ON THE S_n-INVARIANT F-CONJECTURE

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ABSTRACT. By using classical invariant theory, we reduce the S_n-invariant F-conjecture to a feasibility problem in polyhedral geometry. We show by computer that for $n \leq 19$, every integral S_n-invariant F-nef divisor on the moduli space of genus zero stable pointed curves is semi-ample, over arbitrary characteristic. Furthermore, for $n \leq 16$, we show that for every integral S_n-invariant nef (resp. ample) divisor D on the moduli space, $2D$ is base-point-free (resp. very ample). As applications, we obtain the nef cone of the moduli space of stable curves without marked points, and the semi-ample cone that of the moduli space of genus 0 stable maps to Grassmannian for small numerical values.

1. INTRODUCTION

When one studies birational geometric aspects of a projective variety X, the first step is to understand two cones of divisors in $\mathrm{N}^1(X)$: the effective cone $\mathrm{Eff}(X)$ and the nef cone $\mathrm{Nef}(X)$. The first cone contains information on rational contractions of X, and the second cone contains data on regular contractions of X.

In this paper, we study the moduli space $\overline{\mathrm{M}}_{0,n}$ of genus 0 stable pointed curves. Its elementary construction in [Kap93] suggests that its geometry is similar to that of toric varieties, and therefore many people conjectured that $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})$ and $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$ are polyhedral. However, the birational geometric properties of $\overline{\mathrm{M}}_{0,n}$ seem to be very complicated. The cone of effective divisors was conjectured to be generated by boundary divisors. However, now there are many known examples of non-boundary extremal effective divisors ([Ver02, CT13, Opi16]). Doran, Giansiracusa, and Jensen showed that the effectivity of a divisor class depends on the base ring, and there are generators of the Cox ring which do not lie on extremal rays of $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})$ ([DGJ14]). Furthermore, recently it was shown that $\overline{\mathrm{M}}_{0,n}$ is not a Mori dream space for $n \geq 10$ ([CT15, GK16, HKL16]). On the other hand, the S_n-invariant part $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})^{S_n}$ is simplicial and generated by symmetrized boundary divisors $\{B_i = \sum_{|I|=i} B_I\}$ ([KM13, Theorem 1.3]).

For $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$, there has been less progress, but there is an explicit conjectural description. From the analogy with toric varieties again, a natural candidate of the generating set of the Mori cone of $\overline{\mathrm{M}}_{0,n}$ is the set of one-dimensional boundary strata, called *F-curves*.

Definition 1.1. An effective divisor $D = \sum b_I B_I$ on $\overline{\mathrm{M}}_{0,n}$ is *F-nef* if for any F-curve F, $D \cdot F \geq 0$.

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Although this definition uses intersection theory, we can explicitly state the set of linear inequalities with respect to the coefficients $\{b_I\}$ ([GKM02, (0.14)]). Thus we can formally define F-nefness over $\mathrm{Spec}\ \mathbb{Z}$, as well.

Fulton conjectured that the Mori cone of $\overline{\mathrm{M}}_{0,n}$ is generated by F-curves. Dually:

Conjecture 1.2 (F-conjecture)**.** *A divisor on $\overline{\mathrm{M}}_{0,n}$ is nef if and only if it is F-nef.*

This conjecture was shown for $n \le 7$ in [KM13] by Keel and McKernan in characteristic 0. However, as n grows, the Picard number of $\overline{\mathrm{M}}_{0,n}$ grows exponentially, so $\overline{\mathrm{M}}_{0,8}$ is already out of reach.

On the other hand, we may ask the same question for S_n-invariant divisors:

Conjecture 1.3 (S_n-invariant F-conjecture)**.** *An S_n-invariant divisor on $\overline{\mathrm{M}}_{0,n}$ is nef if and only if it is F-nef.*

In characteristic 0, Gibney proved Conjecture 1.3 for $n \le 24$ ([Gib09]). Recently, Fedorchuk showed that the S_n-invariant F-conjecture is true for $n \le 16$ in arbitrary characteristic ([Fed14]).

1.1. **Results.** In this paper, we translate Conjecture 1.3 into a feasibility problem in polyhedral geometry, namely, the nonemptiness of certain polytopes. We use this approach to show the following result.

Theorem 1.4 (Theorem 4.1)**.**
(1) For $n \le 19$, over $\mathrm{Spec}\ \mathbb{Z}$, every S_n-invariant F-nef divisor on $\overline{\mathrm{M}}_{0,n}$ is semi-ample.
(2) For $n \le 16$, over $\mathrm{Spec}\ \mathbb{Z}$, for every S_n-invariant F-nef divisor on $\overline{\mathrm{M}}_{0,n}$, $2D$ is base-point-free.

By using [KT09] or [Tev07], we obtain the following consequence.

Theorem 1.5 (Theorem 4.7)**.** *Suppose $n \le 16$. Over any algebraically closed field, for every integral S_n-invariant ample divisor A on $\overline{\mathrm{M}}_{0,n}$, $2A$ is very ample.*

We immediately obtain the following two corollaries.

Corollary 1.6. *For $n \le 19$, over any algebraically closed field, $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n}/S_n)$ is equal to the F-nef cone and every nef divisor on $\overline{\mathrm{M}}_{0,n}/S_n$ is semi-ample.*

Corollary 1.7. *For $n \le 19$, over any algebraically closed field, the Mori cone of $\overline{\mathrm{M}}_{0,n}/S_n$ is generated by F-curves.*

This result also yields the description of the nef cone of some other moduli spaces. By [GKM02, Theorem 0.3], we obtain the nef cone of $\overline{\mathcal{M}}_g$, whose description was a question raised by Mumford.

Corollary 1.8. *Over any algebraically closed field, for $g \le 19$, the nef cone of $\overline{\mathcal{M}}_g$, the moduli space of genus g stable curves, is equal to the F-nef cone.*

By [CHS09, Theorem 1.1] and [CS06, Theorem 1.1], we obtain the nef cone of the moduli space of genus 0 stable maps to a Grassmannian, including the case of projective space.

