FINITE GENERATION OF THE ALGEBRA OF TYPE A CONFORMAL BLOCKS VIA BIRATIONAL GEOMETRY

HAN-BOM MOON AND SANG-BUM YOO

ABSTRACT. We study birational geometry of the moduli space of parabolic bundles over a projective line, in the framework of Mori's program. We show that the moduli space is a Mori dream space. As a consequence, we obtain the finite generation of the algebra of type A conformal blocks. Furthermore, we compute the H-representation of the effective cone which was previously obtained by Belkale. For each big divisor, the associated birational model is described in terms of moduli space of parabolic bundles.

1. INTRODUCTION

The aim of this paper is twofold. First of all, we prove the following finiteness theorem.

Theorem 1.1 (Theorem 5.3)**.** *The algebra of type A conformal blocks over a projective line is finitely generated.*

The second main result is the completion of Mori's program for $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$, the moduli space of rank r, degree 0, $\mathbf{a}$-semistable parabolic vector bundles on $\mathbb{P}^1$. The rank two case was done in [25].

Theorem 1.2. *Assume $n > 2r$. Let $\mathrm{M} := \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ for a general effective parabolic weight $\mathbf{a}$.*

(1) (Proposition 6.3) If $\rho(\mathrm{M}) = (r-1)n+1$, which is the possible maximum, the effective cone $\mathrm{Eff}(\mathrm{M})$ is the intersection of an explicit finite set of half-planes in $\mathrm{N}^1(\mathrm{M})_{\mathbb{R}}$. In particular, for any collection of partitions $\lambda^1, \cdots, \lambda^n$ of length s, $d \geq 0$ and $0 < s < r$ such that the Gromov-Witten invariant $\langle \omega_{\lambda^1}, \omega_{\lambda^2}, \cdots, \omega_{\lambda^n} \rangle_d$ of Grassmannian $\mathrm{Gr}(s, r)$ is one, there is a hyperplane supporting $\mathrm{Eff}(\mathrm{M})$.

(2) (Proposition 6.1, Section 6.3) For any $D \in \mathrm{intEff}(\mathrm{M})$, $\mathrm{M}(D) = \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{b})$ for some parabolic weight $\mathbf{b}$. The projective models associated to facets of $\mathrm{Eff}(\mathrm{M})$ can be also described in terms of moduli spaces of parabolic bundles.

(3) For a general $D \in \mathrm{intEff}(\mathrm{M})$, the rational contraction $\mathrm{M} \dashrightarrow \mathrm{M}(D)$ is a composition of smooth blow-ups and blow-downs.

Remark 1.3. (1) When n is small, we may think of M as the target of an algebraic fiber space $\mathrm{M}_{\mathbf{q}}(r, 0, \mathbf{a}') \to \mathrm{M}$ where the domain is the moduli space of parabolic bundles with a larger number of parabolic points. Thus the Mori's program for M becomes a part of that for $\mathrm{M}_{\mathbf{q}}(r, 0, \mathbf{a}')$.

Date: November 24, 2018.

(2) When $\rho(\mathrm{M}) < (r-1)n+1$, M is a divisorial contraction of $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}')$ for some $\mathbf{a}'$. Thus the Mori's program for M also becomes a part of that for $\mathrm{M}_{\mathbf{q}}(r, 0, \mathbf{a}')$.
(3) The H-representation in Item 1 of Theorem 1.2 was obtained by Belkale in [4, Theorem 2.8] in a greater generality by a different method. Our approach using wall-crossings is independent of his idea and is elementary. On the other hand, his result indeed tells us a strong positivity: Any integral divisor class in $\mathrm{Eff}(\mathrm{M})$ is effective.
(4) Once Item 2 of Theorem 1.2 is shown, Item 3 of Theorem 1.2 follows from a result of Thaddeus ([29, Section 7]).

1.1. **Conformal blocks.** *Conformal blocks* were introduced by Tsuchiya, Kanie, Ueno and Yamada to construct a two-dimensional chiral conformal field theory (WZW model) ([30, 31, 32]). For each $(C, \mathbf{p}) \in \overline{\mathcal{M}}_{g,n}$, a simple Lie algebra $\mathfrak{g}$, a nonnegative integer ℓ, and a collection of dominant integral weights $\vec{\lambda} := (\lambda^1, \lambda^2, \cdots, \lambda^n)$ such that $(\lambda^i, \theta) \le \ell$ where θ is the highest root, they constructed a finite dimensional vector space $\mathbb{V}^{\dagger}_{\ell, \vec{\lambda}}$ of conformal blocks.

Conformal blocks have several interesting connections in algebraic geometry. It is known that $\mathbb{V}^{\dagger}_{\ell, \vec{\lambda}}$ can be naturally identified with the space of global sections, so-called generalized theta functions, of a certain line bundle on the moduli space of parabolic principal G-bundles ([20, 28]). It can be also regarded as a quantum generalization of the space of invariants ([4]). Recently, conformal blocks have been studied to construct positive vector bundles on $\overline{\mathcal{M}}_{0,n}$ (see [8] and references therein).

We will focus on $C = \mathbb{P}^1$ and $\mathfrak{g} = \mathfrak{sl}_r$ case. There is a map

$$\mathbb{V}^{\dagger}_{\ell, \vec{\lambda}} \otimes \mathbb{V}^{\dagger}_{m, \vec{\mu}} \to \mathbb{V}^{\dagger}_{\ell+m, \vec{\lambda}+\vec{\mu}}$$

which defines a commutative $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r, 0))$-graded $\mathbb{C}$-algebra structure on

$$\mathbb{V}^{\dagger} := \bigoplus_{\ell, \vec{\lambda}} \mathbb{V}^{\dagger}_{\ell, \vec{\lambda}}$$

where $\mathrm{M}_{\mathbf{p}}(r, 0)$ is the moduli stack of rank r, degree 0 quasi parabolic bundles on $\mathbb{P}^1$ (Section 2.2). $\mathbb{V}^{\dagger}$ is called the *algebra of conformal blocks* and naturally identified with the Cox ring ([17]) of $\mathrm{M}_{\mathbf{p}}(r, 0)$.

Many fundamental questions, such as the finite generation of $\mathbb{V}^{\dagger}$, are still widely open. Manon has shown the finite generation for several cases by a degeneration method. In [23], he showed that for $r = 2$ and a generic point configuration $\mathbf{p}$, $\mathbb{V}^{\dagger}$ is generated by 2^{n-1} level one conformal blocks. He showed the same result for $r = 3$ in [21] and extended it to a certain torus invariant subring of $\mathbb{V}^{\dagger}$ for arbitrary rank r ([22]). The finite generation of $\mathbb{V}^{\dagger}$ for $r = 2$ with an arbitrary point configuration was shown in [25] by the authors.

To prove the finite generation of $\mathbb{V}^{\dagger}$, we study birational geometry of $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$, the moduli space of rank r, degree 0, $\mathbf{a}$-semistable parabolic bundles on $\mathbb{P}^1$, in the framework of *Mori's program*.

1.2. **Mori's program.** For a normal $\mathbb{Q}$-factorial projective variety X with trivial irregularity, Mori's program, or log minimal model program, consists of the following three steps.

(1) Compute the effective cone $\mathrm{Eff}(X)$ in $\mathrm{N}^1(X)_{\mathbb{R}}$.
(2) For each integral divisor $D \in \mathrm{Eff}(X)$, compute the projective model

$$X(D) := \mathrm{Proj} \bigoplus_{m \ge 0} \mathrm{H}^0(X, \mathcal{O}(mD)).$$

(3) Study the rational contraction $X \dashrightarrow X(D)$.

However, even among very simple varieties such as a blow-up of $\mathbb{P}^n$ along some points, there are examples that Mori's program cannot be completed because of two kinds of infinities: There may be infinitely many rational contractions, and more seriously, the section ring $\bigoplus_{m \ge 0} \mathrm{H}^0(X, \mathcal{O}(mD))$ may not be finitely generated and thus $X(D)$ is not a variety.

A *Mori dream space* ([17]), MDS for short, is a special kind of variety that has no such technical difficulties to run Mori's program: The effective cone is polyhedral, there is a finite chamber structure which corresponds to finitely many different projective models, and every divisor has a finitely generated section ring. In Section 5, we show that for any parabolic weight $\mathbf{a}$, the moduli space $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ of $\mathbf{a}$-semistable parabolic bundles is a MDS. On the other hand, under some assumption, $\mathbb{V}^{\dagger}$ can be identified with $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}))$. We obtain the finite generation by [17, Proposition 2.9].

Even for a MDS X, practically it is very difficult to complete Mori's program if $\rho(X) \ge 3$ (see [10] for an example). Theorem 1.2 provides another highly non-trivial but completed example.

1.3. **Outline of the proof.** We leave a brief outline of the proof of Theorem 1.1. For simplicity, suppose that n is large enough. The main technique we employ is a careful analysis of wall-crossings. The moduli space $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ depends on a choice of a parabolic weight $\mathbf{a}$. For two general parabolic weights $\mathbf{a}$ and $\mathbf{b}$, there is a birational map $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}) \dashrightarrow \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{b})$ provided stable loci of these two moduli spaces are nonempty. Then the birational map can be decomposed into finitely many explicit blow-ups and blow-downs, and the change can be measured explicitly. Furthermore, when $\mathbf{a}$ is sufficiently small, then $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}) \cong \mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r$ for some explicit linearization L. By analyzing the geometry of the GIT quotient and wall-crossings, we obtain the canonical divisor of $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ where $\mathbf{a}$ is dominant in the sense that $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r, 0)) = \mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}))$. We show that after finitely many flips (not blow-downs), the anticanonical divisor $-K$ becomes big and nef. Because the moduli space is smooth, it is of Fano type. Therefore it is a MDS.

1.4. **Generating set.** The method of the proof does not provide any explicit set of generators. It is an interesting problem to construct a generating set. Based on several results of Manon ([21, 22, 23]), one may ask the question below. Note that the effective cone is polyhedral, so there are finitely many extremal rays.

Question 1.4. Is the algebra $\mathbb{V}^\dagger$ of conformal blocks generated by the set of effective divisors whose numerical classes are the first integral points of extremal rays of $\mathrm{Eff}(\mathrm{M}_{\mathbf{p}}(r,0))$?

1.5. **Organization of the paper.** In Section 2, we recall some preliminaries on the moduli space of parabolic bundles, conformal blocks and deformation theory. In Section 3, we identify moduli spaces of parabolic bundles with small weights with elementary GIT quotients. Section 4 reviews the wall-crossing analysis. In Section 5, we prove Theorem 1.1. Finally in Section 6, we prove Theorem 1.2.

Notations and conventions. We work on an algebraically closed field $\mathbb{C}$ of characteristic zero. Unless there is an explicit statement, we will fix $n \ge 3$ distinct points $\mathbf{p} = (p^1, \cdots, p^n)$ on $\mathbb{P}^1$. These points are called *parabolic points*. $[r]$ denotes the set $\{1, 2, \cdots, r\}$. To minimize the introduction of cumbersome notation, mainly we will discuss parabolic bundles with full flags only, except in Section 6.3 on degenerations of moduli spaces. The readers may easily generalize most part of the paper, to the partial flag cases. In many literatures the dual $\mathbb{V}_{\ell,\vec{\lambda}}$ of $\mathbb{V}^\dagger_{\ell,\vec{\lambda}}$ has been denoted by conformal blocks.

Acknowledgements. The authors thank Prakash Belkale and Young-Hoon Kiem for their comments and suggestions on an earlier draft. The first author thanks Jinhyung Park for helpful conversations. The first author was partially supported by the Minerva Research Foundation.

2. Moduli space of parabolic bundles

In this section, we give a brief review on parabolic bundles and their moduli spaces.

2.1. Parabolic bundles and their moduli spaces.

Definition 2.1. Fix parabolic points $\mathbf{p} = (p^1, p^2, \cdots, p^n)$ on $\mathbb{P}^1$. A rank r *quasi parabolic bundle* over $(\mathbb{P}^1, \mathbf{p})$ is a collection of data $\mathcal{E} := (E, \{W^i_\bullet\})$ where:

(1) E is a rank r vector bundle over $\mathbb{P}^1$;
(2) For each $1 \le i \le n$, $W^i_\bullet \in \mathrm{Fl}(E|_{p^i})$. In other words, $W^i_\bullet$ is a strictly increasing filtration of subspaces $0 \subsetneq W^i_1 \subsetneq W^i_2 \subsetneq \cdots \subsetneq W^i_{r-1} \subsetneq W^i_r = E|_{p^i}$. In particular, $\dim W^i_j = j$.

Let $\mathrm{M}_{\mathbf{p}}(r,d)$ be the moduli stack of rank r, degree d quasi parabolic bundles over $(\mathbb{P}^1, \mathbf{p})$. This moduli stack is a non-separated Artin stack. It is smooth, because it is a fiber bundle with fibers isomorphic to the product of flag varieties, over the moduli stack of bundles on a curve, which is well-known to be smooth. To obtain a proper moduli space, as in the case of moduli spaces of ordinary vector bundles, we introduce the parabolic slope and a stability condition, and collect semi-stable objects only. One major difference here is that there are many different ways to define stability while there is a standard one in the case of ordinary bundles.

Definition 2.2. (1) A *parabolic weight* is a collection $\mathbf{a} = (a^1_\bullet, a^2_\bullet, \cdots, a^n_\bullet)$ of strictly decreasing sequences $a^i_\bullet = (1 > a^i_1 > \cdots > a^i_{r-1} > a^i_r \ge 0)$ of rational numbers of length r. Let $|\mathbf{a}|_j := \sum_{i=1}^n a^i_j$ and $|\mathbf{a}| := \sum_{j=1}^r |\mathbf{a}|_j$.
(2) A *parabolic bundle* is a collection $\mathcal{E} := (E, \{W^i_\bullet\}, \mathbf{a})$.

Definition 2.3. Let $\mathcal{E} = (E, \{W^i_\bullet\}, \mathbf{a})$ be a parabolic bundle.

(1) The *parabolic degree* of $\mathcal{E}$ is

$$\mathrm{pdeg}\,\mathcal{E} := \deg E + |\mathbf{a}|.$$

(2) The *parabolic slope* of $\mathcal{E}$ is $\mu(\mathcal{E}) = \mathrm{pdeg}\,\mathcal{E}/\mathrm{rk}\,E$.

