is to determine all birational models of X. One way to approach this problem is to use divisors that have many sections. For example, for an effective divisor D on X, one can hope to define and learn about a natural projective model

$$X(D) := \mathrm{Proj} \bigoplus_{k \geq 0} H^0(X, \mathcal{O}(\lfloor kD \rfloor)).$$

Many results in birational geometry in last several decades are about overcoming of technical difficulties such as the finite generation of the section ring. This theoretical framework can also be applied to a Deligne-Mumford stack ([HH09, Appendix A]).

The moduli stacks $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ of stable curves and stable pointed curves, are important as they give information about smooth curves and their degenerations. Moreover, as special varieties, their coarse moduli spaces $\overline{M}_g$ and $\overline{M}_{g,n}$ have played a useful role in illustrating and testing the goals of birational geometry for example the *minimal model program*.

In this paper we show the following theorem, describing $\overline{M}_{g,\mathcal{A}}$, the coarse moduli space of Hassett's moduli space $\overline{\mathcal{M}}_{g,\mathcal{A}}$ of weighted pointed stable curves with weight datum $\mathcal{A}$ ([Has03]), as log canonical models of $\overline{\mathcal{M}}_{g,n}$.

Theorem 1.1. *Let* $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ *be a weight datum satisfying* $2g - 2 + \sum_{i=1}^n a_i > 0$. *Then*

$$\overline{\mathcal{M}}_{g,n}(K_{\overline{\mathcal{M}}_{g,n}} + 11\lambda + \sum_{i=1}^n a_i \psi_i) \cong \overline{M}_{g,\mathcal{A}}.$$

This is proved for genus zero in [Moo13]. In this article we establish the result in all genera. A key step is to construct ample divisors on $\overline{\mathcal{M}}_{g,\mathcal{A}}$, for $g > 0$ (Proposition 4.3).

To put these results into context, we next give some history of this problem. By [HM82, Har84], we know that for $g \geq 24$, the canonical divisor $K_{\overline{\mathcal{M}}_g}$ is big, and by [BCHM10], that the canonical model $\overline{\mathcal{M}}_g(K_{\overline{\mathcal{M}}_g})$ exists. The hope is that one will be able to describe

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the canonical model itself as a moduli space, and this problem has attracted a great deal of attention.

One approach has been the *Hassett-Keel program*. By [Mum77, p. 107], it is well known that $\overline{\mathcal{M}}_g(K_{\overline{\mathcal{M}}_g} + D) \cong \overline{M}_g$, where $D = \overline{\mathcal{M}}_g - \mathcal{M}_g$ is the sum of all boundary divisors. So if we figure out the log canonical models $\overline{\mathcal{M}}_g(K_{\overline{\mathcal{M}}_g} + \alpha D)$ for $0 \le \alpha \le 1$ and find the variation of log canonical models as we reduce the coefficient α from 1 to 0, we can finally obtain the canonical model. While this problem is far from complete except small genera cases ([Has05, HL10]), we have understood many different compactifications of $\mathcal{M}_g$. For example, see [HH09, HH13, FS13, Fed12, JCML12, AFSvdW13].

We can carry out a similar program for $\overline{\mathcal{M}}_{g,n}$, the moduli space of pointed stable curves. The first result in this direction is the thesis of M. Simpson ([Sim08]). He studied the log canonical model of $\overline{\mathcal{M}}_{0,n}$ assuming the F-conjecture. He proved that for a suitable range of β, $\overline{\mathcal{M}}_{0,n}(K_{\overline{\mathcal{M}}_{0,n}} + \beta D)$ is isomorphic to $\overline{M}_{0,\mathcal{A}}$ where $\mathcal{A}$ is symmetric weight datum which depends on β. This theorem was later proved without assuming the F-conjecture ([AS12, FS11, KM11]). For $g = 1$, there are results of Smyth ([Smy11]) considering birational models of type $\overline{\mathcal{M}}_{1,n}(s\lambda + t\sum_{i=1}^{n} \psi_i - D)$. In this case, birational models are given by moduli spaces of symmetric weighted curves with even worse singularities. In [Fed11] Fedorchuk showed that for any genus g and weight datum $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ satisfying $2g - 2 + \sum_{i=1}^{n} a_i > 0$, there is a divisor $D_{g,\mathcal{A}}$ on $\overline{\mathcal{M}}_{g,n}$ such that (1) $(\overline{\mathcal{M}}_{g,n}, D_{g,\mathcal{A}})$ is a *lc pair* and (2) $\overline{\mathcal{M}}_{g,n}(K_{\overline{\mathcal{M}}_{g,n}} + D_{g,\mathcal{A}}) \cong \overline{M}_{g,\mathcal{A}}$.

Both formula in Theorem 1.1 and [Fed11] are interesting on their own. Theorem 1.1 says that the *same weight datum* determines the log canonical model of parameterized curves and that of the parameter space itself. Indeed, for any weight datum $\mathcal{A}$, there is a reduction morphism $\varphi_{\mathcal{A}} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ ([Has03, Theorem 4.1]). For a stable curve $(C, x_1, \cdots, x_n) \in \overline{\mathcal{M}}_{g,n}$, its image $\varphi_{\mathcal{A}}(C)$ is given by the *log canonical model*

$$C(\omega_C + \sum_{i=1}^{n} a_i x_i) := \mathrm{Proj} \bigoplus_{k \ge 0} H^0(C, \mathcal{O}(\lfloor k(\omega_C + \sum_{i=1}^{n} a_i x_i)\rfloor))$$

of C.