Corollary 1.9. *Let k, n, and d be positive integers such that $1 \le k \le n-1$ and $d \le 19$. Let $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,n), d)$ be the moduli space of genus 0 stable maps to the Grassmannian $\mathrm{Gr}(k,n)$. Over any algebraically closed field, $\mathrm{Nef}(\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,n), d))$ coincides with the semi-ample cone, and it is polyhedral with explicit generators.*

1.2. **Idea of the proof.** The main ingredient of the proof of Theorem 1.4 is classical invariant theory, in particular *graphical algebras*.

A simple but crucial observation in this paper is that an S_n-invariant F-nef divisor D can be written as $\pi^*(cD_2) - \sum_{i\ge 3} a_i B_i$ where $\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ is a regular contraction and D_2 is an ample divisor. Then the linear system $|D|$ can be identified with a sub linear system $|cD_2|_{\mathbf{a}}$ of $|cD_2|$ on $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ (Proposition 3.7). Thus the study of $|D|$ can be reduced to the study of a non-complete linear system on $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$. Furthermore, there is a sub linear system $|cD_2|_{\mathbf{a},G} \subset |cD_2|_{\mathbf{a}}$ which can be described combinatorially via *graphical algebra*. The Cox ring of $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$, which is the ring of SL_2-invariant divisors on $(\mathbb{P}^1)^n$, is classically known as the graphical algebra since the 19th century. Its generators can be described in terms of finite graphs, thus we may study it by using graph theory. By using the graphical algebra, we obtain a combinatorial description of $|cD_2|_{\mathbf{a},G}$ and its base locus in terms of polytopes (Corollary 3.16).

1.3. **Structure of the paper.** This paper is organized as the following. In Section 2, we recall the definition of the graphical algebra. In Section 3, we translate the F-conjecture into a feasibility problem. Section 4 presents the proof of the main theorem, computational results, and some examples.

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2. Graphical algebra

In this section, we recall the definition and basic properties of graphical algebras, which are key algebraic tools in this approach to the S_n-invariant F-conjecture. We work over $\mathbb{Z}$, but all of the results stated here are valid over any base ring.

The *graphical algebra* is a $\mathbb{Z}$-algebra which was introduced to describe a result in classical invariant theory. Consider $(\mathbb{P}^1)^n$, the space of n points on a projective line. There is a natural diagonal SL_2-action on this space, which is induced by a homomorphism $\mathrm{SL}_2 \to \mathrm{PGL}_2 \cong \mathrm{Aut}(\mathbb{P}^1)$. The graphical algebra is the ring of SL_2-invariant multi-homogeneous polynomials.

Let $[X_i : Y_i]$ be the homogeneous coordinates of i-th factor of $(\mathbb{P}^1)^n$. It is straightforward to check that $Z_{ij} := (X_iY_j - X_jY_i)$ is an SL_2-invariant polynomial, and their products are all invariant polynomials. We can index such polynomials by using finite digraphs. Let $\overrightarrow{\Gamma}$ be a finite directed graph on the vertex set $[n] := \{1, 2, \cdots, n\}$, and let Γ be its underlying undirected graph. We allow multiple edges, but a loop is not allowed. For a vertex i, the *degree* of i (denoted by d_i) is the number of edges incident to i regardless of their directions. The *multidegree* of $\overrightarrow{\Gamma}$ is defined by the degree sequence $(d_1, d_2, \cdots, d_n)$ and denoted by $\mathbf{deg}\ \overrightarrow{\Gamma}$. Let $V_{\overrightarrow{\Gamma}} = V_\Gamma$ be the set of vertices, and let $E_{\overrightarrow{\Gamma}}$ be the set of directed edges. We define $\mathbf{deg}\ \Gamma := \mathbf{deg}\ \overrightarrow{\Gamma}$ and E_Γ is the set of undirected edges in Γ. For any $I \subset [n]$, let w_I be the number of edges connecting vertices in I. For notational simplicity, we set $w_{ij} = w_{\{i,j\}}$, which is the number of edges connecting i and j. So $w_I = \sum_{i,j \in I} w_{ij}$. For $e \in E_{\overrightarrow{\Gamma}}$, $h(e) \in V_{\overrightarrow{\Gamma}}$ is the head and $t(e) \in V_{\overrightarrow{\Gamma}}$ is the tail.

For each $\overrightarrow{\Gamma}$, let

$$Z_{\overrightarrow{\Gamma}} := \prod_{e \in E(\Gamma)} (X_{t(e)}Y_{h(e)} - X_{h(e)}Y_{t(e)}).$$

Then

$$Z_{\overrightarrow{\Gamma}} \in \mathrm{H}^0((\mathbb{P}^1)^n, \mathcal{O}(\mathbf{deg}\ \overrightarrow{\Gamma}))^{\mathrm{SL}_2}.$$

We define the multiplication $\overrightarrow{\Gamma_1} \cdot \overrightarrow{\Gamma_2}$ of two graphs $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$ by a graph with the vertex set $[n]$ and $E_{\overrightarrow{\Gamma_1} \cdot \overrightarrow{\Gamma_2}} := E_{\overrightarrow{\Gamma_1}} \sqcup E_{\overrightarrow{\Gamma_2}}$. Then $\mathbf{deg}\ \overrightarrow{\Gamma_1} \cdot \overrightarrow{\Gamma_2} = \mathbf{deg}\ \overrightarrow{\Gamma_1} + \mathbf{deg}\ \overrightarrow{\Gamma_2}$. Furthermore,

$$Z_{\overrightarrow{\Gamma_1} \cdot \overrightarrow{\Gamma_2}} = Z_{\overrightarrow{\Gamma_1}} \cdot Z_{\overrightarrow{\Gamma_2}}.$$

Note that if we reverse the direction of an edge $e \in E_{\overrightarrow{\Gamma}}$ and make a new graph $\overrightarrow{\Gamma}'$, then $Z_{\overrightarrow{\Gamma}'} = -Z_{\overrightarrow{\Gamma}}$.