Let $\mathcal{E} = (E, \{W^i_\bullet\}, \mathbf{a})$ be a parabolic bundle of rank r. For each subbundle $F \subset E$, there is a natural induced flag structure $W|_{F}{}^i_\bullet$ on $F|_{p^i}$. More precisely, let ℓ be the smallest index such that $\dim(W^i_\ell \cap F|_{p^i}) = j$. Then $W|_{F}{}^i_j = W^i_\ell \cap F|_{p^i}$. Furthermore, we can define the induced parabolic weight $\mathbf{b} = (b^i_\bullet)$ on $F|_{p^i}$ as $b^i_j = a^i_\ell$. This collection of data $\mathcal{F} := (F, \{W|_{F}{}^i_\bullet\}, \mathbf{b})$ is called a *parabolic subbundle* of $\mathcal{E}$. Similarly, one can define the induced flag $W/F^i_\bullet$ on $E/F|_{p^i}$, the inherited parabolic weight $\mathbf{c}$, and the *quotient parabolic bundle* $\mathcal{Q} := (E/F, \{W/F^i_\bullet\}, \mathbf{c})$.

Definition 2.4. A parabolic bundle $\mathcal{E} = (E, \{W^i_\bullet\}, \mathbf{a})$ is $\mathbf{a}$-*(semi)-stable* if for every parabolic subbundle $\mathcal{F}$ of $\mathcal{E}$, $\mu(\mathcal{F})(\le) < \mu(\mathcal{E})$.

Let $\mathrm{M}_{\mathbf{p}}(r, d, \mathbf{a})$ be the moduli space of S-equivalence classes of rank r, topological degree d $\mathbf{a}$-semistable parabolic bundles over $(\mathbb{P}^1, \mathbf{p})$. It is an irreducible normal projective variety of dimension $n\binom{r}{2} - r^2 + 1$ if it is nonempty ([24, Theorem 4.1]).

Remark 2.5. For a general parabolic weight $\mathbf{a}$, we can define the *normalized* weight $\mathbf{a}'$ as ${a'}^i_j = a^i_j - a^i_r$. It is straightforward to check that the map

$$\begin{array}{rcl} \mathrm{M}_{\mathbf{p}}(r, d, \mathbf{a}) & \to & \mathrm{M}_{\mathbf{p}}(r, d, \mathbf{a}') \\ (E, \{W^i_\bullet\}, \mathbf{a}) & \mapsto & (E, \{W^i_\bullet\}, \mathbf{a}') \end{array}$$

is an isomorphism. Thus we will assume that any given parabolic weight is normalized.

Finally, we leave two notions on weight data.

Definition 2.6. (1) A parabolic weight $\mathbf{a}$ is *effective* if $\mathrm{M}_{\mathbf{p}}(r, d, \mathbf{a})$ is nonempty.
(2) An effective parabolic weight $\mathbf{a}$ is *general* if the $\mathbf{a}$-semistability coincides with the $\mathbf{a}$-stability.

If $\mathbf{a}$ is general, $\mathrm{M}_{\mathbf{p}}(r, d, \mathbf{a})$ is a smooth projective variety.

Remark 2.7. The notion of parabolic bundles can be naturally generalized to parabolic bundles with *partial flags*. For notational simplicity, we do not describe them here. Consult [24, Definition 1.5]. In this paper, we use moduli spaces of parabolic bundles with partial flags only in Section 6.3 to describe projective models associated to non-big divisors. A reader who is not interested in this topic can ignore partial flag cases.

2.2. **The algebra of conformal blocks.** For an r-dimensional vector space V, the full flag variety $\mathrm{Fl}(V)$ is embedded into $\prod_{j=1}^{r-1}\mathrm{Gr}(j, V)$, and every line bundle on $\mathrm{Fl}(V)$ is the restriction of $\mathcal{O}(b_\bullet) := \mathcal{O}(b_1, b_2, \cdots, b_{r-1})$. By the Borel-Weil theorem, if all b_i's are non-negative, or equivalently $\mathcal{O}(b_\bullet)$ is effective, then $\mathrm{H}^0(\mathrm{Fl}(V), \mathcal{O}(b_\bullet))$ is the irreducible SL_r-representation V_λ with the highest weight $\lambda = \sum_{i=1}^{r-1} b_{r-i}\omega_i$. F_λ denotes $\mathcal{O}(b_\bullet)$.

Recall that $\mathrm{M}_{\mathbf{p}}(r, 0)$ is the moduli stack of rank r, degree 0 quasi parabolic bundles over $(\mathbb{P}^1, \mathbf{p})$. The Picard group of $\mathrm{M}_{\mathbf{p}}(r, 0)$ is isomorphic to

$$\mathbb{Z}\mathcal{L} \times \prod_{i=1}^{n} \mathrm{Pic}(\mathrm{Fl}(V))$$

([20]). In particular, its Picard number is $(r-1)n + 1$.

The generator $\mathcal{L}$ is the determinant line bundle ([2, Example 3.8]) on $\mathrm{M}_{\mathbf{p}}(r, 0)$ which has the following functorial property: For any family of rank r quasi parabolic bundles $\mathcal{E} = (E, \{W_\bullet^i\})$ over S, consider the determinant bundle $L_S := \det R^1\pi_{S*}E \otimes (\det \pi_{S*}E)^{-1}$, where $\pi_S : X \times S \to S$ is the projection to S. If $p : S \to \mathrm{M}_{\mathbf{p}}(r, 0)$ is the functorial morphism, then $p^*(\mathcal{L}) = L_S$. The line bundle $\mathcal{L}$ has a unique section denoted by Θ. This section Θ vanishes exactly on the locus of $\mathcal{E} = (E, \{W_\bullet^i\})$ such that $E \ne \mathcal{O}^r$ ([7, Section 10.2]).

Any line bundle $F \in \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r, 0))$ can be written uniquely as $\mathcal{L}^\ell \otimes \otimes_{i=1}^n F_{\lambda^i}$ where F_{λ^i} is a line bundle associated to the integer partition λ^i. The space of global sections $\mathrm{H}^0(F)$ is identified with the space of conformal blocks $\mathbb{V}^\dagger_{\ell,\vec{\lambda}} := \mathbb{V}^\dagger_{\ell,(\lambda^1,\cdots,\lambda^n)}$ ([28, Corollary 6.7]). In particular, $\mathbb{V}^\dagger_{1,\vec{0}}$ is generated by Θ. In general, $\mathbb{V}^\dagger_{\ell,\vec{\lambda}}$ is trivial if $\lambda_1^i > \ell$ for some $1 \le i \le n$.

Note that there is a natural injective map $\mathbb{V}^\dagger_{\ell,\vec{\lambda}} \hookrightarrow \mathbb{V}^\dagger_{\ell+1,\vec{\lambda}}$ given by the multiplication of Θ. Moreover, when $\ell \ge (\sum_{i=1}^n \sum_{j=1}^{r-1} \lambda_j^i)/(r+1)$, $\mathbb{V}^\dagger_{\ell,\vec{\lambda}} \cong V_{\vec{\lambda}}^{*\mathrm{SL}_r} := (\bigotimes_{i=1}^n V_{\lambda^i}^*)^{\mathrm{SL}_r}$ ([7, Proposition 1.3]). The last space is a trivial SL_r-subrepresentation of $\bigotimes_{i=1}^n V_{\lambda^i}^*$ and is called the space of (classical) invariants. The spaces of conformal blocks define an increasing filtration on $V_{\vec{\lambda}}^{*\mathrm{SL}_r}$. It could be regarded as a quantum generalization of the space of invariants ([4]).

Definition 2.8. The *algebra of conformal blocks* is a $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r, 0))$-graded algebra

$$\mathbb{V}^\dagger := \bigoplus_{\ell,\vec{\lambda}} \mathbb{V}^\dagger_{\ell,\vec{\lambda}}.$$

Note that $\mathbb{V}^\dagger$ is the space of all sections of all line bundles on $\mathrm{M}_{\mathbf{p}}(r, 0)$. In other words, $\mathbb{V}^\dagger$ is the *Cox ring* $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r, 0))$ of the moduli stack $\mathrm{M}_{\mathbf{p}}(r, 0)$.

The original definition of conformal blocks uses representation theory of affine Lie algebras ([32]). However, for $g = 0$, there is an elementary alternative description using representation theory of semisimple Lie algebras only, obtained by Feigin, Schechtman, and Varchenko ([13]) and independently by Beauville ([1]).

For any collection of finite dimensional irreducible SL_r-representations $V_{\lambda_1}, V_{\lambda_2}, \cdots, V_{\lambda_n}$ parametrized by a sequence of dominant integral weights $\vec{\lambda} := (\lambda_1, \lambda_2, \cdots, \lambda_n)$, let $V_{\vec{\lambda}} := \bigotimes_{i=1}^n V_{\lambda_i}$. We denote its SL_r-invariant subspace by $V_{\vec{\lambda}}^{\mathrm{SL}_r}$. Fix an affine open subset $\mathbb{A}^1 \subset \mathbb{P}^1$

which contains all $\mathbf{p}$. Let $t^i \in \mathbb{C}$ be the coordinate for p^i. Let $T \in \mathrm{End}(V_{\vec{\lambda}})$ is defined by

$$T(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} t^i v_1 \otimes v_2 \otimes \cdots \otimes X_\theta v_i \otimes \cdots \otimes v_n.$$

Then $\mathbb{V}^{\dagger}_{\ell,\vec{\lambda}}$ is the space of SL_r-invariant linear maps $\phi : V_{\vec{\lambda}} \to \mathbb{C}$ such that $\phi \circ T^{\ell+1} = 0$ ([1, Proposition 4.1]). In particular, any conformal block can be regarded as an SL_r-invariant element for some $V_{\vec{\lambda}}$ with a certain vanishing condition depending on ℓ.

2.3. **Deformation theory.** To analyze wall crossings on the moduli space in detail, we employ some results from the deformation theory of parabolic bundles, which was intensively studied by Yokogawa in [33]. In this section we summarize some relevant results.

Let $\mathcal{E} = (E, \{W^i_\bullet\}, \mathbf{a})$ and $\mathcal{F} = (F, \{W'^i_\bullet\}, \mathbf{b})$ be two parabolic bundles. A bundle morphism $f : E \to F$ is called *(strongly) parabolic* if $f(W^i_j) \subset W'^i_k$ whenever $a^i_j\ (\geq) > b^i_{k+1}$. The sheaves of parabolic morphisms and strongly parabolic morphisms are denoted by $\mathcal{P}ar\mathcal{H}om(\mathcal{E},\mathcal{F})$ and $\mathcal{SP}ar\mathcal{H}om(\mathcal{E},\mathcal{F})$ respectively. The spaces of their global sections are denoted by $\mathrm{ParHom}(\mathcal{E},\mathcal{F})$ and $\mathrm{SParHom}(\mathcal{E},\mathcal{F})$.

Yokogawa introduced an abelian category P of parabolic $\mathcal{O}_{\mathbb{P}^1}$-modules which contains the category of parabolic bundles as a full subcategory. P has enough injective objects, so we can define the right derived functor $\mathrm{Ext}^i(\mathcal{E}, -)$ of $\mathrm{ParHom}(\mathcal{E}, -)$. Those cohomology groups can be described in terms of ordinary cohomology groups and behave similarly.

Lemma 2.9 ([33, Theorem 3.6])**.**

$$\mathrm{Ext}^i(\mathcal{E},\mathcal{F}) \cong \mathrm{H}^i(\mathcal{P}ar\mathcal{H}om(\mathcal{E},\mathcal{F})).$$

Lemma 2.10 ([33, Lemma 1.4])**.** *The cohomology* $\mathrm{Ext}^1(\mathcal{E},\mathcal{F})$ *parametrizes isomorphism classes of parabolic extensions, which are exact sequences* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$ *in* P.

Also we have 'Serre duality':

Lemma 2.11 ([33, Proposition 3.7])**.**

$$\mathrm{Ext}^{1-i}(\mathcal{E}, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(n-2)) \cong \mathrm{H}^i(\mathcal{SP}ar\mathcal{H}om(\mathcal{F},\mathcal{E}))^*.$$

3. Small weight case

When a parabolic weight $\mathbf{a}$ is sufficiently 'small', $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ can be constructed as an elementary GIT quotient. This section is devoted to the study of such a small weight case.

3.1. **Moduli of parabolic bundles and GIT quotient.** Let $\pi_i : \mathrm{Fl}(V)^n \to \mathrm{Fl}(V)$ be the projection to the i-th factor. Any line bundle on $\mathrm{Fl}(V)^n$ can be described as a restriction of $L_{\mathbf{b}} := \otimes \pi_i^* \mathcal{O}(b^i_\bullet)$. Note that there is a natural diagonal SL_r-action on $\mathrm{Fl}(V)^n$. The GIT stability with respect to $L_{\mathbf{b}}$ is well-known:

Theorem 3.1 ([12, Theorem 11.1]). *A point $(W^i_\bullet) \in \mathrm{Fl}(V)^n$ is (semi-)stable with respect to $L_{\mathbf{b}}$ if and only if for every proper s-dimensional subspace $V' \subset V$, the following inequality holds:*

$$\frac{1}{s}\sum_{i=1}^{n}\sum_{j=1}^{r-1} b^i_j \dim(W^i_j \cap V')(\le) < \frac{1}{r}\left(\sum_{i=1}^{n}\sum_{j=1}^{r-1} j b^i_j\right). \tag{1}$$

Definition 3.2. Let $L_{\mathbf{b}}$ be a linearization on $\mathrm{Fl}(V)^n$. We say $L_{\mathbf{b}}$ is *effective* if $(\mathrm{Fl}(V)^n)^{ss}(L_{\mathbf{b}}) \ne \emptyset$. An effective linearization $L_{\mathbf{b}}$ is *general* if $(\mathrm{Fl}(V)^n)^{ss}(L_{\mathbf{b}}) = (\mathrm{Fl}(V)^n)^{s}(L_{\mathbf{b}})$.

For a parabolic weight $\mathbf{a}$, let $\mathbf{d} = (d^1_\bullet, d^2_\bullet, \cdots, d^n_\bullet)$ be a new collection of sequences defined by $d^i_j = a^i_j - a^i_{j+1}$. $\mathbf{d}$ is called the associated *difference data*. Let $|\mathbf{d}|_j := \sum_{i=1}^n d^i_j$ and $|\mathbf{d}| = \sum_{j=1}^{r-1} |\mathbf{d}|_j$. Then $|\mathbf{a}|_j = \sum_{k=j}^{r-1} |\mathbf{d}|_k$ and $|\mathbf{a}| = \sum_{j=1}^{r-1} j|\mathbf{d}|_j$.

Theorem 3.3. *Let $\mathbf{a}$ be a parabolic weight and $\mathbf{d}$ be the associated difference data. Suppose that $\mathbf{a}$ is sufficiently small in the sense that*

$$\sum_{j=1}^{s} j(r-s)|\mathbf{d}|_j + \sum_{j=s+1}^{r-1} s(r-j)|\mathbf{d}|_j \le r \tag{2}$$

for all $1 \le s \le r-1$. Then $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}) \cong \mathrm{Fl}(V)^n /\!/_{L_{\mathbf{d}}} \mathrm{SL}_r$, where $L_{\mathbf{d}} := \otimes \pi_i^* \mathcal{O}(d^i_\bullet)$.