In Section 2 we list the definition and computational results of several tautological divisors. In Section 3, we prove the crucial positivity of certain divisor on $\overline{\mathcal{M}}_{g,\mathcal{A}}$. We give the rest of the proof in Section 4.

We work on an algebraic closed field of any characteristic. All divisor classes and their intersection computations are on moduli stacks, not on coarse moduli spaces.

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2. A GLOSSARY OF DIVISORS ON $\overline{\mathcal{M}}_{g,\mathcal{A}}$

In this section, we recall definitions of tautological divisors on the moduli stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$ and their push-forward/pull-back formulas. A rigorous proof can be obtained by checking the change of universal family and using many test curves. Because on many literatures we are able to find the proof for non-weighted cases (or special weight cases), and the proof is just a simple generalization of each proof, we provide several references and leave the computation to the readers.

Definition 2.1. Fix a weight datum $\mathcal{A} = (a_1, a_2, \cdots, a_n)$. Let $[n] := \{1, 2, \cdots, n\}$. For $I \subset [n]$, let $w_I := \sum_{i\in I} a_i$. Let $\pi : \mathcal{U}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ be the universal family and $\sigma_i : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \mathcal{U}_{g,\mathcal{A}}$ for $i = 1, 2, \cdots, n$ be the universal sections. Also let $\omega := \omega_{\mathcal{U}_{g,\mathcal{A}}/\overline{\mathcal{M}}_{g,\mathcal{A}}}$ be the relative dualizing sheaf.

(1) The *kappa class*: $\kappa := \pi_*(c_1^2(\omega))$. Note that our definition is different from several others for example [AC96, AC98, ACG11].
(2) The *Hodge class*: $\lambda := c_1(\pi_*(\omega))$.
(3) The *psi classes*: For $i = 1, 2, \cdots, n$, let $\mathbb{L}_i$ be the line bundle on $\overline{\mathcal{M}}_{g,\mathcal{A}}$, whose fiber over $(C, x_1, x_2, \cdots, x_n)$ is $\Omega_C|_{x_i}$, the cotangent space at x_i in C. The i-*th psi class* ψ_i is $c_1(\mathbb{L}_i)$. On the other hand, ψ_i can be defined in terms of intersection theory. $\psi_i = \pi_*(\omega \cdot \sigma_i) = -\pi_*(\sigma_i^2)$. The *total psi class* is $\psi := \sum_{i=1}^n \psi_i$.
(4) *Boundaries of nodal curves*: Take a pair (j, I) for $0 \le j \le g$ and $I \subset [n]$. Suppose that if $j = 0$, then $w_I > 1$. Let $D_{j,I} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}$ be the closure of the locus of curves with two irreducible components $C_{j,I}$ and C_{g-j,I^c} such that $C_{j,I}$ (resp. C_{g-j,I^c}) is a smooth genus j (resp. $g-j$) curve and $x_i \in C_{j,I}$ if and only if $i \in I$. For a notational convenience, set $D_{j,I} = 0$ when $j = 0$ and $|I| \le 1$. Let D_{irr} be the closure of the locus of irreducible nodal curves. Let D_{nod} be the sum of all $D_{j,I}$ and D_{irr}.
(5) *Boundaries of curves with coincident sections*: Suppose that $I = \{i, j\}$ and $w_I \le 1$. Let $D_{i=j}$ be the locus of curves with $x_i = x_j$. $D_{i=j}$ is equal to $\pi_*(\sigma_i \cdot \sigma_j)$. Let D_{sec} be the sum of all boundaries of curves with coincident sections.

The canonical divisor $K_{\overline{\mathcal{M}}_{g,\mathcal{A}}}$ is computed by Hassett ([Has03, Section 3.3.1]). By Mumford's relation $\kappa = 12\lambda - D_{nod}$ ([Mum77, Theorem 5.10]), it has two different presentations.

Lemma 2.2 ([Has03, Section 3.3.1])**.**

$$K_{\overline{\mathcal{M}}_{g,\mathcal{A}}} = \frac{13}{12}\kappa - \frac{11}{12}D_{nod} + \psi = 13\lambda - 2D_{nod} + \psi.$$

Next, we present the push-forward and pull-back formulas we will often use.

Let $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ and $\mathcal{B} = (b_1, b_2, \cdots, b_n)$ be weight data such that $a_i \ge b_i$ for all $i = 1, 2, \cdots, n$. For $I \subset [n]$, set $w_I^{\mathcal{A}} = \sum_{i\in I} a_i$ and $w_I^{\mathcal{B}} = \sum_{i\in I} b_i$.

Lemma 2.3. *Let $\varphi_{\mathcal{A},\mathcal{B}} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{B}}$ be the reduction morphism (*[Has03, Theorem 4.1]*). Then:*

(1) $\varphi_{\mathcal{A},\mathcal{B}*}(\kappa) = \kappa - \sum_{w^{\mathcal{B}}_{\{j,k\}} \le 1,\ w^{\mathcal{A}}_{\{j,k\}} > 1} D_{j=k}.$
(2) $\varphi_{\mathcal{A},\mathcal{B}*}(\lambda) = \lambda.$
(3) $\varphi_{\mathcal{A},\mathcal{B}*}(\psi_i) = \psi_i + \sum_{w^{\mathcal{B}}_{\{i,j\}} \le 1,\ w^{\mathcal{A}}_{\{i,j\}} > 1} D_{i=j}.$
(4) $\varphi_{\mathcal{A},\mathcal{B}*}(D_{i,I}) = \begin{cases} 0, & i = 0, |I| \ge 3, w_I^{\mathcal{B}} \le 1, \\ D_{j=k}, & i = 0, I = \{j, k\}, w_I^{\mathcal{B}} \le 1, \\ D_{i,I}, & \textit{otherwise}. \end{cases}$