The first fundamental theorem of invariant theory ([Dol03, Theorem 2.1]) says that the ring of SL_2-invariants of $(\mathbb{P}^1)^n$ is generated by the polynomials $Z_{\overrightarrow{\Gamma}}$.

Definition 2.1. The (total) *graphical algebra* R of order n is defined by

$$R := \bigoplus_{L \in \mathrm{Pic}((\mathbb{P}^1)^n)} \mathrm{H}^0((\mathbb{P}^1)^n, L)^{\mathrm{SL}_2} = \bigoplus_{(a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n_{\ge 0}} \mathrm{H}^0((\mathbb{P}^1)^n, \mathcal{O}(a_1, a_2, \cdots, a_n))^{\mathrm{SL}_2}.$$

The support of $Z_{\overrightarrow{\Gamma}}$ is independent of the direction of each edge. Thus we may denote $\mathrm{Supp}(Z_{\overrightarrow{\Gamma}})$ by D_Γ. This SL_2-invariant divisor on $(\mathbb{P}^1)^n$, or a Weil divisor on $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ is called a *graphical divisor*. The support of Z_{ij} is denoted by D_{ij}.

The homogenous coordinate ring of the GIT quotient is a slice of R. Fix an effective linearization (a linearization with a nonempty semistable locus) $L \cong \mathcal{O}(a_1, a_2, \cdots, a_n)$. Then the homogeneous coordinate ring of $(\mathbb{P}^1)^n /\!/_L \mathrm{SL}_2$ is

$$R_L := \bigoplus_{d \ge 0} \mathrm{H}^0((\mathbb{P}^1)^n, L^d)^{\mathrm{SL}_2} \subset R.$$

The ideal of relations is explicitly described in [HMSV09, HMSV12].

From now on, we will use the symmetric linearization $\mathcal{O}(1, 1, \cdots, 1)$ only. In this case, the GIT quotient $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ is a projective variety with a natural S_n-action permuting the n factors. If n is odd, it is regular. If n is even, there are $\binom{n}{n/2}/2$ non-regular closed points which are associated to closed orbits of two distinct points with multiplicities $n/2$.

The generating set of R_L has been well-understood since the 19th century by Kempe ([Kem94]). A combinatorial description, including the relation ideal, is given in [HMSV09]. We summarize the description here.

Theorem 2.2 ([Kem94], [HMSV09, Theorem 2.3]). *The homogeneous coordinate ring R_L is generated by $Z_{\overrightarrow{\Gamma}}$ for $\overrightarrow{\Gamma}$ with* **deg** $\overrightarrow{\Gamma} = (\epsilon, \epsilon, \cdots, \epsilon)$ *where $\epsilon = 2$ if n is odd, and $\epsilon = 1$ if n is even.*

Let e_i be the i-th standard vector in $\mathbb{Z}^n \cong \mathrm{Pic}((\mathbb{P}^1)^n)$. Each D_{ij} on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ is the image of $V(Z_{ij})$ in $(\mathbb{P}^1)^n$ and $Z_{ij} \in \mathrm{H}^0(\mathcal{O}(e_i+e_j))$. Thus $\mathrm{Cl}((\mathbb{P}^1)^n//\mathrm{SL}_2)$ is identified with an index two sub-lattice of $\mathrm{Pic}((\mathbb{P}^1)^n)$, generated by $\deg D_{ij} = e_i + e_j$. Note that a generator D_{ij} of $\mathrm{Cl}((\mathbb{P}^1)^n//\mathrm{SL}_2)$ has a simple moduli theoretic interpretation. Indeed,

$$D_{ij} = \{(p_1, p_2, \cdots, p_n) \in (\mathbb{P}^1)^n//\mathrm{SL}_2 \mid p_i = p_j\}.$$

Let $D_2 = \sum D_{ij}$.

Lemma 2.3. *The S_n-invariant Picard group $\mathrm{Pic}((\mathbb{P}^1)^n//\mathrm{SL}_2)^{S_n}$ is generated by $\frac{1}{n-1}D_2$ (resp. $\frac{2}{n-1}D_2$) when n is even (resp. odd).*

Proof. On $(\mathbb{P}^1)^n//\mathrm{SL}_2$, we denote the descent of the line bundle $\mathcal{O}(a_1, a_2, \cdots, a_n)$ on $(\mathbb{P}^1)^n$ by $\overline{\mathcal{O}}(a_1, a_2, \cdots, a_n)$. By Kempf's descent lemma ([DN89, Theorem 2.3]), we are able to check when a line bundle on $(\mathbb{P}^1)^n$ descends to $(\mathbb{P}^1)^n//\mathrm{SL}_2$. If n is odd, it descends if and only if $\sum a_i$ is even. If n is even, there is one extra constraint: For any $I \subset [n]$ with $|I| = n/2$, $\sum_{i\in I} a_i = \sum_{i\notin I} a_i$. The only line bundles satisfying this condition are the symmetric bundles $\mathcal{O}(a, a, \cdots, a)$. Therefore if n is odd, $\mathrm{Pic}((\mathbb{P}^1)^n//\mathrm{SL}_2)$ is isomorphic to an index two sub-lattice of $\mathrm{Pic}((\mathbb{P}^1)^n) \cong \mathbb{Z}^n$. If n is even, $\mathrm{Pic}((\mathbb{P}^1)^n//\mathrm{SL}_2) \cong \mathbb{Z}$. In particular, $\overline{\mathcal{O}}(1, 1, \cdots, 1)$ (resp. $\overline{\mathcal{O}}(2, 2, \cdots, 2)$) is an integral generator of $\mathrm{Pic}((\mathbb{P}^1)^n//\mathrm{SL}_2)^{S_n} \cong \mathbb{Z}$ when n is even (resp. odd). Because $\mathcal{O}(D_2) = \overline{\mathcal{O}}(n-1, n-1, \cdots, n-1)$, we obtain the desired result. □

3. G-BASE-POINT-FREENESS

In this section, by using graphical algebra, we translate the S_n-invariant F-conjecture to a feasibility problem. We work over $\mathrm{Spec}\,\mathbb{Z}$, unless there is an explicit assumption on the base scheme.