Proof. It is straightforward to check that (2) is equivalent to $r\sum_{j=1}^{s} |\mathbf{a}|_j - s|\mathbf{a}| \le r$.

Let $X = (\mathrm{Fl}(V)^n)^{ss}(L_{\mathbf{d}})$. Consider a family of parabolic bundles $\mathcal{E}$ over X by taking the trivial bundle, the restriction of the universal flag, and the parabolic weight $\mathbf{a}$. We claim that this is a family of $\mathbf{a}$-semistable parabolic bundles. Let $\mathcal{E}_x = (E = V \otimes \mathcal{O}, \{W^i_\bullet\}, \mathbf{a})$ be the fiber over $x \in X$.

Let F be a rank s subbundle of E whose topological degree is negative. Let $\mathcal{F}$ be the induced parabolic subbundle. Then $\mu(\mathcal{F}) \le (-1 + \sum_{j=1}^{s} |\mathbf{a}|_j)/s \le |\mathbf{a}|/r = \mu(\mathcal{E}_x)$. Thus $\mathcal{F}$ is not a destabilizing subbundle. Since E does not have a positive degree subbundle, the only possible destabilizing subbundle is of degree zero. This subbundle must be trivial because it cannot have any positive degree factor.

Suppose that $\mathcal{F} = (V' \otimes \mathcal{O}, \{W'^i_\bullet\}, \mathbf{b})$ is a rank s parabolic subbundle induced by taking an s-dimensional subspace $V' \subset V$. Then

$$\begin{aligned}\mu(\mathcal{F}) &= \frac{1}{s}\sum_{i=1}^{n}\sum_{j=1}^{r} a^i_j \dim(W^i_j \cap F|_{p^i}/W^i_{j-1} \cap F|_{p^i})\\ &= \frac{1}{s}\sum_{i=1}^{n}\sum_{j=1}^{r} a^i_j \dim(W^i_j \cap V'/W^i_{j-1} \cap V') = \frac{1}{s}\sum_{i=1}^{n}\sum_{j=1}^{r-1} d^i_j \dim(W^i_j \cap V')\\ &\le \frac{1}{r}\left(\sum_{i=1}^{n}\sum_{j=1}^{r-1} j d^i_j\right) = \frac{1}{r}\left(\sum_{i=1}^{n}\sum_{j=1}^{r-1} a^i_j\right) = \mu(\mathcal{E}_x).\end{aligned}$$

The inequality is obtained from Theorem 3.1. Therefore $\mathcal{E}_x$ is semistable.

By the universal property, there is an SL_r-invariant morphism $\pi : X \to \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$. Therefore we have the induced map $\bar{\pi} : \mathrm{Fl}(V)^n /\!/_{L_{\mathbf{d}}} \mathrm{SL}_r \to \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$. It is straightforward to check that $\bar{\pi}$ is set-theoretically injective. Since it is an injective map between two normal varieties with the same dimension and the target space is irreducible, it is an isomorphism. □

3.2. **Picard group of GIT quotient.** For the later use, we compute the Picard group of the GIT quotient.

Proposition 3.4. *Suppose that $n > 2r$. There is a general linearization $L_{\mathbf{b}} = \otimes \pi_i^* \mathcal{O}(b_\bullet^i)$ such that $\mathrm{Pic}(\mathrm{Fl}(V)^n//_{L_{\mathbf{b}}}\mathrm{SL}_r)$ is naturally identified with an index r sublattice of $\mathrm{Pic}(\mathrm{Fl}(V)^n) \cong \mathbb{Z}^{(r-1)n}$. In particular, a general linearization which is very close to the symmetric linearization $L_{\mathbf{a}}$ with $a_j^i \equiv 1$ has the property.*

Proof. First of all, we show that for the symmetric linearization $L_{\mathbf{a}}$, the codimension of the non-stable locus $\mathrm{Fl}(V)^n \setminus (\mathrm{Fl}(V)^n)^s(L_{\mathbf{a}})$ is at least two. If $(W_\bullet^i)$ is not stable, then by (1), there is an s-dimensional subspace $V' \subset V$ such that

$$\frac{1}{s}\sum_{i=1}^{n}\sum_{j=1}^{r-1}\dim(W_j^i \cap V') \geq \frac{n(r-1)}{2}. \tag{3}$$

Let $U_s \subset \mathrm{Fl}(V)^n$ be the set of flags $(W_\bullet^i)$ which satisfy (3) for some s-dimensional subspace $V' \subset V$. Let $\widetilde{U}_s \subset \mathrm{Gr}(s, V) \times \mathrm{Fl}(V)^n$ be the space of pairs $(V', (W_\bullet^i))$ such that $(W_\bullet^i)$ satisfies (3) for V'. There are two projections $p_1 : \widetilde{U}_s \to \mathrm{Gr}(s, V)$ and $p_2 : \widetilde{U}_s \to \mathrm{Fl}(V)^n$. The codimension of each fiber of p_1 in $\mathrm{Fl}(V)^n$ is $\lceil n(r-1)/2 - n(s-1)/2 \rceil = \lceil n(r-s)/2 \rceil$ because on (3), the left hand side is $n(s-1)/2$ for a general $(W_\bullet^i)$ and as the value of the left hand side increases by one, the codimension of the locus increases by one, too. Thus the codimension of $\widetilde{U}_s$ is at least $\lceil n(r-s)/2 \rceil$. On the other hand, because $p_2(\widetilde{U}_s) = U_s$, the codimension of U_s is at least $\lceil n(r-s)/2 \rceil - s(r-s)$, which is at least two for any $1 \leq s \leq r-1$. The non-stable locus $\mathrm{Fl}(V)^n \setminus (\mathrm{Fl}(V)^n)^s(L_{\mathbf{a}})$ is $\cup_{s=1}^{r-1} U_s$, so it is of codimension at least two.

By perturbing the linearization slightly, we can obtain a general linearization $L_{\mathbf{b}}$. The unstable locus with respect to $L_{\mathbf{b}}$ is contained in the non-stable locus of $L_{\mathbf{a}}$. Thus it is also of codimension at least two. In particular, $\mathrm{Pic}((\mathrm{Fl}(V)^n)^s(L_{\mathbf{b}})) = \mathrm{Pic}(\mathrm{Fl}(V)^n)$.

Let $\mathrm{Pic}^{\mathrm{SL}_r}((\mathrm{Fl}(V)^n)^s(L_{\mathbf{b}})$ be the group of linearizations. There is an exact sequence

$$0 \to \mathrm{Hom}(\mathrm{SL}_r, \mathbb{C}^*) \to \mathrm{Pic}^{\mathrm{SL}_r}((\mathrm{Fl}(V)^n)^s(L_{\mathbf{b}})) \stackrel{\alpha}{\to} \mathrm{Pic}((\mathrm{Fl}(V)^n)^s(L_{\mathbf{b}})) \to \mathrm{Pic}(\mathrm{SL}_r)$$

([12, Theorem 7.2]). Furthermore, $\mathrm{Hom}(\mathrm{SL}_r, \mathbb{C}^*) = \mathrm{Pic}(\mathrm{SL}_r) = 0$. Thus α is an isomorphism.

By Kempf's descent lemma ([11, Theorem 2.3]), an SL_r-linearized line bundle L on $(\mathrm{Fl}(V)^n)^s(L_{\mathbf{b}})$ descends to $\mathrm{Fl}(V)^n//_{L_{\mathbf{b}}}\mathrm{SL}_r$ if and only if for every closed orbit $\mathrm{SL}_r \cdot x$, the stabilizer $\mathrm{Stab}_x \cong \mu_r$ acts on L_x trivially. (See Lemma 3.5 below.) For any $L_{\mathbf{d}} = \otimes_{i=1}^n \mathcal{O}_{d_\bullet^i} \in \mathrm{Pic}(\mathrm{Fl}(V)^n)$, the stabilizer acts on $L_{\mathbf{d},x}$ as a multiplication of $\zeta^{\sum_{i=1}^n \sum_{j=1}^{r-1}(r-j)d_j^i}$ where ζ is the primitive r-th root of unity. Thus $L_{\mathbf{d}}$ descends to $\mathrm{Fl}(V)^n//_{L_{\mathbf{b}}}\mathrm{SL}_r$ if and only if $\sum_{i=1}^n \sum_{j=1}^{r-1}(r-j)d_j^i = 0$ and this equation defines an index r subgroup of $\mathrm{Pic}(\mathrm{Fl}(V)^n)$. □

Lemma 3.5. *Let $x = (W^i_\bullet)$ be a stable point on $\mathrm{Fl}(V)^n$ with respect to some linearization. Then Stab_x is isomorphic to the group of r-th root of unity.*

Proof. Let $A \in \mathrm{Stab}_x$. A has a finite order because Stab_x is finite. Since $\mathrm{char}\ \mathbb{C} = 0$, the Jordan canonical form of A cannot have any block of size larger than one. Thus we may assume that A is a diagonal matrix. Decompose $V = \oplus_\lambda V_\lambda$ into eigenspaces with respect to A. Note that any invariant space with respect to A has to be of the form $\oplus_\lambda W_\lambda$ where $W_\lambda \subset V_\lambda$. Thus all W^i_j are of these forms. If we take a diagonal matrix B which acts on V_λ as a multiplication by a^λ for some a^λ, then B preserves all W^i_j, so $B \in \mathrm{Stab}_x$. But in this case $\dim \mathrm{Stab}_x$ is the number of distinct eigenvalues minus one. Since x has a finite stabilizer, there is only one eigenvalue. Therefore A is a scalar matrix. □

Definition 3.6. A general linearization $L_{\mathbf{b}}$ is called a linearization *with a maximal stable locus* if $\mathrm{Fl}(V)^n \setminus (\mathrm{Fl}(V)^n)^{ss}(L_{\mathbf{b}})$ is of codimension at least two, (so $\rho(\mathrm{Fl}(V)^n /\!/_{L_{\mathbf{b}}} \mathrm{SL}_r) = (r-1)n$).

Remark 3.7. For small n, Proposition 3.4 is not true. For instance, if $r = 2, n = 4$ or $r = n = 3$, for a general linearization, $\mathrm{Fl}(V)^n /\!/ \mathrm{SL}_r$ is a unirational normal curve. Thus $\mathrm{Fl}(V)^n /\!/ \mathrm{SL}_r \cong \mathbb{P}^1$.

4. Wall crossing analysis

Here we describe how the moduli space changes if one varies the parabolic weight.

4.1. **Walls and chambers.** The space of all valid normalized parabolic weights is an open polytope

$$W^o_{r,n} := \{\mathbf{a} = (a^i_j)_{1 \le j \le r-1, 1 \le i \le n} \mid 1 > a^i_1 > a^i_2 > \cdots > a^i_{r-1} > 0\} \subset \mathbb{R}^{(r-1)n}.$$

Since the weight data is normalized, $a^i_r = 0$. Let $W_{r,n}$ be the closure of $W^o_{r,n}$.

The polytopes $W^o_{r,n}$ and $W_{r,n}$ have a natural wall-chamber structure: If two weights $\mathbf{a}$ and $\mathbf{a}'$ are on the same open chamber, then $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}) = \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}')$. If $\mathbf{a}$ is in one of open chambers, then $\mathbf{a}$ is general thus $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ is smooth. Note that it is possible that $\mathbf{a}$ is not effective, so $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}) = \emptyset$.

A parabolic weight $\mathbf{a}$ is on a wall if there is a strictly semi-stable parabolic bundle $\mathcal{E} = (E, \{W^i_\bullet\}, \mathbf{a})$. Then there is a unique destabilizing parabolic subbundle $\mathcal{F} = (F, \{W|_{F^i_\bullet}\}, \mathbf{b})$. For such an $\mathcal{F}$,

$$\mu(\mathcal{F}) = \frac{\deg F + \sum_{i=1}^n \sum_{j=1}^{r-1} a^i_j \dim((W^i_j \cap F|_{p^i})/(W^i_{j-1} \cap F|_{p^i}))}{\mathrm{rk}\, F} = \frac{|\mathbf{a}|}{r} = \mu(\mathcal{E}).$$

It occurs when there are two integers $d \le 0$ and $1 \le s \le r-1$, n subsets $J^i \subset [r]$ of size s such that

$$\frac{d + \sum_{i=1}^n \sum_{j \in J^i} a^i_j}{s} = \frac{|\mathbf{a}|}{r}.$$

Let $\mathcal{J} := \{J^1, J^2, \cdots, J^n\}$. Thus a stability wall is of the form

$$\Delta(s, d, \mathcal{J}) := \{\mathbf{a} \in W_{r,n}^o \mid r(d + \sum_{i=1}^n \sum_{j \in J^i} a_j^i) = s|\mathbf{a}|\}.$$

This equation is linear with respect to the variables a_j^i. So the polytope $W_{r,n}^o$ is divided by finitely many hyperplanes and each open chamber is a connected component of

$$W_{r,n}^o \setminus \left(\bigcup \Delta(s, d, \mathcal{J}) \right).$$

For $\mathcal{J} = \{J^1, J^2, \cdots, J^n\}$, set $\mathcal{J}^c := \{[r] \setminus J^1, [r] \setminus J^2, \cdots, [r] \setminus J^n\}$. Then $\Delta(s, d, \mathcal{J}) = \Delta(r - s, -d, \mathcal{J}^c)$. $\Delta(s, d, \{J, J, \cdots, J\})$ is denoted by $\Delta(s, d, nJ)$.

A wall-crossing is *simple* if it is a wall-crossing along the relative interior of a wall. Because every wall-crossing can be decomposed into a finite sequence of simple wall-crossings, it is enough to study simple wall-crossings.

Fix a wall $\Delta(s, d, \mathcal{J})$ and take a general point $\mathbf{a} \in \Delta(s, d, \mathcal{J})$. A small open neighborhood of $\mathbf{a}$ is divided into two pieces by the wall. Let $\Delta(s, d, \mathcal{J})^+$ and $\Delta(s, d, \mathcal{J})^-$ be the two connected components such that

$$r(d + \sum_{i=1}^n \sum_{j \in J^i} a_j^i) > s|\mathbf{a}|$$

and

$$r(d + \sum_{i=1}^n \sum_{j \in J^i} a_j^i) < s|\mathbf{a}|$$

respectively. Let $\mathbf{a}^+$ (resp. $\mathbf{a}^-$) be a point on $\Delta(s, d, \mathcal{J})^+$ (resp. $\Delta(s, d, \mathcal{J})^-$).