(5) $\varphi_{\mathcal{A},\mathcal{B}*}(D_{irr}) = D_{irr}$.
(6) $\varphi_{\mathcal{A},\mathcal{B}*}(D_{j=k}) = D_{j=k}$.
(7) $\varphi^*_{\mathcal{A},\mathcal{B}}(\kappa) = \kappa + \sum_{w_I^{\mathcal{B}} \le 1,\ w_I^{\mathcal{A}} > 1} D_{0,I}$.
(8) $\varphi^*_{\mathcal{A},\mathcal{B}}(\lambda) = \lambda$.
(9) $\varphi^*_{\mathcal{A},\mathcal{B}}(\psi_i) = \psi_i - \sum_{i \in I,\ w_I^{\mathcal{B}} \le 1,\ w_I^{\mathcal{A}} > 1} D_{0,I}$.
(10) $\varphi^*_{\mathcal{A},\mathcal{B}}(D_{i,I}) = D_{i,I}$.
(11) $\varphi^*_{\mathcal{A},\mathcal{B}}(D_{irr}) = D_{irr}$.
(12) $\varphi^*_{\mathcal{A},\mathcal{B}}(D_{j=k}) = \begin{cases} D_{j=k} + \sum_{I \supset \{j,k\},\ w_I^{\mathcal{B}} \le 1} D_{0,I}, & w^{\mathcal{A}}_{\{j,k\}} \le 1, \\ \sum_{I \supset \{j,k\},\ w_I^{\mathcal{B}} \le 1} D_{0,I}, & w^{\mathcal{A}}_{\{j,k\}} > 1. \end{cases}$

Proof. All formulas can be shown by looking at the change of universal family carefully. For example, the universal family $\pi : \mathcal{U} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ is changed in codimension one only over $\cup_{w^{\mathcal{B}}_{\{j,k\}} \le 1, w^{\mathcal{A}}_{\{j,k\}} > 1} D_{j=k}$. On this locus, the modification of the family is just contraction of the component containing σ_i. By using test curve method, we obtain item (3) and (9). For the detail, see [FS11, Lemma 2.4, Lemma 2.8]. Items (1), (7) are obtained by the same argument of the proof of [AC96, Section 1]. Items (4), (5), and (6) are simple set-theoretical observations. Since $\varphi_{\mathcal{A},\mathcal{B}}$ is a composition of smooth blow-ups, items (10), (11), and (12) are easily deduced. The rest of them come from Mumford's relation. □

The special case $\varphi_{(1,1,\cdots,1),\mathcal{A}} : \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,(1,1,\cdots,1)} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ is particularly important so we treat this case separately. For notational convenience, let $\varphi_{\mathcal{A}} := \varphi_{(1,1,\cdots,1),\mathcal{A}}$.

Corollary 2.4. *For* $\varphi_{\mathcal{A}} : \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,(1,1,\cdots,1)} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$*,*

(1) $\varphi_{\mathcal{A}*}(\kappa) = \kappa - D_{sec}$.
(2) $\varphi_{\mathcal{A}*}(\lambda) = \lambda$.
(3) $\varphi_{\mathcal{A}*}(\psi_i) = \psi_i + \sum_{w_{\{i,j\}} \le 1} D_{i=j}$.
(4) $\varphi_{\mathcal{A}*}(D_{i,I}) = \begin{cases} 0, & i = 0, |I| \ge 3, w_I \le 1, \\ D_I, & i = 0, |I| = 2, w_I \le 1, \\ D_{i,I}, & \textit{otherwise}. \end{cases}$
(5) $\varphi_{\mathcal{A}*}(D_{irr}) = D_{irr}$.
(6) $\varphi^*_{\mathcal{A}}(\kappa) = \kappa + \sum_{w_I \le 1} D_{0,I}$.
(7) $\varphi^*_{\mathcal{A}}(\lambda) = \lambda$.
(8) $\varphi^*_{\mathcal{A}}(\psi_i) = \psi_i - \sum_{i \in I,\ w_I \le 1} D_{0,I}$.
(9) $\varphi^*_{\mathcal{A}}(D_{i,I}) = D_{i,I}$.
(10) $\varphi^*_{\mathcal{A}}(D_{irr}) = D_{irr}$.
(11) $\varphi^*_{\mathcal{A}}(D_{j=k}) = \sum_{I \supset \{j,k\},\ w_I \le 1} D_{0,I}$.

Lemma 2.5. *Let* $\rho : \overline{\mathcal{M}}_{g,\mathcal{A}\cup\{a_p\}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ *be the forgetful morphism* ([Has03, Theorem 4.3]).