The following result explains an explicit connection between $\overline{\mathrm{M}}_{0,n}$ and $(\mathbb{P}^1)^n//\mathrm{SL}_2$.

Theorem 3.1 ([Kap93], [Has03, Theorem 4.1 and 8.3]). *There is a birational contraction morphism*

$$\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n//\mathrm{SL}_2.$$

For any $I \subset [n]$ with $2 \le I \le n/2$, let $B_I \subset \overline{\mathrm{M}}_{0,n}$ be the associated boundary divisor, and let $B_i := \sum_{|I|=i} B_I$. The image of $B_{ij} := B_{\{i,j\}}$ is D_{ij}. For $I \subset [n]$ with $3 \le |I| < n/2$, B_I is contracted by π, and its image is

$$\pi(B_I) = \{(p_1, p_2, \cdots, p_n) \mid p_i = p_j \text{ for all } i, j \in I\}.$$

If $p : ((\mathbb{P}^1)^n)^{ss} \to (\mathbb{P}^1)^n//\mathrm{SL}_2$ is the GIT quotient map, then $\pi(B_I)$ is the image $p(W_I)$ of $W_I := V(Z_{ij})_{i,j\in I} \subset ((\mathbb{P}^1)^n)^{ss}$.

When n is even and $|I| = n/2$, $B_I = B_{I^c}$. Then $\pi(B_I) = \pi(B_{I^c})$ is an isolated singular point on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ and the associated closed orbit is

$$\{(p_1, p_2, \cdots, p_n) \mid p_i = p_j \text{ for all } i, j \in I \text{ or } i, j \in I^c\}.$$

Thus $\pi(B_I)$ is the image $p(W_I)$ of $W_I := V(Z_{ij})_{i,j\in I} \cap V(Z_{ij})_{i,j\in I^c}$. We denote $\pi(B_I) = p(W_I)$ by V_I for all I.

By Theorem 2.2, D_2 is very ample on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ since $\mathcal{O}(D_2) = \overline{\mathcal{O}}(n-1, n-1, \cdots, n-1)$. We have

$$\pi^*(D_2) = \sum_{i\ge 2}^{\lfloor n/2\rfloor} \binom{i}{2} B_i \tag{1}$$

([KM11, Lemma 5.3]). Since π is a regular contraction, the complete linear system $|\pi^*(D_2)|$ is base-point-free on $\overline{\mathrm{M}}_{0,n}$. Indeed, $\pi^*(D_2)$ is an extremal ray of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})^{S_n}$.

The following is a very simple but important observation.

Lemma 3.2. *Every non-trivial S_n-invariant F-nef $\mathbb{Q}$-divisor on $\overline{\mathrm{M}}_{0,n}$ can be written uniquely as*

$$\pi^*(cD_2) - \sum_{i\ge 3} a_i B_i$$

for some rational numbers $c > 0$ and $0 \le a_i < c\binom{i}{2}$. Furthermore, it is integral if and only if $a_i \in \mathbb{Z}$ and $c \in \frac{1}{n-1}\mathbb{Z}$ (resp. $c \in \frac{2}{n-1}\mathbb{Z}$) when n is even (resp. n is odd).

For notational simplicity, we set $a_1 = a_2 = 0$.

Proof. Since F-nefness is defined formally, it is sufficient to prove the result over any algebraically closed field. In $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$, by [KM13, Theorem 1.3], $\{B_i\}$ forms a $\mathbb{Q}$-basis. So is $\{\pi^*(D_2), B_i\}_{3\le i\le \lfloor n/2\rfloor}$. Thus we have the existence and the uniqueness of the expression. Since $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})^{S_n}$ is generated by B_i for $2 \le i \le \lfloor n/2\rfloor$ and every S_n-invariant F-nef divisor is big ([KM13, Theorem 1.3]), an S_n-invariant F-nef divisor is a strictly positive linear combination of B_i's. From (1), $a_i < c\binom{i}{2}$ and $c > 0$. Let F_j be any F-curve class whose associated partition has parts $\{1, 1, j, n-2-j\}$ for $1 \le j \le \lfloor n/2\rfloor - 2$. Then since F_j is contracted by π,

$$0 \le F_j \cdot (\pi^*(cD_2) - \sum_{i\ge 3} a_i B_i) = a_j + a_{j+2} - 2a_{j+1},$$

so the sequence $\{a_j\}$ is convex. From $a_1 = a_2 = 0$, inductively we obtain $a_i \ge 0$ for all i.

The last assertion follows from Lemma 2.3, since each B_i are all integral. □

Remark 3.3. A natural question is whether the computational approach in this paper extends to non-symmetric F-nef divisors. It is unclear, at least to the authors, that every (non-necessarily S_n-invariant) F-nef divisor on $\overline{\mathrm{M}}_{0,n}$ can be written as a sum of divisors of the form in Lemma 3.2.

Definition 3.4. For a non-trivial integral S_n-invariant F-nef divisor $D = \pi^*(cD_2) - \sum_{i\ge 3} a_i B_i$, let $|cD_2|_{\mathbf{a}}$ be the sub linear system of $|cD_2|$ on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ consisting of the divisors whose multiplicity along V_I is at least $a_{|I|}$.

Lemma 3.5. *Let $D = \pi^*(cD_2) - \sum_{i\ge 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. The complete linear system $|D|$ is identified with $|cD_2|_{\mathbf{a}}$ on $(\mathbb{P}^1)^n//\mathrm{SL}_2$.*

Proof. Since D is non-trivial, $c > 0$ by Lemma 3.2. For any $E \in |cD_2|_{\mathbf{a}}$, $\pi^*E = E' + \sum_{i\geq 3} a_iB_i$ and $E' \in |D|$. Thus we have an injective map $|cD_2|_{\mathbf{a}} \to |D|$. Any divisor $F \in |D|$ can be written as $F + \sum_{i\geq 3} a_iB_i = \pi^*F'$ for some $F' \in |cD_2|$, and from the expression $F' \in |cD_2|_{\mathbf{a}}$. Thus we have a map $|D| \to |cD_2|_{\mathbf{a}}$. It is straightforward to see that they are inverses of each other. □

Definition 3.6. For a divisor $E \in |cD_2|_{\mathbf{a}}$, let $\widetilde{E} \in |D|$ be $\pi^*(E) - \sum_{i\geq 3} a_iB_i$, the divisor identified with E via the isomorphism $|D| \cong |cD_2|_{\mathbf{a}}$.