There are two functorial morphisms ([9, Theorem 3.1], [29, Section 7])

$$\begin{array}{ccccc} \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}^-) & & & & \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}^+) \\ & \searrow^{\phi^-} & & {}^{\phi^+}\swarrow & \\ & & \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}). & & \end{array}$$

Let $Y \subset \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ be the locus that one of $\phi^{\pm} : Y^{\pm} := {\phi^{\pm}}^{-1}(Y) \to Y$ is not an isomorphism. That means $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}^-) \setminus Y^- \cong \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}) \setminus Y \cong \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}^+) \setminus Y^+$. We call $Y^{\pm}$ as the *wall-crossing center*. Suppose that $\mathcal{E} = (E, \{W_\bullet^i\}, \mathbf{a})$ is on Y. Then there is a rank s destabilizing subbundle $\mathcal{E}^+ = (E^+, \{W|_{E^+}{}_\bullet^i\}, \mathbf{b})$ with $\mu(\mathcal{E}^+) = \mu(\mathcal{E})$. We have a short exact sequence $0 \to \mathcal{E}^+ \to \mathcal{E} \to \mathcal{E}^- \to 0$ where $\mathcal{E}^- = (E^- := E/E^+, \{W/E^+{}_\bullet^i\}, \mathbf{c})$. Then Y parametrizes S-equivalence classes of $\mathcal{E}^+ \oplus \mathcal{E}^-$. Therefore $Y \cong \mathrm{M}_{\mathbf{p}}(s, d, \mathbf{b}) \times \mathrm{M}_{\mathbf{p}}(r-s, -d, \mathbf{c})$.

For the same data, define $\mathbf{b}^{\pm}$ and $\mathbf{c}^{\pm}$ by using $\mathbf{a}^{\pm}$. Let $(\mathbf{E}^+, \mathbf{b}^{\pm})$ be the universal family on $\mathrm{M}_{\mathbf{p}}(s, d, \mathbf{b}^{\pm})$. Let $(\mathbf{E}^-, \mathbf{c}^{\pm})$ be the universal family on $\mathrm{M}_{\mathbf{p}}(r - s, -d, \mathbf{c}^{\pm})$. Let $\pi^+ : \mathrm{M}_{\mathbf{p}}(s, d, \mathbf{b}^+) \times \mathrm{M}_{\mathbf{p}}(r-s, -d, \mathbf{c}^+) \times \mathbb{P}^1 \to \mathrm{M}_{\mathbf{p}}(s, d, \mathbf{b}^+) \times \mathrm{M}_{\mathbf{p}}(r-s, -d, \mathbf{c}^+)$ and $\pi^- : \mathrm{M}_{\mathbf{p}}(s, d, \mathbf{b}^-) \times$

$\mathrm{M}_{\mathbf{p}}(r-s,-d,\mathbf{c}^-)\times\mathbb{P}^1\to\mathrm{M}_{\mathbf{p}}(s,d,\mathbf{b}^-)\times\mathrm{M}_{\mathbf{p}}(r-s,-d,\mathbf{c}^-)$ be projections. The exceptional fibers of ϕ^- and ϕ^+ are projective bundles

$$Y^- = \mathbb{P}R^1\pi^-_*\mathcal{P}ar\mathcal{H}om((\mathbf{E}^-,\mathbf{c}^-),(\mathbf{E}^+,\mathbf{b}^-))$$

and

$$Y^+ = \mathbb{P}R^1\pi^+_*\mathcal{P}ar\mathcal{H}om((\mathbf{E}^+,\mathbf{b}^+),(\mathbf{E}^-,\mathbf{c}^+))$$

respectively. Fiberwisely, $\phi^{-\,-1}([\mathcal{E}^+\oplus\mathcal{E}^-]) = \mathbb{P}\mathrm{Ext}^1((E^-,\{W/E^{+i}_\bullet\},\mathbf{c}^-),(E^+,\{W|_{E^+\bullet}^{\;i}\},\mathbf{b}^-))$ and $\phi^{+-1}([\mathcal{E}^+\oplus\mathcal{E}^-]) = \mathbb{P}\mathrm{Ext}^1((E^+,\{W|_{E^+\bullet}^{\;i}\},\mathbf{b}^+),(E^-,\{W/E^{+i}_\bullet\},\mathbf{c}^+))$.

Remark 4.1. An element $\mathcal{E} = (E,\{W^i_\bullet\},\mathbf{a}^-)\in Y^-$ fits into a short exact sequence $0\to(E^+,\{V^i_\bullet\},\mathbf{b}^-)\to(E,\{W^i_\bullet\},\mathbf{a}^-)\to(E^-,\{X^i_\bullet\},\mathbf{c}^-)\to 0$ of parabolic bundles with parabolic weights. Their underlying bundles also fit into $0\to E^+\to E\to E^-\to 0$. After the wall-crossing, $\phi^{+-1}(\phi^-([\mathcal{E}]))$ is the set of parabolic bundles $\mathcal{F}$ which fits into the sequence $0\to(E^-,\{X^i_\bullet\},\mathbf{c}^+)\to\mathcal{F}\to(E^+,\{V^i_\bullet\},\mathbf{b}^+)\to 0$ of parabolic bundles with parabolic weights $\mathbf{c}^+$, $\mathbf{a}^+$. Again, their underlying bundles fit into the exact sequence $0\to E^-\to F\to E^+\to 0$.

Proposition 4.2 ([29, Section 7])**.** *Suppose that* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^\pm)$ *are nonempty. The blow-up of* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^-)$ *along* Y^- *is isomorphic to the blow-up of* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^+)$ *along* Y^+*. In paticular,* $\dim Y^- + \dim Y^+ - \dim Y = \dim \mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}) - 1$.

Note that for some weight data $\mathbf{a}$, the moduli space $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$ may be empty.

Proposition 4.3. *Suppose that* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^+)=\emptyset$*. Then* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^-)$ *has a projective bundle structure over* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}) = \mathrm{M}_{\mathbf{p}}(s,d,\mathbf{b}^-)\times\mathrm{M}_{\mathbf{p}}(r-s,-d,\mathbf{c}^-)$.

Proof. It is sufficient to show that $Y^- = \mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^-)$. If Y^- is a proper subvariety of $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^-)$, $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^-)\setminus Y^-\cong\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}^+)\setminus Y^+\neq\emptyset$. □

4.2. **Scaling up.** In this section, we examine a special kind of wall-crossing. Let $\mathbf{a}$ be a general parabolic weight. For a positive real number $c>0$, define a parabolic weight $\mathbf{a}(c)$ as $a(c)^i_j := ca^i_j$. When $c=\epsilon\ll 1$, $\mathbf{a}(c)$ satisfies the smallness condition in Theorem 3.3, so $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}(\epsilon)) = \mathrm{Fl}(V)^n/\!/_{L_{\mathbf{a}(\epsilon)}}\mathrm{SL}_r$. As c increases, we may cross several walls. By perturbing if it is necessary, we may assume that all wall-crossings are simple. We call this type of wall-crossings *scaling wall-crossings*.

Suppose that $\Delta(s,d,\mathcal{J})$ is a wall we can meet and $\mathbf{a}^0 := \mathbf{a}(c)\in\Delta(s,d,\mathcal{J})$ is a general point. Let $\mathbf{a}^\pm = \mathbf{a}(c\pm\epsilon)$. For a parabolic bundle $\mathcal{E} = (E,\{W^i_\bullet\},\mathbf{a}^0)\in Y$, let $\mathcal{E}^+$ be the destabilizing subbundle with respect to $\mathbf{a}^0$ and $\mathcal{E}^-$ be the quotient $\mathcal{E}/\mathcal{E}^+$. The induced weight data of $\mathcal{E}^+$ with respect to $\mathbf{a}^\pm$ is denoted by $\mathbf{b}^\pm$, as before.

Here we would like to compute $\dim\mathrm{Ext}^1(\mathcal{E}^-,\mathcal{E}^+)$ with respect to $\mathbf{a}^-$. By Serre duality (Lemma 2.11), $\mathrm{Ext}^1(\mathcal{E}^-,\mathcal{E}^+)\cong\mathrm{SParHom}(\mathcal{E}^+\otimes\mathcal{O}(-(n-2)),\mathcal{E}^-)^*$. Consider the following short exact sequence of sheaves

$$\begin{aligned}&0\to\mathcal{SP}ar\mathcal{H}om(\mathcal{E}^+\otimes\mathcal{O}(-(n-2)),\mathcal{E}^-)\to\mathcal{H}om(E^+\otimes\mathcal{O}(-(n-2)),E^-)\\ &\quad\to\frac{\bigoplus_{i=1}^n\mathrm{Hom}(E^+\otimes\mathcal{O}(-(n-2))|_{p^i},E^-|_{p^i})}{\bigoplus_{i=1}^n N_{p^i}(\mathcal{E}^+\otimes\mathcal{O}(-(n-2)),\mathcal{E}^-)}\to 0\end{aligned}\tag{4}$$

where $N_p(\mathcal{E}_1, \mathcal{E}_2)$ is the subspace of strongly parabolic maps in $\mathrm{Hom}(E_1|_p, E_2|_p)$ at $p \in \mathbb{P}^1$. For $\mathbf{a}^-$, $\mu(\mathcal{E}^-) > \mu(\mathcal{E}^+)$. Because $(\mathcal{E}^-, \mathbf{b}^-)$ and $(\mathcal{E}^+, \mathbf{c}^-)$ are stable,

$$\mathrm{H}^1(\mathcal{SP}ar\mathcal{H}om(\mathcal{E}^+ \otimes \mathcal{O}(-(n-2)), \mathcal{E}^-)) = \mathrm{Ext}^0(\mathcal{E}^-, \mathcal{E}^+)^* = \mathrm{ParHom}(\mathcal{E}^-, \mathcal{E}^+)^* = 0$$

by Lemma 2.11. Thus we have an exact sequence of vector spaces

$$\begin{aligned}(5)\qquad & 0 \to \mathrm{SParHom}(\mathcal{E}^+ \otimes \mathcal{O}(-(n-2)), \mathcal{E}^-) \to \mathrm{Hom}(E^+ \otimes \mathcal{O}(-(n-2)), E^-) \\ & \to \frac{\bigoplus_{i=1}^n \mathrm{Hom}(E^+ \otimes \mathcal{O}(-(n-2))|_{p^i}, E^-|_{p^i})}{\bigoplus_{i=1}^n N_{p^i}(\mathcal{E}^+ \otimes \mathcal{O}(-(n-2)), \mathcal{E}^-)} \to 0.\end{aligned}$$

Recall that at each parabolic point p^i, the intersection of E^+ with $E|_{p^i}$ is described by an s-subset $J^i \subset [r]$.

Lemma 4.4. *Suppose that* $\mathrm{rk}\, E^+ = s$. *In the above situation,*

$$\dim N_{p^i}(\mathcal{E}^+ \otimes \mathcal{O}(-(n-2)), \mathcal{E}^-) = \dim \omega_{J^i}$$

where ω_{J^i} *is the Schubert class in* $\mathrm{H}^*(\mathrm{Gr}(s, r))$ *associated to the increasing sequence* J^i.

Proof. Note that $\dim N_{p^i}(\mathcal{E}^+ \otimes \mathcal{O}(-(n-2)), \mathcal{E}^-) = \dim N_{p^i}(\mathcal{E}^+, \mathcal{E}^-)$. At the fiber of p^i, take an ordered basis $\{e_j\}$ of $E|_{p^i}$ by choosing a nonzero vector e_j for each $W_j^i \setminus W_{j-1}^i$. Then $E^+|_{p^i}$ (resp. $E^-|_{p^i}$) is spanned by $\{e_j\}_{j \in J^i}$ (resp. $\{e_j\}_{j \in [r] \setminus J^i}$). Now to construct a map in $N_{p^i}(\mathcal{E}^+, \mathcal{E}^-)$, $e_j \in E^+|_{p^i}$ can be mapped into the subspace of generated by e_k where $k < j$ and $k \notin J^i$. Therefore the dimension is $\sum_{j=1}^s (J_j^i - j)$ and this is equal to $\dim \omega_{J^i}$. □

From (5) and Lemma 4.4, we obtain the following result.

Proposition 4.5. *Suppose that* $n \gg 0$. *Let* $\mathcal{E}$ *be a parabolic bundle on* $Y \subset \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ *for a scaling wall* $\Delta(s, d, \mathcal{J})$. *Let* $\mathcal{E}^+$ *be the destabilizing subbundle and* $\mathcal{E}^- = \mathcal{E}/\mathcal{E}^+$. *With respect to the parabolic weight* $\mathbf{a}^-$,

$$(6)\qquad \dim \mathrm{Ext}^1(\mathcal{E}^-, \mathcal{E}^+) = \dim \mathrm{Hom}(E^+ \otimes \mathcal{O}(-(n-2)), E^-) - ns(r-s) + \sum_{i=1}^n \dim \omega_{J^i}.$$

Let a be a general parabolic weight and consider the scaling wall-crossing. For the weight data $\mathbf{a}(\epsilon)$, $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}(\epsilon))$ is the GIT quotient $\mathrm{Fl}(V)^n /\!/_{L_{\mathbf{a}(\epsilon)}} \mathrm{SL}_r$, and any underlying vector bundle E of $\mathcal{E} \in \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}(\epsilon))$ is trivial.

We show that the first wall we meet while the scaling wall-crossing is of the form $\Delta(s, -1, n[s])$. Let $\Delta(s, d, \mathcal{J})$ be the first wall. Because E does not have any positive degree subbundle, $d \le 0$. A wall of the form $\Delta(s, 0, \mathcal{J})$ does not appear while scaling. (These walls are GIT walls.) The maximal parabolic slope we can obtain occurs when $|d|$ is the smallest one and $J^i = [s]$.

Furthermore, the walls $\Delta(s, -1, n[s])$ for $2 \le s \le r-2$ do not intersect $W_{r,n}^o$.