(1) $\rho^*(\kappa) = \kappa + \sum_{w_{\{i,p\}}>1} D_{0,\{i,p\}}$.
(2) $\rho^*(\lambda) = \lambda$.
(3) $\rho^*(\psi_i) = \begin{cases} \psi_i, & w_{\{i,p\}} \le 1, \\ \psi_i - D_{0,\{i,p\}}, & w_{\{i,p\}} > 1. \end{cases}$
(4) $\rho^*(D_{i,I}) = D_{i,I} + D_{i,I\cup\{p\}}$.
(5) $\rho^*(D_{irr}) = D_{irr}$.
(6) $\rho^*(D_{j=k}) = \begin{cases} D_{j=k}, & w_{\{j,k,p\}} \le 1, \\ D_{j=k} + D_{0,\{j,k,p\}}, & w_{\{j,k,p\}} > 1. \end{cases}$

Proof. The readers may find a proof of (1) for non-weighted cases in [AC96, Section 1]. Items (4), (5) and (6) are obvious. (3) is proved in [AC98, Lemma 3.1]. □

For $I = \{j_1, j_2, \cdots, j_r\} \subset [n]$, let $D_{i,I}$ be a boundary of nodal curves. Set $I^c = \{k_1, k_2, \cdots, k_s\}$. Then $D_{i,I}$ is isomorphic to $\overline{\mathcal{M}}_{i,\mathcal{A}_I} \times \overline{\mathcal{M}}_{g-i,\mathcal{A}_{I^c}}$ where $\mathcal{A}_I = (a_{j_1}, a_{j_2}, \cdots, a_{j_r}, 1)$ and $\mathcal{A}_{I^c} = (a_{k_1}, a_{k_2}, \cdots, a_{k_s}, 1)$. Let $\eta_{i,I} : \overline{\mathcal{M}}_{i,\mathcal{A}_I} \times \overline{\mathcal{M}}_{g-i,\mathcal{A}_{I^c}} \cong D_{i,I} \hookrightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$ be the inclusion. Let π_ℓ for $\ell = 1, 2$ be the projection from $\overline{\mathcal{M}}_{i,\mathcal{A}_I} \times \overline{\mathcal{M}}_{g-i,\mathcal{A}_{I^c}}$ to the ℓ-th component.

Lemma 2.6. *Let* p *(resp.* q*) be the last index of* $\mathcal{A}_I$ *(resp.* $\mathcal{A}_{I^c}$*) with weight one.*

(1) $\eta_{i,I}^*(\kappa) = \pi_1^*(\kappa + \psi_p) + \pi_2^*(\kappa + \psi_q)$.
(2) $\eta_{i,I}^*(\lambda) = \pi_1^*(\lambda) + \pi_2^*(\lambda)$.
(3) $\eta_{i,I}^*(\psi_j) = \begin{cases} \pi_1^*(\psi_j), & j \in I, \\ \pi_2^*(\psi_j), & j \in I^c. \end{cases}$
(4) $\eta_{i,I}^*(D_{j,J}) = \begin{cases} -\pi_1^*(\psi_p) - \pi_2^*(\psi_q), & D_{i,I} = D_{j,J}, \\ \pi_1^*(D_{j,J}), & j \le i, J \subset I, D_{i,I} \ne D_{j,J}, \\ \pi_1^*(D_{g-j,J^c}), & g-j \le i, J^c \subset I, D_{i,I} \ne D_{j,J}, \\ \pi_2^*(D_{j,J}), & j \le g-i, J \subset I^c, D_{i,I} \ne D_{j,J}, \\ \pi_2^*(D_{g-j,J^c}), & i \le j, I \subset J, D_{i,I} \ne D_{j,J}, \\ 0, & \textit{otherwise}. \end{cases}$
(5) $\eta_{i,I}^*(D_{irr}) = \pi_1^*(D_{irr}) + \pi_2^*(D_{irr})$.
(6) $\eta_{i,I}^*(D_{j=k}) = \begin{cases} \pi_1^*(D_{j=k}), & j, k \in I, \\ \pi_2^*(D_{j=k}), & j, k \notin I, \\ 0, & \textit{otherwise}. \end{cases}$

Proof. The proof is similar to the case of $\eta : \overline{\mathcal{M}}_{i,|I|+1} \times \overline{\mathcal{M}}_{g-i,|I^c|+1} \to \overline{\mathcal{M}}_{g,n}$, which is proved in [AC98, p.106]. □

Let $(C, x_1, x_2, \cdots, x_n, p, q)$ be a genus $g-1$, $\mathcal{A}\cup\{1,1\}$-stable curve. By gluing p and q, we obtain an $\mathcal{A}$-stable curve of genus g. Since this gluing operation is extended to families of curves and functorial, we obtain a morphism $\xi : \overline{\mathcal{M}}_{g-1,\mathcal{A}\cup\{1,1\}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$. Moreover, ξ is an embedding and the image is precisely D_{irr}.

Lemma 2.7. *Let ξ be the gluing map and let p, q be two identified sections with weight 1.*

(1) $\xi^*(\kappa) = \kappa + \psi_p + \psi_q$.
(2) $\xi^*(\lambda) = \lambda$.
(3) $\xi^*(\psi_i) = \psi_i$.
(4) $\xi^*(D_{i,I}) = D_{i,I} + D_{i-1,I\cup\{p,q\}}$.
(5) $\xi^*(D_{irr}) = D_{irr} - \psi_p - \psi_q + \sum_{p\in I,\ q\notin I} D_{i,I}$.
(6) $\xi^*(D_{j=k}) = D_{j=k}$.

Proof. See [AC98, Lemma 3.2]. □

Finally, for a nonempty subset $I \subset [n]$, assume that $w_I \le 1$. Let $\mathcal{A}'$ be a new weight datum defined by replacing $(a_i)_{i\in I}$ with a single rational number $w_I = \sum_{i\in I} a_i$. We can define an embedding $\chi_I : \overline{\mathcal{M}}_{g,\mathcal{A}'} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ which sends an $\mathcal{A}'$-stable curve to the $\mathcal{A}$-stable curve obtained by replacing the point of weight w_I with $|I|$ points of weight $(a_i)_{i\in I}$ on the same position.