Note that $\widetilde{E}$ is *not* the proper transform of E in general.

Although the non complete sub linear system $|cD_2|_{\mathbf{a}}$ has the key to understand the base-point-freeness or the semi-ampleness of $|D|$, still it is hard to analyze. We will define a sub linear system of $|cD_2|_{\mathbf{a}}$ which can be studied in purely combinatorial terms.

Recall that for any $I \subset [n]$, w_I is the number of edges connecting vertices in I. Recall also that for notational simplicity, we set $a_2 = 0$.

Proposition 3.7. *Let $D = \pi^*(cD_2) - \sum_{i\geq 3} a_iB_i$ be an integral S_n-invariant F-nef divisor. Let D_Γ be a graphical divisor associated to a graph Γ. Then $D_\Gamma \in |cD_2|_{\mathbf{a}}$ if and only if*

(1) $\mathbf{deg}\, \Gamma = (c(n-1), c(n-1), \cdots, c(n-1))$;
(2) For every $I \subset [n]$ with $2 \leq |I| \leq n/2$, $w_I \geq a_{|I|}$;

Proof. The condition (1) is exactly $D_\Gamma \in |cD_2|$, because $\mathcal{O}(cD_2) = \overline{\mathcal{O}}(c(n-1), c(n-1), \cdots, c(n-1))$. Note that for $3 \leq |I| < n/2$, a general point of V_I is smooth. Thus D_Γ vanishes on V_I with multiplicity at least $a_{|I|}$ if and only if $p^*(D_\Gamma)$ vanishes on $W_I = V(Z_{ij})_{i,j\in I}$ with multiplicity at least $a_{|I|}$ if and only if Γ has at least a_i edges connecting vertices in I.

When n is even and $|I| = n/2$, V_I is an isolated singular point and it is the image of $W_I = V(Z_{ij})_{i,j\in I} \cap V(Z_{ij})_{i,j\in I^c}$ on $((\mathbb{P}^1)^n)^{ss}$. Suppose that Z_Γ satisfies the following condition:

$$\text{(C) For every } I \subset [n] \text{ with } |I| = n/2, w_I + w_{I^c} \geq 2a_{n/2}.$$

We claim that this condition is equivalent to the condition that multiplicity along $B_{n/2}$ is at least $a_{n/2}$.

Consider the following commutative diagram:

$$\begin{tikzcd}
 & (\mathrm{Bl}_{W_I}(\mathbb{P}^1)^n)^s \arrow[r, "q"] \arrow[d, "\tilde{p}"'] & ((\mathbb{P}^1)^n)^{ss} \arrow[d, "p"] \\
\overline{\mathrm{M}}_{0,n} \arrow[r] & \mathrm{Bl}_{W_I}(\mathbb{P}^1)^n/\!/\mathrm{SL}_2 \arrow[r, "\bar{q}"] & (\mathbb{P}^1)^n/\!/\mathrm{SL}_2
\end{tikzcd}$$

The vertical arrows are GIT quotients and q is the blow-up along W_I. The superscript ss (resp. s) denotes the semistable (resp. stable) locus. In [KM11, Theorem 1.1], it was shown that Hassett's contraction $\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n/\!/\mathrm{SL}_2$ is decomposed into $\overline{\mathrm{M}}_{0,n} \to \mathrm{Bl}_{W_I}(\mathbb{P}^1)^n/\!/\mathrm{SL}_2 \xrightarrow{\bar{q}} (\mathbb{P}^1)^n/\!/\mathrm{SL}_2$ and $\bar{q}$ is Kirwan's partial desingularization.

For any $D_G \in |cD_2|$, $p^*(D_G)$ is an SL_2-invariant divisor on $((\mathbb{P}^1)^n)^{ss}$. Its multiplicity along W_I is at least $2a_{|I|} = 2a_{n/2}$. Thus $q^*p^*(D_G)$ has the multiplicity at least $2a_{n/2}$ along the exceptional divisor E_I. But since $-\mathrm{Id} \in \mathrm{SL}_2$ acts on E_I nontrivially, $2E_I$ descends to B_I. Therefore the multiplicity along B_I on $\mathrm{Bl}_{W_I}(\mathbb{P}^1)^n//\mathrm{SL}_2$, which is equal to the multiplicity along B_I on $\overline{\mathrm{M}}_{0,n}$, is at least $a_{n/2}$. The converse is similar.

Since the degree of each vertex is $c(n-1)$, $w_I = cn(n-1)/2 - \sum_{i \in I, j \in I^c} w_{ij} = w_{I^c}$. Thus we may reduce Condition (C) to (2). ☐

Definition 3.8. Let $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ be a non-trivial S_n-invariant F-nef divisor with $c, , a_i \in \mathbb{Z}$. Let $|cD_2|_{\mathbf{a},G} \subset |cD_2|_{\mathbf{a}}$ be the sub linear system generated by D_Γ satisfying two conditions in Proposition 3.7. Let $|D|_G \subset |D|$ be the sub linear system which is identified with $|cD_2|_{\mathbf{a},G}$ via the identification

$$|D| \cong |cD_2|_{\mathbf{a}}.$$

In other words, $|D|_G$ is generated by $\widetilde{D}_\Gamma$ for Γ in Proposition 3.7.

Remark 3.9. In general, $|D|_G$ is not equal to $|D|$. Equivalently, $|cD_2|_{\mathbf{a},G}$ is not equal to $|cD_2|_{\mathbf{a}}$. See Example 4.6.

The following lemma is straightforward.