Lemma 4.6. *The first wall is either* $\Delta(1, -1, n[1])$ *or* $\Delta(r-1, -1, n[r-1])$. *Moreover, only one of them occurs during the scaling wall-crossing.*

Proof. Suppose that $\Delta(s,-1,n[s]) \cap W_{r,n}^o$ is nonempty and the wall actually provides a nontrivial wall-crossing for $2 \le s \le r-2$. For notational simplicity, we may assume that $\mathbf{a} = \mathbf{a}(1) \in \Delta(s,-1,n[s])$. $\mathcal{E} = (E = \mathcal{O}^r, \{W_\bullet^i\}, \mathbf{a}^-) \in Y^-$ has a parabolic subbundle $\mathcal{F} = (F \cong \mathcal{O}(-1) \oplus \mathcal{O}^{s-1}, \{W|_{F_\bullet^i}\}, \mathbf{b}^-)$ such that $W|_{F_j^i} = W_j^i$ for all i and $1 \le j \le s$, and $\mu(\mathcal{F}) = \mu(\mathcal{E})$ with respect to $\mathbf{a}$. Then there is an $(s+1)$-dimensional vector space V' such that $F \to E$ factors through $F \to V' \otimes \mathcal{O} \to E$. Let $\mathcal{F}'$ be the induced parabolic subbundle of $\mathcal{E}$ whose underlying bundle is $V' \otimes \mathcal{O}$. With respect to $\mathbf{a}^- = \mathbf{a}(1-\epsilon)$, $\mathcal{E}$ is stable. Thus we have

$$\frac{1}{s+1}\sum_{i=1}^n \sum_{j=1}^s a_j^i = \mu(\mathcal{F}') < \mu(\mathcal{E}) = \frac{1}{r}|\mathbf{a}|.$$

On the other hand, let $F'' \subset F$ be any rank one trivial subbundle and let $\mathcal{F}''$ be the parabolic subbundle induced by F''. Then we have

$$\sum_{i=1}^n a_s^i \le \mu(\mathcal{F}'') < \mu(\mathcal{E}) = \frac{1}{r}|\mathbf{a}|.$$

By taking the weighted average of left sides, we have

$$\frac{1}{r}|\mathbf{a}| > \frac{1}{r}\left((s+1)\frac{1}{s+1}\sum_{i=1}^n\sum_{j=1}^s a_j^i + (r-s-1)\sum_{i=1}^n a_s^i\right) = \frac{1}{r}\sum_{i=1}^n\left(\sum_{j=1}^{s-1} a_j^i + (r-s)a_s^i\right) > \frac{1}{r}|\mathbf{a}|$$

and this is a contradiction.

Now suppose that the first wall is $\Delta(1,-1,n[1])$. We may assume that $\mathbf{a} \in \Delta(1,-1,n[1])$. $\mathcal{E} = (E = \mathcal{O}^r, \{W_\bullet^i\}, \mathbf{a}^-) \in Y^-$ has a subbundle $\mathcal{F}$ whose underlying bundle F is $\mathcal{O}(-1)$ such that $F|_{p^i} = W_1^i$. We can take a 2-dimensional V' and $\mathcal{F}'$ as before. Then

$$\frac{1}{2}\sum_{i=1}^n a_1^i \le \mu(\mathcal{F}') < \mu(\mathcal{E}) = \frac{1}{r}|\mathbf{a}|.$$

Now assume that $\mathcal{E}$ is also in the wall-crossing center for $\Delta(r-1,-1,n[r-1])$. Then there is a subbundle $G \cong \mathcal{O}(-1) \oplus \mathcal{O}^{r-2}$ of E such that $G|_{p^i} = W_{r-1}^i$. Let $\mathcal{G}$ be the induced parabolic subbundle whose underlying bundle is G. Let V' be the $(r-2)$-dimensional vector space such that $G = \mathcal{O}(-1) \oplus (V' \otimes \mathcal{O})$ and let $\mathcal{G}'$ be the parabolic subbundle associated to $V' \otimes \mathcal{O}$. Then

$$\frac{1}{r-2}\sum_{i=1}^n \sum_{j=2}^{r-1} a_j^i \le \mu(\mathcal{G}') < \mu(\mathcal{E}) = \frac{1}{r}|\mathbf{a}|.$$

An weighted average of the left hand sides is $\mu(\mathcal{E})$. This makes a contradiction. □

Remark 4.7. The proof tells us that two wall-crossing centers Y_1^- for $\Delta(1,-1,n[1])$ and Y_{r-1}^- for $\Delta(r-1,-1,n[r-1])$ cannot be simultaneously stable on $\mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r$ for any linearization L.

The first wall-crossing is always a blow-up.

Lemma 4.8. *The scaling wall-crossing along* $\Delta(1,-1,n[1])$ *(resp.* $\Delta(r-1,-1,n[r-1])$*) is a blow-up along* $Y^- \cong \mathrm{M}_{\mathbf{p}}(r-1,1,\mathbf{c})$ *(resp.* $\mathrm{M}_{\mathbf{p}}(r-1,-1,\mathbf{b})$*).*

Proof. By Proposition 4.2, the wall-crossing is a blow-up if and only if $Y^- = Y$ if and only if $\dim \mathrm{Ext}^1(\mathcal{E}^-,\mathcal{E}^+)$ with respect to $\mathbf{a}^-$ is one for $\mathcal{E} \in Y^-$.

Consider the case of $\Delta(1,-1,n[1])$. The underlying bundle E^+ is $\mathcal{O}(-1)$ and $E^- = \mathcal{O}(1)\oplus\mathcal{O}^{r-2}$. Thus $\mathrm{Hom}(E^+\otimes\mathcal{O}(-(n-2)),E^-) \cong \mathrm{H}^0(\mathcal{O}(n)\oplus\mathcal{O}(n-1)^{r-2})$. Because $J^i = [1]$, $\dim \omega_{J^i} = 0$. By using Proposition 4.5, it is straightforward to see that $\dim \mathrm{Ext}^1(\mathcal{E}^-,\mathcal{E}^+) = 1$. The blow-up center is obtained from the description in Section 4.1 and the fact that $\mathrm{M}_{\mathbf{p}}(1,d,\mathbf{b})$ is a point. The other case is similar. □

Now suppose that $\mathbf{a}$ is a general small weight such that $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}) = \mathrm{Fl}(V)^n//_{L_{\mathbf{a}}}\mathrm{SL}_r$ and $\rho(\mathrm{Fl}(V)^n//_{L_{\mathbf{a}}}\mathrm{SL}_r) = (r-1)n$. Such a weight exists by Theorem 3.3 and Proposition 3.4. By scaling up the weight, for some $c > 1$, the weight data $\mathbf{a}(c)$ crosses either $\Delta(1,-1,n[1])$-wall or $\Delta(r-1,-1,n[r-1])$-wall. Then for $\mathbf{a}(c+\epsilon)$, $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}(c+\epsilon))$ has Picard number $(r-1)n+1$, which is maximal because the moduli stack $\mathrm{M}_{\mathbf{p}}(r,0)$ has the same Picard number.

Definition 4.9. A general parabolic weight $\mathbf{a}$ is *dominant* if $\rho(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})) = (r-1)n+1$.

Such a weight is called dominant because any other $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{b})$ can be obtained as a rational contraction of $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$ (See Section 6).

When $\mathbf{a}$ is general, then the moduli stack $\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a})$ of $\mathbf{a}$-stable parabolic bundles is a Deligne-Mumford stack. We have the following diagram:

$$\begin{array}{ccc} \mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a}) & \xrightarrow{\iota} & \mathrm{M}_{\mathbf{p}}(r,0) \\ \downarrow{\scriptstyle p} & & \\ \mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}) & & \end{array}$$

Suppose further that $\mathbf{a}$ is dominant. Since $\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a})$ is an open substack of $\mathrm{M}_{\mathbf{p}}(r,0)$, the pull-back morphism $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0)) \to \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a}))$ is surjective ([16, Corollary 3.4]) and the latter is a quotient of a free abelian group of rank $(r-1)n+1$. But since we have a morphism $p^* : \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})) \to \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a}))$, which is injective ([27, Theorem 11.1.2]), $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$ is a rank $(r-1)n+1$ sub free abelian group of $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a}))$. We can conclude that $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0)) \cong \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a}))$ and we may identify $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$ with a finite index subgroup of $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0))$.

It also follows that any global section of $\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a})$ is uniquely extended to $\mathrm{M}_{\mathbf{p}}(r,0)$. Then for any $L \in \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$,

$$\mathrm{H}^0(\mathrm{M}_{\mathbf{p}}(r,0),p^*L) \cong \mathrm{H}^0(\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a}),p^*L) \cong \mathrm{H}^0(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}),L).$$

The last isomorphism follows from [27, Theorem 11.1.2].

Thus $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$ is identified with $\bigoplus_{L\in\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))}\mathrm{H}^0(\mathrm{M}_{\mathbf{p}}(r,0),p^*L)$. Since $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$ is a finite index subgroup of $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0))$, we can conclude that $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r,0)) \cong \mathbb{V}^\dagger$ is finitely generated if and only if $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$ is finitely generated.

Indeed, $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r,0))$ can be identified with that of a projective moduli space.

Proposition 4.10. *Let* $\mathbf{a}$ *be a dominant weight. Then* $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})) = \mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r,0))$.

Proof. Since a flip does not affect to the Cox ring, we may assume that $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$ is a blow-up of $\mathrm{Fl}(V)^n/\!/_L\mathrm{SL}_r$. Let $\pi : \mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}) \to \mathrm{Fl}(V)^n/\!/_L\mathrm{SL}_r$ be the blow-up. The exceptional divisor Y^+ parametrizes the parabolic bundles with nontrivial underlying bundles (Remark 4.1). Thus Y^+ is the generalized theta divisor Θ described in Section 2.2.

We already showed that for any $L \in \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$, $\mathrm{H}^0(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}),L) \cong \mathrm{H}^0(\mathrm{M}_{\mathbf{p}}(r,0),p^*L)$ for $p^* : \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})) \to \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0)^s(\mathbf{a})) \cong \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0))$. Thus it is sufficient to show that a line bundle $L \in \mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0))$ has a nonzero section only if L is in the image of p^*. Suppose that $\sigma \in \mathbb{V}^\dagger_{\ell,\vec{\lambda}}$ is a nonzero section. Note that $\mathbb{V}^\dagger_{\ell,\vec{\lambda}} \hookrightarrow \mathbb{V}^\dagger_{\ell',\vec{\lambda}}$ for $\ell' \ge \ell$ and for some $\ell' \gg 0$ determined by $\vec{\lambda}$, $\mathbb{V}^\dagger_{\ell',\vec{\lambda}} \cong V_{\vec{\lambda}}^{*\mathrm{SL}_r}$, the space of classical invariants. Then $V_{\vec{\lambda}}^{*\mathrm{SL}_r} \cong \mathrm{H}^0(\mathrm{Fl}(V)^n/\!/_L\mathrm{SL}_r, L_{\vec{\lambda}}) \cong \mathrm{H}^0(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}),\pi^*L_{\vec{\lambda}})$ for a certain line bundle $L_{\vec{\lambda}}$. Then σ is a section of $\pi^*L_{\vec{\lambda}} - (\ell'-\ell)\Theta$, which is a line bundle on $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$. □

5. THE MODULI SPACE IS A MORI DREAM SPACE

In this section, we prove Theorem 5.1. The finite generation of $\mathbb{V}^\dagger$ (Theorem 5.3) follows immediately.

Theorem 5.1. *For any rank* r *and a general parabolic weight* $\mathbf{a}$*,* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$ *is of Fano type.*

Recall that a $\mathbb{Q}$-factorial normal varieity X is of *Fano type* if there is an effective $\mathbb{Q}$-divisor Δ such that $-(K_X+\Delta)$ is ample and (X,Δ) is a klt pair.

By [3, Corollary 1.3.2], a $\mathbb{Q}$-factorial normal variety of Fano type is a Mori dream space.

Corollary 5.2. *For any general parabolic weight* $\mathbf{a}$*,* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$ *is a Mori dream space.*

Theorem 5.3. *The algebra* $\mathbb{V}^\dagger$ *of conformal blocks is finitely generated.*

Proof. By Proposition 4.10, the Cox ring of the moduli stack $\mathrm{M}_{\mathbf{p}}(r,0)$ is the same with that of $\mathrm{M}_{\mathbf{a}}(r,0,\mathbf{a})$ if $\mathbf{a}$ is dominant. If $n > 2r$, by Theorem 3.3 and Proposition 3.4, there is a small weight $\mathbf{b}$ such that $\rho(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{b})) = (r-1)n$. During the scaling wall-crossing, the first wall-crossing is a blow-up. Thus there is a dominant weight $\mathbf{a} = \mathbf{b}(c)$. Then $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r,0)) = \mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$ is finitely generated by Corollary 5.2 and [17, Theorem 2.9].

When n is small, by Lemma 5.4, for a sufficiently large point configuration $\mathbf{q} \supset \mathbf{p}$, there is a morphism $\mathrm{M}_{\mathbf{q}}(r,0,\mathbf{a}') \to \mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$ for some $\mathbf{a}'$. Then $\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$ is a subgroup of $\mathrm{Pic}(\mathrm{M}_{\mathbf{q}}(r,0,\mathbf{a}'))$. Let $H := \mathrm{Hom}(\mathrm{Pic}(\mathrm{M}_{\mathbf{q}}(r,0,\mathbf{a}'))/\mathrm{Pic}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})),\mathbb{C}^*)$. There is a natural action of H on $\mathrm{Cox}(\mathrm{M}_{\mathbf{q}}(r,0,\mathbf{a}'))$ and by propagation of vacua ([32, Theorem 3.15]),

$$\mathbb{V}^\dagger \cong \bigoplus_{\ell,\vec{\lambda}} \mathbb{V}^\dagger_{\ell,(\lambda^1,\lambda^2,\cdots,\lambda^n)} \cong \bigoplus_{\ell,\vec{\lambda}} \mathbb{V}^\dagger_{\ell,(\lambda^1,\lambda^2,\cdots,\lambda^n,0,0,\cdots,0)} \cong \mathrm{Cox}(\mathrm{M}_{\mathbf{q}}(r,0,\mathbf{a}'))^H.$$

A torus-invariant subring of a finitely generated algebra is finitely generated, too ([12, Theorem 3.3]). □

Lemma 5.4. *Let* $\mathbf{a}$ *be a general effective parabolic weight. Then for any finite point configuration* $\mathbf{q} \supset \mathbf{p}$*, there is a parabolic weight* $\mathbf{a}'$ *such that there is a morphism* $\mathrm{M}_{\mathbf{q}}(r, 0, \mathbf{a}') \to \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$*.*

Proof. Suppose that $\mathbf{p} = (p^1, p^2, \cdots, p^n)$ and $\mathbf{q} = (p^1, p^2, \cdots, p^{n+m})$. Let $\mathbf{a}'$ be a parabolic weight such that $a'^i_\bullet = a^i_\bullet$ for $i \le n$ and a'^i_j are sufficiently small for $i > n$. There is a natural 'forgetful' map

$$\begin{array}{rcl} \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}') & \to & \mathrm{M}_{\mathbf{q}}(r, 0, \mathbf{a}) \\ (E, \{W^i_\bullet\}, \mathbf{a}') & \mapsto & (E, \{W^i_\bullet\}_{i \le n}, \mathbf{a}). \end{array}$$

This map is regular, because small weights $(a^i_\bullet)_{i>n}$ do not affect on the inequalities for the stability. □

Remark 5.5. The proof of Theorem 5.3 does not provide any explicit set of generators. When $r \le 3$ and $\mathbf{p}$ is a generic configuration of points, by using a degeneration method, Manon showed that the set of r^{n-1} level one conformal blocks generates $\mathbb{V}^\dagger$ ([21, Theorem 3], [23, Theorem 1.5]). For $r \ge 4$, the set of level one conformal blocks is insufficient to generate $\mathbb{V}^\dagger$. We expect that the generic configuration assumption is not essential.