Lemma 2.8. *Let $\chi_I : \overline{\mathcal{M}}_{g,\mathcal{A}'} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ be the replacing morphism. Let p be the unique index of $\mathcal{A}'$ replacing indices in I.*

(1) $\chi_I^*(\kappa) = \kappa$.
(2) $\chi_I^*(\lambda) = \lambda$.
(3) $\chi_I^*(\psi_i) = \begin{cases} \psi_i, & i \notin I, \\ \psi_p, & i \in I. \end{cases}$
(4) $\chi_I^*(D_{nod}) = D_{nod}$.
(5) $\chi_I^*(D_{irr}) = D_{irr}$.
(6) $\chi_I^*(D_{j=k}) = \begin{cases} D_{j=k}, & j,k \notin I, \\ D_{j=p}, & j \notin I, k \in I, \\ -\psi_p, & j,k \in I. \end{cases}$

Proof. This is a restatement of [FS11, Lemma 2.9]. □

3. A POSITIVITY RESULT ON FAMILIES OF CURVES

A key step of the proof of Theorem 1.1 is to construct an ample divisor on $\overline{\mathcal{M}}_{g,\mathcal{A}}$. In this section, we prove the following technical positivity result of a divisor, which will be used in the proof of the main theorem.

Proposition 3.1. *Fix a weight datum $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ and a positive genus g. Let B be a complete curve. Let $\pi : (\mathcal{U}, \sigma_1, \sigma_2, \cdots, \sigma_n) \to B$ be a flat family of $\mathcal{A}$-stable genus g curves. Suppose that a general fiber of π is smooth. Then there exists a positive rational number $\epsilon_{g,\mathcal{A}} > 0$ which depends only on g and $\mathcal{A}$ such that*

$$(2\kappa + \psi) \cdot B \ge \epsilon_{g,\mathcal{A}} \cdot \mathrm{mult}_x B \tag{1}$$

for any point $x \in B$.

Remark 3.2. (1) Proposition 3.1 is not true when $g = 0$. Indeed, $K_{\overline{\mathcal{M}}_{0,n}} \equiv 2\kappa + \psi$ ([Moo13, Lemma 2.6]). It is well-known that for $n = 4, 5$, $K_{\overline{\mathcal{M}}_{0,n}}$ is anti-ample. So it intersects negatively with every curve.

(2) Proposition 3.1 does *not* imply that $2\kappa + \psi$ is ample on $\overline{\mathcal{M}}_{g,\mathcal{A}}$ even though the statement is similar to Seshadri's ampleness criterion (Theorem 3.3). For example, for $n \geq 3$ and $I \subset [n]$ with $|I| = 3$, consider $\eta_{0,I} : \overline{\mathcal{M}}_{0,3+1} \times \overline{\mathcal{M}}_{g,n-3+1} \to \overline{\mathcal{M}}_{g,n}$. Then by Lemma 2.6, $\eta_{0,I}^*(2\kappa + \psi) = \pi_1^*(2\kappa + \psi + \psi_p) + \pi_2^*(2\kappa + \psi + \psi_q) = \pi_1^*(K_{\overline{\mathcal{M}}_{0,4}} + \psi_p) + \pi_2^*(2\kappa + \psi + \psi_q) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(-1)) + \pi_2^*(2\kappa + \psi + \psi_q)$. Therefore $2\kappa + \psi$ intersects negatively with a boundary curve.

We will use following positivity results.

Theorem 3.3 (Seshadri's criterion, [Laz04, Theorem 1.4.13]). *Let X be a projective variety and D be a divisor on X. Then D is ample if and only if there exists a positive number $\epsilon > 0$ such that*

$$D \cdot C \geq \epsilon \cdot \mathrm{mult}_x C$$

for every point $x \in C$ and every complete curve $C \subset X$.

Theorem 3.4 ([Cor93, Lemma 3.2]). *There are positive integers h and M depending on g, r and d, such that the following statement holds for any flat family $\pi : \mathcal{U} \to B$ of nodal curves over any complete curve B. Let L be a relative degree d line bundle on $\mathcal{U}$. Suppose that $\pi : (\mathcal{U}, L) \to B$ is not isotrivial as a family of* polarized *curves. Moreover, assume that*

1. *a general fiber is smooth,*
2. $R^1\pi_*(L^i) = 0$ *for $i \gg 0$ and $r := \dim H^0(\mathcal{U}_b, L_{\mathcal{U}_b})$ is independent of $b \in B$,*
3. *For a general $b \in B$, $L_{\mathcal{U}_b}$ is base-point-free, very ample and embeds $\mathcal{U}_b$ in $\mathbb{P}^{r-1}$ as a Hilbert stable subscheme.*

Then

$$\left(\frac{r}{2}(L \cdot L) - d(\deg \pi_*(L))\right) h^2 + \left((g-1)(\deg \pi_*(L)) - \frac{r}{2}(L \cdot \omega)\right) h + r \deg \lambda \geq \frac{1}{M}\mathrm{mult}_x B$$

for every point $x \in B$.

Remark 3.5. (1) In [Cor93], Cornalba assumed that $g \geq 2$, but in the proof of the theorem, he did not use the genus condition. Thus this result is true without the assumption. See [Cor93, Section 3].

(2) If $d > 2g > 0$, then by [Mum77, Theorem 4.15] a smooth curve is Chow stable and hence Hilbert stable as well ([Mor80, Corollary 3.5]). Therefore if $d \gg 0$, then the stability condition is automatic.