Lemma 3.10. *Let $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. Let $B = \bigcap_{I\in T} B_I$ be a boundary stratum indexed by a nonempty subset $T \subset \{I \subset [n] \mid 2 \leq |I| \leq \lfloor n/2 \rfloor\}$. Then a general point in B is not in the support of $\widetilde{D}_\Gamma \in |D|_G$ if and only if for every $I \in T$ with $|I| \leq n/2$, $w_I = a_{|I|}$.*

Definition 3.11. An integral S_n-invariant F-nef divisor $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ is called *G-base-point-free* if $|D|_G$ is base-point-free. It is *G-semi-ample* if $|mD|_G$ is base-point-free for some $m \gg 0$.

This sub linear system is particularly nice because the base locus can be described in a combinatorial way.

Lemma 3.12. *Let $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. Then the base locus $\mathrm{Bs}(|D|_G)$ is a union of closures of boundary strata.*

Proof. For $\widetilde{D}_\Gamma \in |D|_G$,

$$\mathrm{Supp}(\widetilde{D}_\Gamma) = \bigcup_I B_I$$

where the sum is taken over all I where the number of edges connecting vertices in I is strictly larger than $a_{|I|}$. Since $|D|_G$ is generated by $\widetilde{D}_\Gamma$,

$$\mathrm{Bs}(|D|_G) = \bigcap_{\widetilde{D}_\Gamma \in |D|_G} \mathrm{Supp}(\widetilde{D}_\Gamma).$$

☐

Proposition 3.13. *Let $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. Then D is G-base-point-free if and only if for every F-point $F = \cap_{I\in T} B_I$, there is a graph Γ such that*

(1) **deg** $\Gamma = (c(n-1), c(n-1), \cdots, c(n-1))$;
(2) *For each* $I \subset [n]$ *with* $2 \le |I| \le n/2$, $w_I \ge a_{|I|}$;
(3) *For each* $J \in T$ *with* $|J| \le n/2$, $w_J = a_{|J|}$;

Proof. The above condition implies that the base locus of $|D|_G$ does not contain any F-point. Since $\mathrm{Bs}(|D|_G)$ is a union of boundary strata by Lemma 3.12, if it is non-empty, then there must be at least on F-point in it. Thus $|D|_G$ is base-point-free. □

Since the G-base-point-freeness implies the base-point-freeness, Proposition 3.13 provides a purely combinatorial/computational sufficient condition for being a base-point-free divisor.

Note that sums and scalar multiples of graphical divisors are graphical divisors, too. So it is straightforward to see that for any two G-base-point-free (resp. G-semi-ample) divisors D and D', $D+D'$ and their nonnegative scalar multiples are all G-base-point-free (resp. G-semi-ample).

Definition 3.14. Let $\mathrm{GS}(\overline{\mathrm{M}}_{0,n})$ be the convex cone generated by G-semi-ample divisors in $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$.

We have the following obvious implications:

$$G\text{-base-point-free} \Rightarrow G\text{-semi-ample} \Rightarrow \text{semi-ample} \Rightarrow \text{nef} \Rightarrow F\text{-nef}. \tag{2}$$

Theorem 3.15. *The cone* $\mathrm{GS}(\overline{\mathrm{M}}_{0,n})$ *of G-semi-ample divisors of* $\overline{\mathrm{M}}_{0,n}$ *is closed and polyhedral.*

Proof. We may consider a multigraph Γ as a graph weighting $w : E_{K_n} \to \mathbb{Q}$ where E_{K_n} is the set of edges on the complete graph K_n, by setting $w(\overline{ij}) = w_{ij}$, the number of edges between i and j. Consider $V := \mathbb{Q}^{\binom{n}{2}}$ with coordinates $\{w_{ij}\}_{1\le i<j\le n}$. This space can be regarded as the space of graph weightings. By representing any non-trivial S_n-invariant F-nef divisor in the form $\pi^*(cD_2) - \sum_{i\ge 3} a_iB_i$, we may identify $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ with $\mathbb{Q}^{\lfloor n/2\rfloor -1}$ whose coordinates are $(c, a_i)_{3\le i\le \lfloor n/2\rfloor}$. For each F-point $F = \bigcap_{J\in T} B_J$, we can define a polyhedral cone $Q(n,F) \subset V \times \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ by the following inequalities and equations:

(1) $c, a_i, w_{ij} \ge 0$;
(2) $\sum_{j\ne i} w_{ij} = c(n-1)$;
(3) $\sum_{i,j\in I} w_{ij} \ge a_{|I|}$ for each I with $3 \le |I| \le n/2$;
(4) $\sum_{i,j\in J} w_{ij} = a_{|J|}$ for each $J \in T$ with $|J| \le n/2$.

Let $\rho : V \times \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n} \to \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ be the projection defined by

$$\rho(w_{ij}, c, a_i) = \pi^*(cD_2) - \sum_{i\ge 3} a_iB_i = (c, a_i). \tag{3}$$

For an integral S_n-invariant F-nef divisor $D = \pi^*(cD_2) - \sum_{i\ge 3} a_iB_i$, $\mathrm{Bs}(|D|_G)$ does not contain an F-point $F := \bigcap_{J\in T} B_J$ if and only if $\rho^{-1}(D) \cap Q(n,F)$ has an integral point. So $\mathrm{Bs}(|mD|_G)$ does not contain F for some m if and only if $\rho^{-1}(D) \cap Q(n,F)$ has a rational point, if and only if $D \in \rho(Q(n,F))$.

Therefore $|mD|_G$ is base-point-free for some $m \gg 0$ if and only if $D \in \bigcap_F \rho(Q(n,F))$ where the intersection is taken over all F-points. In particular, a priori, the G-semi-ample cone is the intersection of the S_n-invariant F-nef cone and $\bigcap_F \rho(Q(n,F))$. Therefore it is polyhedral and closed.