Remark 5.6. In [26], Mukai showed that a certain $\mathbf{G}_a^n$-invariant ring of a polynomial ring is finitely generated. He identified the invariant subring with the Cox ring of the moduli space $\mathrm{M}_{\mathbf{p}}(2, -1, \mathbf{a})$ for a certain $\mathbf{a}$. The outline of our proof of the finite generation is a generalization of [26]. Note that the finite generation of $\mathrm{Cox}(\mathrm{M}_{\mathbf{p}}(2, -1, \mathbf{a}))$ follows from Corollary 5.2, because Proposition 6.7 implies that $\mathrm{M}_{\mathbf{p}}(2, -1, \mathbf{a})$ is a rational contraction of $\mathrm{M}_{\mathbf{q}}(2, 0, \mathbf{a}')$.

The remaining of this section is devoted to the proof of Theorem 5.1. We start with the computation of the canonical divisor. $\overline{\mathcal{O}}(D)$ denotes the descent of a line bundle $\mathcal{O}(D)$ on X to the GIT quotient $X/\!/G$.

Lemma 5.7. *Let* L *be a general linearization on* $\mathrm{Fl}(V)^n$ *with a maximal stable locus. The canonical divisor* K *of* $\mathrm{Fl}(V)^n/\!/_L \mathrm{SL}_r$ *is*

$$\bigotimes_{i=1}^n \pi_i^* \overline{\mathcal{O}}(-2, -2, \cdots, -2).$$

Proof. Because the canonical divisor is S_n-invariant, we have $K = \bigotimes_{i=1}^n \pi_i^* \overline{\mathcal{O}}(b_1, \cdots, b_{r-1})$. Let $\widetilde{C}^i_j \cong \mathbb{P}^1 \subset \mathrm{Fl}(V)^n$ be a Schubert curve that is obtained by taking a family of parabolic bundles $(\mathcal{O}^r, \{W^i_\bullet\})$ such that $W^l_\bullet$ for $l \ne i$ and W^i_k for $k \ne j$ are fixed and general, but W^i_j is varying as a one-dimensional family of subspaces in W^i_{j+1} containing W^i_{j-1}.Since L is a linearization with a maximal stable locus, $\widetilde{C}^i_j$ does not intersect the unstable locus whose codimension is at least two. Thus by taking its image in $\mathrm{Fl}(V)^n/\!/_L \mathrm{SL}_r$, we have a curve C^i_j. By projection formula, $C^i_j \cdot \bigotimes_{i=1}^n \pi_i^* \overline{\mathcal{O}}(b_1, \cdots, b_{r-1}) = b_j$.

Note that $\widetilde{C}^i_j$ is indeed a curve on a fiber of the projection map $p : \mathrm{Fl}(V)^n \to \mathrm{Fl}(V)^{n-1}$ which forgets the i-th factor. The tangent bundle of the i-th factor $\mathrm{Fl}(V)$ is the quotient

$$0 \to \mathrm{Hom}_{F.}(V, V) \otimes \mathcal{O} \to \mathrm{Hom}(V, V) \otimes \mathcal{O} \to \mathcal{T}_{\mathrm{Fl}(V)} \to 0$$

where $\mathrm{Hom}_{F.}(V,V)$ is the space of endomorphisms which preserve the flag. The restriction of the sequence to $\widetilde{C}_j^i$ is isomorphic to

$$0 \to \mathcal{O}^{(r^2+r-4)/2} \oplus \mathcal{O}(-1)^2 \to \mathcal{O}^{r^2} \to \mathcal{T}_{\mathrm{Fl}(V)}|_{\widetilde{C}_j^i} \to 0.$$

So $\deg \mathcal{T}_{\mathrm{Fl}(V)}|_{\widetilde{C}_j^i} = 2$, and thus $\deg \mathcal{T}_{\mathrm{Fl}(V)^n}|_{\widetilde{C}_j^i} = 2$. By the SL_r-action, $\widetilde{C}_j^i$ deforms without fixed points. Thus along the fiber of the quotient map, the restriction of the tangent bundle is trivial. Therefore $\deg \mathcal{T}_{\mathrm{Fl}(V)^n/\!/_L\mathrm{SL}_r}|_{C_j^i} = 2$. So $b_j = C_j^i \cdot K = -2$. □

Corollary 5.8. *For a general small weight* $\mathbf{a}$, $\mathrm{H}^0(-K) = \mathbb{V}^\dagger_{(r-1)n,(\lambda,\lambda,\cdots,\lambda)}$ *where* $\lambda = 2(\sum_{j=1}^{r-1}\omega_j)$.

Proof. By Lemma 5.7, $-K = \otimes_{i=1}^n \pi_i^*\overline{\mathcal{O}}(2,2,\cdots,2)$. This is a product of $(r-1)n$ line bundles of the form $\pi_i^*\overline{\mathcal{O}}(e_a) \otimes \pi_j^*\overline{\mathcal{O}}(e_{r-a})$ where e_k is the standard k-th vector. Each $\mathrm{H}^0(\pi_i^*\overline{\mathcal{O}}(e_a) \otimes \pi_j^*\overline{\mathcal{O}}(e_{r-a}))$ is identified with $\mathbb{V}^\dagger_{1,(0,\cdots,0,\omega_{r-a},0,\cdots,0,\omega_a,0,\cdots)}$ where ω_{r-a} is on the i-factor and ω_a is on the j-th factor ([7, Proposition 1.3]). Now by taking the tensor product of them, we obtain the statement. □

Proposition 5.9. *Let* $\mathbf{a}$ *be a dominant parabolic weight. Let* $\mathrm{M} = \mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$. *Then* $\mathrm{H}^0(-K_{\mathrm{M}}) = \mathbb{V}^\dagger_{2r,(\lambda,\lambda,\cdots,\lambda)}$ *where* $\lambda = (2\sum_{j=1}^{r-1}\omega_j)$.

Proof. We may assume that $\mathbf{a}$ is the weight data right after the first wall-crossing while the scaling wall-crossing from $\mathbf{a}(\epsilon)$. By Lemma 4.8, the first wall-crossing is the blow-up along $\mathrm{M}_{\mathbf{p}}(r-1,1,\mathbf{c})$ or $\mathrm{M}_{\mathbf{p}}(r-1,-1,\mathbf{b})$. In particular, the codimension of the blow-up center is $(r-1)n-2r+1$. By the blow-up formula of canonical divisors, if K denotes the canonical divisor of $\mathrm{Fl}(V)^n/\!/_{L_{\mathbf{a}}}\mathrm{SL}_r$ and if $\pi : \mathrm{M} \to \mathrm{Fl}(V)^n/\!/_{L_{\mathbf{a}}}\mathrm{SL}_r$ is the blow-up morphism, $-K_{\mathrm{M}} = \pi^*(-K) - ((r-1)n-2r)Y^+$. Since Y^+ is the theta divisor Θ, $\mathrm{H}^0(-K_{\mathrm{M}}) = \mathbb{V}^\dagger_{(r-1)n-((r-1)n-2r),(\lambda,\lambda,\cdots,\lambda)} = \mathbb{V}^\dagger_{2r,(\lambda,\lambda,\cdots,\lambda)}$. □

A key theorem is the following classical result of Pauly.

Theorem 5.10 ([28, Theorem 3.3, Corollary 6.7])**.** *Let* $\mathbf{a} = (a^i_\bullet)$ *be a parabolic weight. Then there is an ample line bundle* $\Theta_{\mathbf{a}}$ *on* $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$ *such that* $\mathrm{H}^0(\Theta_{\mathbf{a}})$ *is canonically identified with* $\mathbb{V}^\dagger_{\ell,(\lambda^1,\lambda^2,\cdots,\lambda^n)}$ *where* ℓ *is the smallest positive integer such that* $\ell a^i_j \in \mathbb{Z}$ *and* $\lambda^i_j = \ell a^i_j$.

Let $\mathbf{a}_c$ be a parabolic weight such that $({a_c}^i_\bullet) = \frac{1}{r}(r-1,r-2,\cdots,1)$. By Theorem 5.10, $-K_{\mathrm{M}} \in \mathbb{V}^\dagger_{2r,(\lambda,\lambda,\cdots,\lambda)}$ is an ample divisor on $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}_c)$. Thus $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}_c)$ is a Fano variety.

But there are two technical problems. First of all, in many cases (even for rank two), $\mathbf{a}_c$ lies on a wall, so it is not general. To avoid this technical difficulty, we perturb the weight data slightly. Let $\mathbf{a}_d$ be a general small perturbation of $\mathbf{a}_c$ such that the set of walls that we meet while the scaling wall crossing from $\mathbf{a}_c(\epsilon)$ to $\mathbf{a}_c$ is equal to that for the scaling wall crossing from $\mathbf{a}_d(\epsilon)$ to $\mathbf{a}_d$. Since $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}_d(\epsilon)) \cong \mathrm{Fl}(V)^n/\!/_{L_{\mathbf{a}_d(\epsilon)}}\mathrm{SL}_r$ and $L_{\mathbf{a}_d(\epsilon)}$ is sufficiently close to the symmetric linearization, by Proposition 3.4, $\rho(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}_d(\epsilon))) = (r-1)n$ if $n > 2r$.

The second issue is that, in general, $\mathbf{a}_d$ is not dominant. Then the fact that $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}_d)$ is a MDS does not imply the finite generation of $\mathbb{V}^{\dagger}$. We show that if n is sufficiently large, however, $\mathbf{a}_d$ is always dominant.

Proposition 5.11. *Let $\Delta(s, d, \mathcal{J})$ be a wall one meets while the scaling wall-crossing from $\mathbf{a}_d(\epsilon)$ to $\mathbf{a}_d$. Suppose that $n \gg 0$. Then the wall-crossing is not a blow-down. In particular, $\mathbf{a}_d$ is dominant.*

Proof. Let $\mathbf{a} = \mathbf{a}_d(c)$ be the weight on the wall, and $\mathbf{a}^{\pm}$ are weights near the wall as before. Recall that the wall-crossing center Y^- is isomorphic to $\mathbb{P}\mathrm{Ext}^1(\mathcal{E}^-, \mathcal{E}^+)$-bundle over $\mathrm{M}_{\mathbf{p}}(s, d, \mathbf{b}) \times \mathrm{M}_{\mathbf{p}}(r-s, -d, \mathbf{c})$.

Suppose that a general point on Y^- parametrizes a parabolic bundle with a non-trivial underlying bundle. Then $Y^- \subset \Theta$. If the wall-crossing is blow-down, then $Y^- = \Theta$ because Θ is an irreducible divisor. But Θ is not contracted by scaling-up by Lemma 6.2. Thus we may assume that a general point of Y^- parametrizes a parabolic bundle with a trivial underlying bundle.

Let $\mathcal{E} = (E, \{W^i_\bullet\}, \mathbf{a}^-)$ be a general point on Y^- and $\mathcal{E}^+ = (E^+, \{W|_{E^i_\bullet}\}, \mathbf{b}^-)$ (resp. $\mathcal{E}^- = (E^-, \{W/E^i_\bullet\}, \mathbf{c}^-)$) be the destabilizing subbundle (resp. quotient bundle). Since E is trivial, E^+ (resp. E^-) is a direct sum of line bundles with nonpositive (resp. nonnegative) degrees. Thus $\dim \mathrm{Hom}(E^+ \otimes \mathcal{O}(-(n-2)), E^-) = -dr + (n-1)s(r-s)$. By Proposition 4.5, with respect to $\mathbf{a}^-$,

$$\dim \mathrm{Ext}^1(\mathcal{E}^-, \mathcal{E}^+) = -dr - s(r-s) + \sum_{i=1}^{n} \dim \omega_{J^i}.$$

If the wall-crossing is a blow-down,

$$\dim \mathrm{Ext}^1(\mathcal{E}^-, \mathcal{E}^+) + \dim \mathrm{M}_{\mathbf{p}}(s, d, \mathbf{b}) + \dim \mathrm{M}_{\mathbf{p}}(r-s, -d, \mathbf{c}) = \dim \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}).$$

Since $\dim \mathrm{M}_{\mathbf{p}}(r, d, \mathbf{a}) = nr(r-1)/2 - r^2 + 1$, this is equivalent to

$$\sum_{i=1}^{n} \dim \omega_{J^i} = (n-1)s(r-s) + dr - 1.$$

If $\Delta(s, d, \mathcal{J})$ is a wall that we cross while scaling, then there is a constant $0 < c \leq 1$ such that $\mu(\mathcal{E}^+) = \mu(\mathcal{E})$ for the weight $\mathbf{a} = \mathbf{a}_c(c)$ on $\Delta(s, d, \mathcal{J})$.

Note that the weight data $\mathbf{a}_c(c)$ is defined as $a_c(c)^i_\bullet = \frac{c}{r}(r-1, r-2, \cdots, 1)$. Thus

$$\begin{aligned}
\mu(\mathcal{E}^+) &= \frac{1}{s}\left(d + \sum_{i=1}^n \sum_{j \in J^i} \frac{c}{r}(r-j)\right) = \frac{1}{s}\left(d + \sum_{i=1}^n \sum_{k=1}^s \left(c - \frac{c}{r}k - \frac{c}{r}(J^i_k - k)\right)\right) \\
&= \frac{1}{s}\left(d + cn\left(s - \frac{s(s+1)}{2r}\right) - \frac{c}{r}\sum_{i=1}^n \dim \omega_{J^i}\right) \\
&= \frac{1}{s}\left(d + cn\left(s - \frac{s(s+1)}{2r}\right) - \frac{c}{r}\left((n-1)s(r-s) + dr - 1)\right)\right) \\
&= \frac{1}{s}\left(cn\left(s - \frac{s(s+1)}{2r} - \frac{s(r-s)}{r}\right) + \frac{c}{r}(1 + s(r-s)) + (1-c)d\right).
\end{aligned}$$

On the other hand,

$$\mu(\mathcal{E}) = \frac{1}{r}\left(\sum_{i=1}^n \sum_{j=1}^{r-1} \frac{c}{r}(r-j)\right) = \frac{cn(r-1)}{2r}.$$

From $\mu(\mathcal{E}^+) = \mu(\mathcal{E})$, we have

$$\frac{csn(r-1)}{2r} = cn\left(s - \frac{s(s+1)}{2r} - \frac{s(r-s)}{r}\right) + \frac{c}{r}(1 + s(r-s)) + (1-c)d,$$

which is equivalent to

$$c\left(ns\frac{s-r}{2r} + \frac{1 + s(r-s)}{r}\right) = -(1-c)d.$$

If $n \gg 0$, then the left hand side is a negative number, but the right hand side is non-negative because $d < 0$. Thus there is no such $0 < c \le 1$. □

We are ready to prove the main theorem.