(3) Even if a given family of curves is isotrivial as a family of *abstract* curves, we can apply the theorem if the family is not isotrivial as a family of *polarized* curves.

Proof of Proposition 3.1. We will divide the proof into several steps.

Step 1. It is sufficient to show the result for a weight datum $n \cdot \tau = (\tau, \tau, \cdots, \tau)$ for sufficiently small $\tau > 0$ satisfying $\tau \leq 1/n$.

Let $\varphi_{\mathcal{A},n\cdot\tau} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,n\cdot\tau}$ be the reduction morphism. (We may take τ with the reduction morphism as $\tau = \min\{a_1, a_2, \cdots, a_n, \frac{1}{n}\}$.) By the assumption that a general fiber of π is smooth, $\overline{B} := \varphi_{\mathcal{A},n\cdot\tau}(B)$ is a curve in $\overline{\mathcal{M}}_{g,n\cdot\tau}$. By the projection formula and Lemma 2.3,

$$\begin{aligned}(2\kappa+\psi)\cdot\overline{B} = \varphi^*_{\mathcal{A},n\cdot\tau}(2\kappa+\psi)\cdot B &= (2\kappa + 2\sum_{w_I>1} D_{0,I} + \psi - |I|\sum_{w_I>1} D_{0,I})\cdot B\\ &= (2\kappa+\psi)\cdot B - (|I|-2)\sum_{w_I>1} D_{0,I}\cdot B \le (2\kappa+\psi)\cdot B,\end{aligned}$$

because $|I| \ge 2$ and $D_{0,I}\cdot B \ge 0$. Thus if the result is true for the weight datum $n\cdot\tau$, then

$$(2\kappa+\psi)\cdot B \ge (2\kappa+\psi)\cdot\overline{B} \ge \epsilon_{g,n\cdot\tau}\cdot \mathrm{mult}_{\varphi_{\mathcal{A},n\cdot\tau}(x)}\overline{B} \ge \epsilon_{g,n\cdot\tau}\cdot\mathrm{mult}_x B.$$

Therefore if we define $\epsilon_{g,\mathcal{A}} := \epsilon_{g,n\cdot\tau}$, the proposition holds.

Step 2. We can reduce the number of sections.

Let $\rho : \overline{\mathcal{M}}_{g,n\cdot\tau} \to \overline{\mathcal{M}}_{g,(n-1)\cdot\tau}$ be the forgetful morphism where $n \ge 2$. There are two possible cases. If $\overline{B} := \rho(B)$ is a curve, then by Lemma 2.5,

$$(2\kappa+\psi)\cdot\overline{B} = \rho^*(2\kappa+\psi)\cdot B = (2\kappa+\psi)\cdot B.$$

Thus $(2\kappa+\psi)\cdot B = (2\kappa+\psi)\cdot\overline{B} \ge \epsilon_{g,(n-1)\cdot\tau}\cdot\mathrm{mult}_{\pi(x)}\overline{B} \ge \epsilon_{g,(n-1)\cdot\tau}\cdot\mathrm{mult}_x B$.

If $\rho(B)$ is a point, then the family $\pi : \mathcal{U} \to B$ is isotrivial as a family of abstract pointed curves after forgetting the last section. Note that in our situation, $\deg\lambda = \lambda\cdot B = 0$ and $D_{nod}\cdot B = 0$, $\psi_i\cdot B = 0$ for $i = 1, 2, \cdots, n-1$. Thus $\kappa\cdot B = 0$ by Mumford's relation. Also $(2\kappa+\psi)\cdot B = \psi_n\cdot B$.

We will use Theorem 3.4 with $L = (\omega(\sigma_n))^k$ for sufficiently large k. Then $d = 2k(g-1)+k$, $r = (2k-1)(g-1)+k$ and L satisfies all assumptions in Theorem 3.4. Note that $\pi : (\mathcal{U}, L) \to B$ is not isotrivial as a family of polarized curves because the last section σ_n is not a constant section. By Riemann-Roch theorem,

$$\deg\pi_*(L) = \frac{(L\cdot L)}{2} - \frac{(L\cdot\omega)}{2} + \deg\lambda,$$

since $R^1\pi_*(L) = 0$. Because $(L\cdot L) = k^2\psi_n$ and $(L\cdot\omega) = k\psi_n$, $\deg\pi_*(L) = \frac{k(k-1)}{2}\psi_n$. Now it is straightforward to check that

$$\left(\frac{r}{2}(L\cdot L) - d(\deg\pi_*(L))\right)h^2 + \left((g-1)(\deg\pi_*(L)) - \frac{r}{2}(L\cdot\omega)\right)h + r\deg\lambda = \left(\frac{gk^2}{2}h^2 + O(h)\right)\psi_n$$

is a positive scalar multiple of ψ_n for sufficiently large h. Therefore by Theorem 3.4,

$$(2\kappa+\psi)\cdot B = \psi_n\cdot B \ge \alpha\cdot\mathrm{mult}_x B$$

for some $\alpha > 0$.

Thus we can find $\epsilon_{g,n\cdot\tau} > 0$ by taking the minimum of α and $\epsilon_{g,(n-1)\cdot\tau}$.

Step 3. For $\overline{\mathcal{M}}_{g,(\tau)} \cong \overline{\mathcal{M}}_{g,1}$, the proposition holds.