Indeed, $\bigcap_F \rho(Q(n,F))$ is a subcone of the S_n-invariant F-nef cone. For any boundary stratum $B = \bigcap_{J \in T} B_J$, we can define a cone $Q(n,B) \subset V \times \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ in the same way. If B and B' are two boundary strata such that $B \subset B'$, then $Q(n,B) \subset Q(n,B')$. Therefore the intersection $\bigcap_B \rho(Q(n,B))$ for all boundary strata is equal to $\bigcap_F \rho(Q(n,F))$ where the intersection is taken over F-points. In particular, if D is a divisor in $\bigcap_F \rho(Q(n,F))$, then for each F-curve F_I, there is a section in $\mathrm{H}^0(\mathcal{O}(mD))$ for some $m \gg 0$ which is nonzero on F_I. This implies $D \cdot F_I \ge 0$ and D is F-nef. $\square$

Thus we obtain a polyhedral lower bound of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})^{S_n}$.

By the proof of Theorem 3.15, we obtain the following corollary, which provides a computational approach to the S_n-invariant F-conjecture.

Corollary 3.16. *Let $\rho : V \times \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n} \to \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ be the projection in (3). Then an S_n-invariant F-nef divisor $\pi^*(cD_2) - \sum a_i B_i$ is G-semi-ample (hence semi-ample) if and only if $D \in \bigcap_F \rho(Q(n,F))$.*

4. Computational results

In this section we list several computational results. The calculations can be found on the webpage [MS16].

Theorem 4.1. *For $n \le 19$, over $\mathrm{Spec}\ \mathbb{Z}$, the S_n-invariant F-nef cone coincides with the G-semi-ample cone.*

Proof. Since $\mathrm{GS}(\overline{\mathrm{M}}_{0,n})$ is a convex subcone of the S_n-invariant F-nef cone, it is sufficient to show that every integral generator of an extremal ray of the S_n-invariant F-nef cone is G-semi-ample. By Corollary 3.16, it is sufficient to show the feasibility of the polytope $\rho^{-1}(D) \cap Q(n,F)$ for each integral generator D and an F-point F. By using Sage [S$^+$] and Gurobi [Gur16], we checked that for $n \le 19$, such a polytope is nonempty. $\square$

Indeed a more efficient result is true. By checking G-base-point-freeness of the Hilbert basis of S_n-invariant F-nef cone, we obtain the following result.

Theorem 4.2. *(1) For $n \le 16$, over $\mathrm{Spec}\ \mathbb{Z}$, for any integral S_n-invariant F-nef divisor D, $2D$ is G-base-point-free.*

(2) For $n \le 11$ or 13, over $\mathrm{Spec}\ \mathbb{Z}$, for any integral S_n-invariant F-nef divisor D, D is G-base-point-free.

Remark 4.3. This computation was faster than we predicted. To check the G-base-point-freeness of a divisor D we need to do the following computation.

1. Take a representative of an F-point F from each S_n-orbit. Let P be the set of the representatives.
2. For each $F \in P$, compute the nonemptyness of $Q(n,F)$.

But when n is small, for most of integral divisors, $\cap_{F \in P} Q(n, F)$ is nonempty. Thus it is sufficient to solve the feasibility problem once per each divisor. Even when this is not the case, the number of feasibility problems we need to solve is significantly small compare to the cardinality of P.

Conjecture 4.4. *For any integral S_n-invariant F-nef divisor D, $2D$ is G-base-point-free. In particular, $2D$ is base-point-free.*

Remark 4.5. Most of the integral divisors are G-base-point-free. More precisely, for $n = 12$, there are only two integral S_n-invariant F-nef divisors which are not base-point-free. For $n = 14, 15$, there is only one for each n.

Example 4.6. Let $n = 12$. Consider the divisor class

$$D = \frac{1}{11}(4B_2 + 12B_3 + 13B_4 + 18B_5 + 16B_6) = \pi^*(\frac{4}{11}D_2) - B_4 - 2B_5 - 4B_6.$$

Here we give an example of an integral S_n-invariant base-point-free divisor which is not G-base-point-free. Then the base locus $\mathrm{Bs}(|D|_G)$ contains the S_{12}-orbit of an F-point

$$F = B_{\{1,2\}} \cap B_{\{1,2,3\}} \cap B_{\{4,5\}} \cap B_{\{4,5,6\}} \cap B_{\{1,2,3,4,5,6\}} \cap B_{\{7,8\}} \cap B_{\{7,8,9\}} \cap B_{\{10,11\}} \cap B_{\{10,11,12\}}.$$

One can check that the locus of curves having four tails with three marked points is the base locus of $|D|_G$. The base locus is isomorphic to a disjoint union of 15400 disjoint unions of $(\overline{\mathrm{M}}_{0,4})^5$.

On the other hand, by using a computer and Kapranov's model, we constructed a divisor $E \in |D|$ such that $F \notin E$. Therefore $|D|_G \neq |D|$. See [MS16].

Now the very ampleness of S_n-invariant ample divisors is an immediate consequence of the result of Keel and Tevelev.

Theorem 4.7. *Let $n \leq 16$. Over any algebraically closed field, for every integral S_n-invariant ample divisor A on $\overline{\mathrm{M}}_{0,n}$, $2A$ is very ample.*

Proof. By [Tev07, Theorem 1.5] or [KT09, Theorem 1.1], the log canonical divisor $K_{\overline{\mathrm{M}}_{0,n}} + B$ is very ample. Note that for every F-curve F, $(K_{\overline{\mathrm{M}}_{0,n}} + B) \cdot F = 1$. So if A is an S_n-invariant integral ample divisor, then $A - (K_{\overline{\mathrm{M}}_{0,n}} + B)$ is an S_n-invariant nef divisor, so $2(A - (K_{\overline{\mathrm{M}}_{0,n}} + B))$ is base-point-free by Theorem 4.2. Therefore

$$2A = 2(A - (K_{\overline{\mathrm{M}}_{0,n}} + B)) + 2(K_{\overline{\mathrm{M}}_{0,n}} + B)$$

is a sum of a base-point-free divisor and a very ample divisor, which is very ample. $\square$

Remark 4.8. By Remark 4.5, when $n \leq 11$ or $n = 13$, every integral S_n-invariant nef divisor is base-point-free. So for those n, every integral S_n-invariant ample divisor is very ample.

5. Comparison to other cones

There are several lower bounds of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. In this section we compare S_n-invariant part of them with the cone $\mathrm{GS}(\overline{\mathrm{M}}_{0,n})$ of G-semi-ample divisors.