Proof of Theorem 5.1. First of all, suppose that n is sufficiently large. By Proposition 5.11, $\mathrm{M} := \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}_d)$ has Picard number $(r-1)n + 1$. Then, $-K_{\mathrm{M}}$ is nef because it is a limit of ample divisors. If the anticanonical divisor is not big, then the wall-crossing center is the whole M, and $\dim \mathrm{Ext}^1(\mathcal{E}^-, \mathcal{E}^+) = \dim \mathrm{M} - \dim \mathrm{M}_{\mathbf{p}}(s, d, \mathbf{b}) - \dim \mathrm{M}_{\mathbf{p}}(r-s, -d, \mathbf{c}) + 1$, or equivalently, $\sum_{i=1}^n \dim \omega_{J^i} = (n-1)s(r-s) + dr$. By a similar computation as in the proof of Proposition 5.11, one can check that such a boundary wall-crossing does not occur as long as n is large. Thus the anticanonical divisor is also big and M is a smooth weak Fano variety. Since $-K$ is big, there is an ample divisor A and an effective divisor D such that $-cK = A + D$ for some positive integer c. So for another positive integer $d > 0$, $-dK = -(d-c)K + A + D$ and $-(K + \frac{1}{d}D) = \frac{1}{d}(-(d-c)K + A)$. The right hand side is ample because it is sum of a nef divisor and an ample divisor. If $d \gg 0$, $(\mathrm{M}, \frac{1}{d}D)$ is a klt pair. Thus M is of Fano type.

For a general non-necessarily dominant weight $\mathbf{a}$, because $\mathbf{a}_d$ is dominant, $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ is obtained from $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}_d)$ by taking several flips and blow-downs, but no blow-ups. By [15, Theorem 1.1, Corollary 1.3], $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ is also of Fano type. When n is small, by

Lemma 5.4, $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ is an image of $\mathrm{M}_{\mathbf{q}}(r, 0, \mathbf{a}')$ for some large $\mathbf{q}$. Thus it is of Fano type by [15, Corollary 1.3]. □

6. MORI'S PROGRAM OF THE MODULI SPACE

We are ready to run Mori's program of $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$. In this section, $n > 2r$ and $\mathbf{a}$ is a dominant weight.

6.1. **Birational models.** Recall that for an integral divisor D on a projective variety X,

$$X(D) := \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(X, \mathcal{O}(mD))$$

be the associated projective model. The following observation is an immediate consequence of Pauly's theorem (Theorem 5.10).

Proposition 6.1. *Let* $D \in \mathrm{intEff}(\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}))$. *Then* $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})(D) \cong \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{b})$ *for some parabolic weight* $\mathbf{b}$.

In particular, all *birational* models of $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ obtained from Mori's program are again moduli spaces of parabolic bundles with some weight data.

Proof. For notational simplicity, set $\mathrm{M} = \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$. We may assume that M is the blow-up of $\mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r$ along $\mathrm{M}_{\mathbf{p}}(r-1, -1, \mathbf{b})$ or $\mathrm{M}_{\mathbf{p}}(r-1, 1, \mathbf{c})$. Let $\pi : \mathrm{M} \to \mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r$ be the blow-up morphism, and $Y^+ = \Theta$ be the exceptional divisor. With respect to such an L, by Proposition 3.4, $\mathrm{Pic}(\mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r)$ is identified with an index r sublattice of $\mathrm{Pic}(\mathrm{Fl}(V)^n)$. Thus any line bundle on $\mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r$ can be uniquely written as $\otimes_{i=1}^n \pi_i^* \overline{F}_{\lambda^i}$ where $\overline{F}_{\lambda^i}$ is the descent of F_{λ^i} on $\mathrm{Fl}(V)$ and $\pi_i : \mathrm{Fl}(V)^n \to \mathrm{Fl}(V)$ is the i-th projection. Similarly, any line bundle $\mathcal{O}(D)$ on M can be uniquely written as $\pi^* \otimes_{i=1}^n \pi_i^* \overline{F}_{\lambda^i} - k\Theta$ for some $k \in \mathbb{Z}$.

When $k = 0$,

$$\begin{aligned}\mathrm{M}(D) &= (\mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r)(\otimes_{i=1}^n \pi_i^* \overline{F}_{\lambda^i}) = \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(\mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r, \otimes_{i=1}^n \pi_i^* \overline{F}^m_{\lambda^i}) \\ &= \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(\mathrm{Fl}(V)^n, \otimes_{i=1}^n \pi_i^* F^m_{\lambda^i})^{\mathrm{SL}_r} = \mathrm{Fl}(V)^n /\!/_{\otimes_{i=1}^n \pi_i^* F_{\lambda^i}} \mathrm{SL}_r,\end{aligned}$$

which is $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{b})$ for some $\mathbf{b}$ by Theorem 3.3.

If $k < 0$, then $\mathrm{M}(D) = \mathrm{M}(\pi^* \otimes_{i=1}^n \pi_i^* \overline{F}_{\lambda^i} - k\Theta) = \mathrm{M}(\pi^* \otimes_{i=1}^n \pi_i^* \overline{F}_{\lambda^i})$ because Θ is an exceptional divisor of the rational contraction $\mathrm{M} \dashrightarrow \mathrm{M}(\pi^* \otimes_{i=1}^n \pi_i^* \overline{F}_{\lambda^i})) = \mathrm{Fl}(V)^n /\!/_{\otimes_{i=1}^n \pi_i^* F_{\lambda^i}} \mathrm{SL}_r$.

Suppose that $k > 0$.

$$\begin{aligned}\mathrm{H}^0(\mathrm{M}, \pi^* \otimes_{i=1}^n \pi_i^* \overline{F}_{\lambda^i}) &= \mathrm{H}^0(\mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r, \otimes_{i=1}^n \pi_i^* \overline{F}_{\lambda^i}) = \mathrm{H}^0(\mathrm{Fl}(V)^n, \otimes_{i=1}^n \pi_i^* F_{\lambda^i})^{\mathrm{SL}_r} \\ &= \mathbb{V}^\dagger_{N, (\lambda^1, \lambda^2, \cdots, \lambda^n)}\end{aligned}$$

for some $N > 0$. Thus $\mathrm{H}^0(\mathrm{M}, D) = \mathbb{V}^\dagger_{N-k, (\lambda^1, \lambda^2, \cdots, \lambda^n)}$. If $N - k > \lambda^i_1$ for all i, then Theorem 5.10 implies that $\mathbb{V}^\dagger_{N-k, (\lambda^1, \lambda^2, \cdots, \lambda^n)}$ is an ample linear system on $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{b})$ for some $\mathbf{b}$.

Suppose that $N-k=\lambda_1^i$ for some i. Then for any m,

$$\mathrm{H}^0(mD-\Theta)=\mathbb{V}^\dagger_{m(N-k)-1,(m\lambda^1,m\lambda^2,\cdots,m\lambda^n)}=0$$

because $m(N-k)-1<m\lambda_1^i$. Thus D is on the boundary of the effective cone. □

We close this section with a lemma which was used in the proof of Proposition 5.11.

Lemma 6.2. *During a scaling wall-crossing, Θ is not contracted.*

Proof. Let $\mathbf{a}$ be a dominant weight. We may assume that $\mathrm{M}:=\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$ is the blow-up of $\mathrm{Fl}(V)^n/\!/_L\mathrm{SL}_r$. By Proposition 6.1, the scaling wall-crossing is the computation of $\mathrm{M}(D-c\Theta)$ where $D=\pi^*\otimes_{i=1}^n\pi_i^*\overline{F}_{\lambda^i}$, from $c=0$ to $c\gg 0$.

Suppose that for some $c>0$, $\mathrm{M}(D-c\Theta)$ is a contraction of Θ. Then $\mathrm{M}(D-c\Theta+d\Theta)=\mathrm{M}(D-c\Theta)$ for any $d>0$. In particular, $\mathrm{M}(D-\epsilon\Theta)$ is a contraction of Θ for $0<\epsilon\ll 1$. But $D-\epsilon\Theta$ is an ample divisor on M and we have a contradiction. □

6.2. **Effective cone.** The first step of Mori's program is the computation of the effective cone.

For some weight data $\mathbf{b}$, $\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{b})$ may be empty. By combining this observation with Proposition 6.1, we can compute an H-representation of $\mathrm{Eff}(\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a}))$. This result was obtained by Belkale in [4] in a greater generality and with a different idea.

Set $\mathrm{M}:=\mathrm{M}_{\mathbf{p}}(r,0,\mathbf{a})$. Since M is a MDS, $\mathrm{Eff}(\mathrm{M})$ is a closed polyhedral cone. For each $D\in\mathrm{Eff}(\mathrm{M})$, $\mathrm{H}^0(D)$ is identified with $\mathbb{V}^\dagger_{\ell,(\lambda^1,\lambda^2,\cdots,\lambda^n)}$. There are two classes of linear inequalities for the non-vanishing of conformal blocks:

(1) $\lambda^i_j\ge\lambda^i_{j+1}$ (it includes $\lambda^i_{r-1}\ge 0$ by our normalization assumption);
(2) $\lambda^i_1\le\ell$.

The first class of inequalities comes from the effectiveness of $\otimes_{i=1}^n\pi_i^*F_{\lambda^i}$ on $\mathrm{Fl}(V)^n$. For the second class, see [7, Section 4] for an explanation. Here we construct extra linear inequalities.

Recall that the (genus zero) *Gromov-Witten invariant* counts the number of rational curves intersecting several subvarieties. Here we employ the definition in [6] using quot scheme, which is different from the definition using moduli spaces of stable maps ([14]). For a partition $\lambda=(r\ge\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_s\ge 0)$ and a complete flag $W_\bullet$ of an r-dimension vector space V, we obtain a Schubert subvariety $\Omega_\lambda(W_\bullet)\subset\mathrm{Gr}(s,V)=\mathrm{Gr}(s,r)$. Its numerical class is independent of the choice of $W_\bullet$, and is denoted by $\omega_\lambda\in\mathrm{H}^*(\mathrm{Gr}(s,r))$. For a collection of general complete flags $W^i_\bullet$ of V and a nonnegative integer d, the Gromov-Witten invariant

$$\langle\omega_{\lambda^1},\omega_{\lambda^2},\cdots,\omega_{\lambda^n}\rangle_d$$

is the number of maps $f:(\mathbb{P}^1,\mathbf{p}=(p^i))\to\mathrm{Gr}(s,r)$ of degree d such that $f(p^i)\in\Omega_{\lambda^i}(W^i_\bullet)$ if the number is finite, and otherwise it is zero. Since the moduli space of maps from $\mathbb{P}^1$ to $\mathrm{Gr}(s,r)$ is not proper, a rigorous definition requires a compactified space of maps, the quot scheme over $\mathbb{P}^1$, but by Moving Lemma ([6, Lemma 2.2A]), the number is equal to the number of genuine maps from $\mathbb{P}^1$ to $\mathrm{Gr}(s,r)$.

Proposition 6.3. *For each collection of partitions $\lambda^1, \lambda^2, \cdots, \lambda^n$ of length s and a nonpositive integer d such that the Gromov-Witten invariant $\langle \omega_{\lambda^1}, \omega_{\lambda^2}, \cdots, \omega_{\lambda^n}\rangle_{-d}$ on $\mathrm{Gr}(s,r)$ is one, there is a linear inequality*

$$\frac{1}{s}\left(d\ell + \sum_{i=1}^{n}\sum_{j\in J^i}\lambda_j^i\right) \le \frac{1}{r}\left(\sum_{i=1}^{n}\sum_{j=1}^{r-1}\lambda_j^i\right). \tag{7}$$

which defines $\mathrm{Eff}(\mathrm{M})$, where $J^i = \{r-s+j-\lambda_j^i | j \in [s]\}$. Moreover, these inequalities and two classes (1) and (2) of inequalities provide the H-represenation of $\mathrm{Eff}(\mathrm{M})$.

Proof. Let D be a general point on a facet of $\mathrm{Eff}(\mathrm{M})$, which is not one of facets described above. Take an embedding of a small line segment $\gamma : [-\epsilon, \epsilon] \to \mathrm{N}^1(\mathrm{M})_{\mathbb{R}}$ such that $\gamma(0) = D$ and $\gamma(x) \in \mathrm{int}\ \mathrm{Eff}(\mathrm{M})$ when $x < 0$. Let $D^{\pm} := \gamma(\pm\epsilon)$.

Note that each $x \in [-\epsilon, \epsilon]$ defines an $\mathbb{R}$-divisor $D_x = \mathcal{L}^{\ell} \otimes \otimes_{i=1}^n F_{\lambda^i}$, and hence a parabolic weight $\mathbf{a}_x$ by setting $(a_x)_i = \frac{1}{\ell}(\lambda_j^i)$. We may assume that all $\mathbf{a}_x$ are general except $\mathbf{a}_0$. Because the moduli space becomes empty after changing the weight from $\mathbf{a}_{-\epsilon}$ to $\mathbf{a}_{\epsilon}$, there is a boundary wall $\Delta(s, d, \mathcal{J})$ at $\mathbf{a}_0$. A wall-crossing is a boundary one if and only if a general point $\mathcal{E} = (\mathcal{O}^r, \{W_\bullet^i\}, \mathbf{a}_{-\epsilon})$ of $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}_{-\epsilon})$ has the unique detabilizing bundle $\mathcal{E}^+ = (E^+, \{W|_{E^+}{}_\bullet^i\}, \mathbf{b}_{-\epsilon})$ of rank s such that $\mu(\mathcal{E}^+) = \mu(\mathcal{E})$ with respect to $\mathbf{a}_0$. This implies that there is a short exact sequence $0 \to E^+ \to E \to E^- \to 0$ of bundles such that $E^+|_{p^i} \in \Omega_{\lambda^i}(W_\bullet^i)$. Therefore there is a map $f : (\mathbb{P}^1, \mathbf{p}) \to \mathrm{Gr}(s,r)$ of degree $-d$ such that $f(p^i) \in \Omega_{\lambda^i}(W_\bullet^i)$. Thus $\langle \omega_{\lambda^1}, \omega_{\lambda^2}, \cdots, \omega_{\lambda^n}\rangle_{-d} = 1$. In particular, to have a nonempty moduli space, $\mu(\mathcal{E}^+) \le \mu(\mathcal{E})$, which is (7).

Now suppose that D is a divisor satisfies all of the given strict linear inequalities of the form (7) for every collection of partitions $\lambda^1, \lambda^2, \cdots, \lambda^n$ with $\langle \omega_{\lambda^1}, \omega_{\lambda^2}, \cdots, \omega_{\lambda^n}\rangle_{-d} = 1$. Let $\mathbf{a}$ be the associated parabolic weight data. Then for a general parabolic bundle $\mathcal{E} = (\mathcal{O}^r, \{W_\bullet^i\}, \mathbf{a})$, there is no possible destabilizing bundle. Therefore $\mathcal{E} \in \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ and the moduli space is nonempty. Because D is an ample divisor on $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$, $|mD| \ne \emptyset$ for some $m > 0$. Therefore $D \in \mathrm{intEff}(\mathrm{M})$. By taking the closure, we can obtain the effective cone. □

Remark 6.4. The computation of the V-representation from the H-representation is highly nontrivial. In [5], Belkale explains how to compute the extremal rays of the effective cone for the quotient stack $[\mathrm{Fl}(V)^3/\mathrm{SL}_r]$. He informed to the authors that this computation can be generalized to the case of arbitrary n and for $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$, too.