First of all, suppose that $g \geq 2$. Let $\rho : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ be the forgetting morphism. If $\overline{B} = \rho(B)$ is a curve, then

$$(2\kappa + \psi) \cdot B = 2\rho^*(\kappa) \cdot B + \psi \cdot B = 2\kappa \cdot \overline{B} + \pi_*(\omega \cdot \sigma_1).$$

The divisor κ is ample on $\overline{\mathcal{M}}_g$ by [CH88, Theorem 1.3]. By Seshadri's criterion (Theorem 3.3), there is a positive number $\alpha > 0$ such that $\kappa \cdot \overline{B} \geq \alpha \cdot \mathrm{mult}_x \overline{B}$ for every irreducible curve $\overline{B}$ and $x \in \overline{B}$.

On the other hand, let $\pi' : \mathcal{U}' \to \overline{B}$ be the corresponding family of stable curves. Then there is a stabilization morphism $\tilde{\rho} : \mathcal{U} \to \mathcal{U}'$. On $\mathcal{U}'$, ω is ample by [Ara71, Proposition 3.2]. So if $\tilde{\rho}$ is an isomorphism, then $\pi_*(\omega \cdot \sigma_1) > 0$. If $\tilde{\rho}$ is a contraction, then $\omega = \tilde{\rho}^*(\omega) + E$ where E is an exceptional curve. E is a rational curve and $E \cdot \sigma_1 = 1$. Now

$$\pi_*(\omega \cdot \sigma_1) = \pi_*((\tilde{\rho}^*(\omega) + E) \cdot \sigma_1) > 0.$$

If $\pi : \mathcal{U} \to B$ is isotrivial after forgetting the section σ_1, then by exactly same argument with Step 2, we can obtain the inequality (1).

On $\overline{\mathcal{M}}_{1,1}$, $\kappa = 0$ and $\psi_1 = \frac{1}{12}\lambda$ is ample ([AC98, Theorem 2.2], note that κ_1 in [AC98] is $\kappa + \psi_1$.) Therefore we obtain $\epsilon_{1,(1)} > 0$ and the inequality (1) by Seshadri's criterion. $\square$

4. Proof of the main theorem

In this section, we prove our main result.

Theorem 4.1. *Let $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ be a weight datum satisfying $2g - 2 + \sum_{i=1}^n a_i > 0$. Then*

$$\overline{\mathcal{M}}_{g,n}(K_{\overline{\mathcal{M}}_{g,n}} + 11\lambda + \sum_{i=1}^n a_i\psi_i) \cong \overline{M}_{g,\mathcal{A}}.$$

Remark 4.2. Theorem 4.1 is a generalization of [Moo13, Theorem 1.4] because when $g = 0$, the Hodge class λ is trivial.

A key step of the proof is to construct an ample divisor on $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

Proposition 4.3. *Let*

$$\Delta_{\mathcal{A}} := K_{\overline{\mathcal{M}}_{g,n}} + 11\lambda + \sum_{i=1}^n a_i\psi_i = 2\kappa + \sum_{i=1}^n (1 + a_i)\psi_i.$$

Then the push-forward $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is ample.

Proof. By using definitions of tautological divisors and several formulas in Section 2, it is straightforward to see that

$$\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) = 2\kappa + \sum_{i=1}^n (1 + a_i)\psi_i + \sum_{w_{\{j,k\}} \leq 1} w_{\{j,k\}} D_{j=k} = \pi_*((\omega + \sum_{i=1}^n a_i\sigma_i)(2\omega + \sum_{i=1}^n \sigma_i)).$$

A key feature of $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is that if we restrict it to boundaries, the result is also described by the same formula. More precisely, by Lemma 2.6,

$$\eta_{i,I}^*(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \pi_1^*(\varphi_{\mathcal{A}_I*}(\Delta_{\mathcal{A}_I})) + \pi_2^*(\varphi_{\mathcal{A}_{I^c}*}(\Delta_{\mathcal{A}_{I^c}})). \tag{2}$$

Also by Lemma 2.7,

$$\xi^*(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \varphi_{\mathcal{A}\cup\{1,1\}*}(\Delta_{\mathcal{A}\cup\{1,1\}}). \tag{3}$$

Finally, for $I \subset [n]$ such that $w_I \le 1$, if we write $J := I^c \cup \{p\}$ for the index set for $\mathcal{A}$ and p for the replaced marked point, then with the notations for new weight datum $\mathcal{A}' := (a_i')$ and $w_K' = \sum_{i\in K} a_i'$ (so $a_p' = w_I = \sum_{i\in I} a_i$),

$$\begin{aligned}
&\chi_I^*(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}))\\
&= 2\kappa + \sum_{i\in J}(1+a_i')\psi_i + \sum w'_{\{j,k\}}D_{j=k} + (|I|-1)((1-\sum_{i\in I} a_i)\psi_p + \sum_{i\in I^c, w_{\{i,p\}}\le 1} a_i' D_{i=p})\\
&= \varphi_{\mathcal{A}'*}(\Delta_{\mathcal{A}'}) + (|I|-1)((1-a_i')\psi_p + \sum_{i\in I^c} a_i' D_{j=p})\\
&= \varphi_{\mathcal{A}'*}(\Delta_{\mathcal{A}'}) + (|I|-1)\pi_*((\omega + \sum_{i\in J} a_i'\sigma_i)\cdot\sigma_p)
\end{aligned} \tag{4}$$

(See the notation for Lemma 2.8. The computation is identical to [Moo13, (15)].)