In [GM12], Gibney and Maclagan defines a lower bound of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$, by using an embedding of $\overline{\mathrm{M}}_{0,n}$ into a non-proper toric variety. Let X_Δ be a toric variety whose associated fan is $\Delta \subset \mathbb{R}^n$. Suppose that there is a projective toric variety X_Σ with $\Delta \subset \Sigma$.

Definition 5.1. The cone $\mathcal{G}_\Delta \subset \mathrm{Pic}(X_\Delta)_\mathbb{Q}$ is the semi-ample cone of X_Δ.

Let Y be a projective variety with an embedding $i : Y \hookrightarrow X_\Delta$. Then we obtain a lower bound

$$i^*(\mathcal{G}_\Delta) \subset \mathrm{Nef}(Y).$$

The cone $i^*(\mathcal{G}_\Delta)$ is polyhedral and can be described combinatorially. These properties follow from the corresponding properties of $\mathcal{G}_\Delta$:

Proposition 5.2 ([GM12, Proposition 2.3])**.** *Let $D = \sum_{i\in\Delta(1)} a_i D_i \in \mathrm{Pic}(X_\Delta)$. Then the following are equivalent:*

(1) $D \in \mathcal{G}_\Delta$;
(2) $D \in \bigcap_{\sigma\in\Delta} \mathrm{pos}(D_i \mid i \notin \sigma)$;
(3) there is a piecewise linear convex function $\psi : N_\mathbb{R} \to \mathbb{R}$ and $\psi(v_i) = a_i$ where v_i is the first integral vector in the ray i.
(4) $D \in \bigcup_\Sigma i_\Sigma^*(\mathrm{Nef}(X_\Sigma))$ where the union is over all projective toric varieties X_Σ with $\Delta \subset \Sigma$ and $i_\Sigma : X_\Delta \to X_\Sigma$ is the inclusion. Furthermore, we may assume that $\Delta(1) = \Sigma(1)$.

It is well-known that $\overline{\mathrm{M}}_{0,n}$ can be embedded into a non-proper toric variety X_Δ where Δ is the space of phylogenetic trees. Thus we obtain a lower bound $i^*(\mathcal{G}_\Delta)$ of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$.

On the other hand, in [Fed14], Fedorchuk introduced another combinatorial notion of *boundary semi-ampleness*.

Definition 5.3. Let D be a divisor on $\overline{\mathrm{M}}_{0,n}$. D is *boundary semi-ample* if for every $x \in \overline{\mathrm{M}}_{0,n}$, there exists an effective boundary $\mathbb{Q}$-divisor $E \in |D|$ such that $x \notin \mathrm{Supp}(E)$.

The following result was pointed out by Fedorchuk.

Proposition 5.4. *Suppose that D is an S_n-invariant divisor on $\overline{\mathrm{M}}_{0,n}$. Then the following three conditions are equivalent:*

(1) $D \in \mathrm{GS}(\overline{\mathrm{M}}_{0,n})$;
(2) D is boundary semi-ample;
(3) $D \in i^*(\mathcal{G}_\Delta)$.

Proof. Clearly a G-semi-ample divisor D is boundary semi-ample.

Because $\mathrm{Pic}(X_\Delta) \cong \mathrm{Pic}(\overline{\mathrm{M}}_{0,n})$ and each boundary divisor B_I is a restriction of a toric boundary, from item (2) of Proposition 5.2, it is straightforward to see that D is boundary semi-ample if and only if $D \in i^*(\mathcal{G}_\Delta)$.

So it is sufficient to show that every S_n-invariant boundary semi-ample divisor is G-semi-ample. Suppose that D is boundary semi-ample and S_n-invariant. By [Fed14, Lemma 3.2.3],

$$D = \mathcal{D}(\mathbb{Z}_n, f; (1)_n) = f(1)\psi - \sum_{2\le i\le n/2} f(i)B_i$$

for some symmetric F-nef function $f : \mathbb{Z}_n \to \mathbb{Q}$ (See [Fed14, Section 3] for the notation). On the other hand, D can be uniquely written as $\pi^*(cD_2)-\sum_{i\ge 3} a_iB_i$. If we set $a_1 = a_2 = 0$, then by using $\pi^*(D_2) = \frac{c(n-1)}{2}(-\psi + \sum_{i\ge 2} iB_i)$, one can see that $f(i) = (-ci(n-1) + a_i)/2$.

Then by [Fed14, Lemma 2.3.3], for an F-point $F = \cap_{I\in T} B_I$, there is an effective sum of boundary $E \in |D|$ such that $F \notin \mathrm{Supp}(E)$ if and only if there is a graph weighting $m : E_{K_n} \to \mathbb{Q}$ such that

(1) $\sum_{j\neq i} m(ij) = f(1)$;
(2) $\sum_{i\in I, j\in I^c} m(ij) \ge f(i)$;
(3) For $I \in T$, $\sum_{i\in I, j\in I^c} m(ij) = f(i)$.

Now set $w(ij) := -2m(ij)$. Then from (1), $\sum_{j\neq i} w(ij) = -2(\sum_{j\neq i} m(ij)) = c(n-1)$. Note that if $|I| = i$, then $2\sum_{i,j\in I} w(ij) + \sum_{i\in I,j\in I^c} w(ij) = ic(n-1)$. So

$$\begin{aligned} ic(n-1) &= 2\sum_{i,j\in I} w(ij) + \sum_{i\in I, j\in I^c} w(ij) = 2\sum_{i,j\in I} w(ij) - 2\sum_{i\in I, j\in I^c} w(ij) \le 2\sum_{i,j\in I} w(ij) - 2f(i) \\ &= 2\sum_{i,j\in I} w(ij) + ci(n-1) - a_i, \end{aligned}$$

and this implies $\sum_{i,j\in I} w(ij) \ge a_i$. Similarly, Item (3) is $\sum_{i,j\in I} w(ij) = a_i$ for all $I \in T$. Therefore we obtain precisely, the inequalities and equalities for G-semi-ampleness. Therefore D is G-semi-ample. □

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