6.3. **Projective models and wall-crossing.** The remaining steps of Mori's program are the computation of projective models $\mathrm{M}(D)$ for $\mathrm{M} := \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ and the study of the rational contraction $\mathrm{M} \dashrightarrow \mathrm{M}(D)$. For $D \in \mathrm{intEff}(\mathrm{M})$, Proposition 6.1 already provides the answer. It remains to find projective models associated to $D \in \partial\mathrm{Eff}(\mathrm{M})$. We content ourselves with a description for facets of $\partial\mathrm{Eff}(\mathrm{M})$.

The first type of facets are that associated to Gromov-Witten invariants, as described in Section 6.2. We call this type of facets as *GW facets*. In this case, the boundary wall-crossing in Proposition 4.3 gives a contraction.

Proposition 6.5. *Suppose that D is a general point on a GW facet of $\mathrm{Eff}(\mathrm{M})$. Then $\mathrm{M}(D) = \mathrm{M}_{\mathbf{p}}(s, -d, \mathbf{b}) \times \mathrm{M}_{\mathbf{p}}(r-s, d, \mathbf{c})$ for some $0 < s < r$, $d \ge 0$, and $\mathbf{b}$ and $\mathbf{c}$.*

The second type of facets are of the form $\lambda_j^k = \lambda_{j+1}^k$. This case is related to moduli spaces of parabolic bundles with degenerated flags, which forgets j-th flag on p^k. In [28], Pauly proved Theorem 5.10 for such degenerated flags, too. The proof of the next proposition is essentially same to that of Proposition 6.1.

Proposition 6.6. *Suppose that D is a general point of the facet of $\mathrm{Eff}(\mathrm{M})$ which is given by $\lambda_j^k = \lambda_{j+1}^k$. Then $\mathrm{M}(D) = \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{b})$, which is the moduli space of parabolic bundles where its k-th flag is a partial flag of type $(1, 2, \cdots, \hat{j}, \cdots, r-1)$.*

The last type of facets are of the type $\lambda_1^k = \ell$.

Proposition 6.7. *Suppose that D is a general point on the facet $\lambda_1^k = \ell$. Then $\mathrm{M}(D) = \mathrm{M}_{\mathbf{p}}(r, -1, \mathbf{b})$ where $\mathbf{b}$ is a parabolic weight such that $b^i = \frac{1}{\ell}(\lambda_1^i, \lambda_2^i, \cdots, \lambda_{r-1}^i)$ for $i \ne k$ and $b^k = \frac{1}{\ell}(\lambda_1^k - \lambda_{r-1}^k, \lambda_2^k - \lambda_{r-1}^k, \cdots, \lambda_{r-2}^k - \lambda_{r-1}^k)$ (the last flag is of type $(2, 3, \cdots, r-1)$).*

Proof. By symmetry, we may assume that $k = n$. Let D' be a big divisor which is sufficiently close to D. Then $\mathrm{M}(D) = \mathrm{M}(D')(D)$. Thus we may replace M by $\mathrm{M}(D')$. Equivalently, after Theorem 5.10, we may assume that $\mathbf{a}$ is sufficiently close to $(\frac{1}{\ell}\lambda^i)$.

Let $\mathcal{E} = (E, \{W_\bullet^i\}, \mathbf{a}) \in \mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$. Consider the quotient map $E \to E|_{p^n}/W_{r-1}^n \to 0$ and let E' be the kernel. Then E' is a vector bundle of degree -1. For p^i with $i < n$, let ${W'}_j^i = W_j^i$. Over p^n, let ${W'}_j^n = r^{-1}(W_j^n)$ where $r : E'|_{p^n} \to E|_{p^n}$ is the restriction of $E' \hookrightarrow E$. Note that $\dim {W'}_j^n = j+1$. Thus we have a quasi parabolic bundle $\mathcal{E}' := (E', \{{W'}_\bullet^i\})$ whose last flag over p^n is of type $(2, 3, \cdots, r-1)$.

We claim that $\mathcal{E}'$ is stable with respect to $\mathbf{b}$. Let $\mathcal{F}' = (F', \{{V'}_\bullet^i\}, \mathbf{c})$ be a parabolic subbundle of $\mathcal{E}'$. To avoid a confusion, the slope with respect to $\mathbf{b}$ is denoted by $\mu_{\mathbf{b}}$. Because $b_j^n = a_j^n - a_{r-1}^n$,

$$\begin{aligned}\mu_{\mathbf{b}}(\mathcal{E}') &= \frac{1}{r}\left(-1 + \sum_{i<n}\sum_{j=1}^{r-1} a_j^i + b_1^n + \sum_{j=1}^{r-2} b_j^n\right) = \mu(\mathcal{E}) - \frac{1}{r} - \frac{1}{r}\sum_{j=1}^{r-1} a_j^n + \frac{1}{r}\left(b_1^n + \sum_{j=1}^{r-2} b_j^n\right)\\ &= \mu(\mathcal{E}) - \frac{1}{r} + \frac{1}{r}a_1^n - a_{r-1}^n.\end{aligned}$$

Suppose that F' is a rank s subbundle of E. Then $\ker r \cap F'|_{p^n} = 0$. If J^i denotes the subset of indices $j \in [r]$ such that $W'^i_j \cap F'|_{p^i} \neq W'^i_{j-1} \cap F'|_{p^i}$,

$$\begin{aligned}\mu_{\mathbf{b}}(\mathcal{F}') &= \frac{1}{s}\left(\deg F' + \sum_{i<n}\sum_{j \in J^i} a^i_j + \sum_{j \in J^n} b^n_j\right)\\ &= \frac{1}{s}\left(\deg F' + \sum_{i}\sum_{j \in J^i} a^i_j\right) + \frac{1}{s}\left(\sum_{j \in J^n}(b^n_j - a^n_j)\right) = \mu(\mathcal{F}') - a^n_{r-1}.\end{aligned} \tag{8}$$

Therefore $\mu_{\mathbf{b}}(\mathcal{E}') - \mu_{\mathbf{b}}(\mathcal{F}') = \mu(\mathcal{E}) - \mu(\mathcal{F}') - \frac{1}{r} + \frac{1}{r}a^n_1$. Since a^n_1 is sufficiently close to one, $\mu_{\mathbf{b}}(\mathcal{E}') - \mu_{\mathbf{b}}(\mathcal{F}') > 0$.

If F' is not a subbundle of E, then there is a subbundle F of E which contains F' and F/F' is a torsion sheaf. Since E/E' is of length one, F/F' is also of length one. Therefore $\deg F' = \deg F - 1$. Let $\mathcal{F}$ be the parabolic subbundle of $\mathcal{E}$ whose underlying bundle is F. By a similar computation with (8), we have

$$\mu_{\mathbf{b}}(\mathcal{F}') = \mu(\mathcal{F}) - \frac{1}{s} + \frac{1}{s}(a^n_1 - a^n_{r-1}) - a^n_{r-1}$$

and it is straightforward to see that $\mu_{\mathbf{b}}(\mathcal{E}') - \mu_{\mathbf{b}}(\mathcal{F}') > 0$.

Therefore the map $\mathrm{M} \to \mathrm{M}_{\mathbf{p}}(r, -1, \mathbf{b})$ which sends $\mathcal{E}$ to $\mathcal{E}'$ is a well-defined morphism. This is a $\mathbb{P}^1$-fibration and it is of relative Picard number one. It is clear that this projective model is associated to the facet $\lambda^n_1 = \ell$, or equivalently, $a^n_1 = 1$. ☐

Because all of the birational models and projective models can be described in terms of moduli spaces of parabolic bundles, the wall-crossings in Section 4.1 are building blocks of the rational contraction $\mathrm{M} \dashrightarrow \mathrm{M}(D)$. This is in some sense very satisfactory, because all of them are smooth blow-ups/downs and projective bundle morphisms.

6.4. **Rationality.** It is an old open problem determining whether the moduli space of (parabolic) bundles with a fixed determinant over a Riemann surface is rational or not ([18, 19]). The wall-crossing toward a boundary wall was applied to show the fact that $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ is rational in [9]. Here we leave a sketch, for a reader's convenience.

It is sufficient to prove for the case that $\mathbf{a}$ is sufficiently small, so $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a}) = \mathrm{Fl}(V)^n /\!/_L \mathrm{SL}_r$ for some L. Cross several walls of type $\Delta(s, 0, \mathcal{J})$, which are indeed GIT walls. If $\mathbf{a}$ is sufficiently close to the boundary, then by Proposition 4.3, $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ is a projective bundle over $\mathrm{M}_{\mathbf{p}}(s, 0, \mathbf{b}) \times \mathrm{M}_{\mathbf{p}}(r-s, 0, \mathbf{c})$. Thus the problem is reduced to a lower rank case. If $r = 1$, the moduli space is a point, so it is trivial.

Proposition 6.8 ([9, Proposition 5.1])**.** *The moduli space* $\mathrm{M}_{\mathbf{p}}(r, 0, \mathbf{a})$ *is rational.*

REFERENCES

[1] Arnaud Beauville. Conformal blocks, fusion rules and the Verlinde formula. *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry* (Ramat Gan, 1993), 75–96, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996. 6, 7

[2] Arnaud Beauville and Yves Laszlo. Conformal blocks and generalized theta functions. *Commun. Math. Phys.*, 164:385–419, 1994. 6

[3] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010. 16

[4] Prakash Belkale. Quantum generalization of the Horn conjecture. *J. Amer. Math. Soc.*, 21(2):365–408, 2008. 2, 6, 22

[5] Prakash Belkale. Extremal rays in the hermitian eigenvalue problem. arXiv:1705.10580, 2017. 23

[6] Aaron Bertram. Quantum Schubert calculus. *Adv. Math.*, 128(2):289–305, 1997. 22, 23

[7] Prakash Belkale, Angela Gibney, and Swarnava Mukhopadhyay. Vanishing and identities of conformal blocks divisors. *Algebr. Geom.*, 2(1):62–90, 2015. 6, 18, 22

[8] P. Belkale, A. Gibney, and S. Mukhopadhyay. Nonvanishing of conformal blocks divisors on $\overline{M}_{0,n}$. *Transform. Groups*, 21(2):329–353, 2016. 2

[9] Hans U. Boden and Yi Hu. Variations of moduli of parabolic bundles. *Math. Ann.*, 301(3):539–559, 1995. 11, 25

[10] Kiryong Chung and Han-Bom Moon. Mori's program for the moduli space of conics in grassmannian. *Taiwanese. J. Math.*, 21(3):621–652, 2017. 3

[11] J.-M. Drezet and M. S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.*, 97(1):53–94, 1989. 9

[12] Igor Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003. 8, 9, 16

[13] Boris Feigin, Vadim Schechtman, Alexander Varchenko. On algebraic equations satisfied by hypergeometric correlators in WZW models. II. *Comm. Math. Phys.* 170 (1995), no. 1, 219–247. 6

[14] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997. 22

[15] Yoshinori Gongyo, Shinnosuke Okawa, Akiyoshi Sannai, and Shunsuke Takagi. Characterization of varieties of Fano type via singularities of Cox rings. *J. Algebraic Geom.*, 24(1):159–182, 2015. 20, 21

[16] Jochen Heinloth, Lectures on the moduli stack of vector bundles on a curve. *Affine flag manifolds and principal bundles*, 123–153, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010. 15

[17] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday. 2, 3, 16

[18] Norbert Hoffmann. Rationality and Poincaré families for vector bundles with extra structure on a curve. *Int. Math. Res. Not. IMRN*, (3):Art. ID rnm010, 30, 2007. 25

[19] Alastair King and Aidan Schofield. Rationality of moduli of vector bundles on curves. *Indag. Math. (N.S.)*, 10(4):519–535, 1999. 25

[20] Yves Laszlo and Christoph Sorger. The line bundles on the moduli of parabolic G-bundles over curves and their sections. *Ann. Sci. École Norm. Sup. (4)*, 30(4):499–525, 1997. 2, 6

[21] Christopher Manon. The algebra of $SL_3(\mathbb{C})$ conformal blocks. *Transform. Groups*, 18(4):1165–1187, 2013. 2, 3, 17

[22] Christopher Manon. Cox rings of moduli of quasi-parabolic principal bundles and the K-Pieri rule. *J. Combin. Theory Ser. A*, 139:1–29, 2016. 2, 3

[23] Christopher Manon. The algebra of conformal blocks. *J. Eur. Math. Soc. (JEMS)* 20 (2018), no. 11, 2685–2715. 2, 3, 17

[24] V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Math. Ann.*, 248(3):205–239, 1980. 5

[25] Han-Bom Moon and Sang-Bum Yoo. Birational Geometry of the Moduli Space of Rank 2 Parabolic Vector Bundles on a Rational Curve. *Int. Math. Res. Not. IMRN*, (3):827–859, 2016. 1, 2

[26] Shigeru Mukai. Finite generation of the Nagata invariant rings in A-D-E cases. RIMS preprint 1502, 2005. 17

[27] Martin Olsson, *Algebraic spaces and stacks*. American Mathematical Society Colloquium Publications, 62. American Mathematical Society, Providence, RI, 2016. xi+298 pp. 15

[28] Christian Pauly. Espaces de modules de fibrés paraboliques et blocs conformes. *Duke Math. J.*, 84(1):217–235, 1996. 2, 6, 18, 24

[29] Michael Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996. 2, 11, 12

[30] Akihiro Tsuchiya and Yukihiro Kanie. Vertex operators in conformal field theory on $\mathbf{P}^1$ and monodromy representations of braid group. In *Conformal field theory and solvable lattice models (Kyoto, 1986)*, volume 16 of *Adv. Stud. Pure Math.*, pages 297–372. Academic Press, Boston, MA, 1988. 2

[31] Akihiro Tsuchiya, Kenji Ueno, and Yasuhiko Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 459–566. Academic Press, Boston, MA, 1989. 2

[32] Kenji Ueno. *Conformal field theory with gauge symmetry*, volume 24 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2008. 2, 6, 16

[33] Kôji Yokogawa. Infinitesimal deformation of parabolic Higgs sheaves. *Internat. J. Math.*, 6(1):125–148, 1995. 7

DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, NEW YORK, NY 10023, USA

Email address: hmoon8@fordham.edu

DIVISION OF GENERAL STUDIES, UNIST, ULSAN 44919, REPUBLIC OF KOREA

Email address: sangbum.yoo@gmail.com