We will use Seshadri's criterion (Theorem 3.3) to show the ampleness of $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$. For $\overline{\mathcal{M}}_{1,1}$, $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) = 2\kappa + 2\psi = \frac{1}{12}\lambda$ is ample by [AC98, Theorem 2.2]. The case of $g = 0$ is shown in [Moo13]. So we can use induction on the dimension of $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

If B is contained in a boundary of nodal curves, then $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})\cdot B \ge \epsilon\cdot \mathrm{mult}_x B$ by (2) and (3). If B is in a boundary of coincident sections, on the last line of (4), $\varphi_{\mathcal{A}'*}(\Delta_{\mathcal{A}'})$ is ample by the induction hypothesis. And $\pi_*((\omega + \sum_{i\in J} a_i'\sigma_i)\cdot\sigma_p)$ is nef because on the family $\mathcal{U}$ over B, $\omega + \sum_{i\in J} a_i'\sigma_i$ is nef ([Fed11, Proposition 2.1]) and σ_p is effective. Thus $\chi_I^*(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}))$ is ample by (4) and we can find $\epsilon > 0$ on the statement of Seshadri's criterion.

So it is sufficient to check the case where $B \cap \mathcal{M}_{g,\mathcal{A}} \ne \emptyset$. Let $\pi : \mathcal{U} \to B$ with $\sigma_i : B \to \mathcal{U}$ for $i = 1, 2, \cdots, n$ be a family of $\mathcal{A}$-stable curves. We rewrite $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ as

$$\begin{aligned}
&\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})\\
&= \varphi_{\mathcal{A}*}((\omega + \sum_{a_i=1}\sigma_i + \sum_{a_i<1} a_i\sigma_i)(2\omega + \sum_{i=1}^n \sigma_i))\\
&= \varphi_{\mathcal{A}*}(((1-\delta)\omega + \sum_{a_i=1}(1-\delta)\sigma_i + \sum_{a_i<1} a_i\sigma_i)(2\omega + \sum_{i=1}^n \sigma))\\
&\quad + \delta(\sum_{a_i=1}\sigma_i)(2\omega + \sum_{i=1}^n \sigma_i) + \delta\omega(2\omega + \sum_{i=1}^n \sigma_i)\\
&= \varphi_{\mathcal{A}*}(((1-\delta)\omega + \sum_{a_i=1}(1-\delta)\sigma_i + \sum_{a_i<1} a_i\sigma_i)(2\omega + \sum_{i=1}^n \sigma_i)) + \delta\sum_{a_i=1}\psi_i + \delta(2\kappa+\psi).
\end{aligned}$$

On the third line, $(\sum_{a_i=1}\sigma_i)(\sum_{i=1}^n \sigma_i) = \sum_{a_i=1}\sigma_i^2$ because if $a_i = 1$, $\sigma_i\cdot\sigma_j = 0$ for every $j \ne i$.

Note that for there is $\delta > 0$ which depends on g and $\mathcal{A}$ such that

$$\omega + \sum_{a_i=1}\sigma_i + \sum_{a_i<1}\frac{1}{1-\delta}a_i\sigma_i$$

satisfies the assumption of [Fed11, Proposition 2.1]. So it is nef on $\mathcal{U}$. By [Moo13, Lemma 3.4], $2\omega + \sum_{i=1}^{n} \sigma_i$ is effective. Thus

$$\varphi_{\mathcal{A}*}(((1-\delta)\omega + \sum_{a_i=1}(1-\delta)\sigma_i + \sum_{a_i<1} a_i\sigma_i)(2\omega + \sum_{i=1}^{n}\sigma_i))$$

is nef on B. For the forgetful map $\rho : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_g$, let $\pi' : \mathcal{U}' \to \rho(B)$ be the corresponding family and σ_i' be the image of section σ_i on $\mathcal{U}'$. Then $\psi_i = -\sigma_i^2 \geq -\sigma_i'^2 = \omega \cdot \sigma_i' \geq 0$ by [Ara71, Proposition 3.2]. Finally, by Proposition 3.1, there exists $\epsilon' > 0$ depending only on g and $\mathcal{A}$ such that $(2\kappa + \psi) \cdot B \geq \epsilon' \cdot \mathrm{mult}_x B$. For $\epsilon := \epsilon'\delta$, we have

$$\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) \cdot B \geq \epsilon \cdot \mathrm{mult}_x B$$

for all $x \in B$.

There are only *finitely* many boundary strata on $\overline{\mathcal{M}}_{g,\mathcal{A}}$. Therefore we can find the minimum of ϵ for all strata of $\overline{\mathcal{M}}_{g,\mathcal{A}}$ and we obtain an $\epsilon > 0$ for all curves in $\overline{\mathcal{M}}_{g,\mathcal{A}}$. □

Now Theorem 4.1 is an immediate consequence of Proposition 4.3.

Proof of Theorem 4.1. By Corollary 2.4, it is straightforward to check that

$$\Delta_{\mathcal{A}} = \varphi_{\mathcal{A}}^*\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) + \sum_{w_I \leq 1}(|I|-2)(1-w_I)D_{0,I}.$$

Note that $D_{0,I}$ with $|I| \geq 3$ and $w_I \leq 1$ is an exceptional divisor for $\varphi_{\mathcal{A}}$. Therefore $\Delta_{\mathcal{A}}$ is a sum of the pull-back of an ample divisor and $\varphi_{\mathcal{A}}$-exceptional effective divisors. Hence we obtain

$$\overline{\mathcal{M}}_{g,n}(\Delta_{\mathcal{A}}) = \overline{\mathcal{M}}_{g,n}(\varphi_{\mathcal{A}}^*\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \overline{\mathcal{M}}_{g,\mathcal{A}}(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \overline{\mathrm{M}}_{g,\mathcal{A}}.$$

See [Moo13, Proof of Theorem 3.1] for the detail. □

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