GIT compactifications of $M_{0,n}$

Han-Bom Moon

University of Georgia
Joint work with N. Giansiracusa and D. Jensen

July 2, 2013

Busan

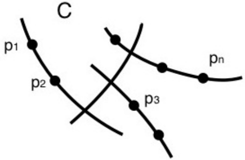
Moduli space of pointed rational curves

$M_{0,n} := \{(C, p_1, \cdots, p_n) \mid C \cong \mathbb{P}^1,\ p_i \neq p_j\}/\sim$

$\cdots$ moduli space of n-pointed smooth rational curves.

It is an affine open subset of $\mathbb{A}^{n-3}$, so it is smooth but not compact.

$\overline{M}_{0,n} := \Bigg\{$



$\Bigg| \quad \begin{array}{c} p_a(C) = 0 \\ (C, \sum p_i) : \text{ slc pair} \\ \omega_C + \sum p_i \text{ is ample} \end{array} \Bigg\}/\sim$

$\cdots$ Deligne-Mumford compactification, or moduli space of n-pointed stable rational curves.

Moduli space of pointed rational curves

$\overline{M}_{0,n}$ has many nice properties.

- $\overline{M}_{0,n}$ is $n-3$ dimensional smooth variety.
- $\overline{M}_{0,n}$ is projective. So it is compact.
- A smooth n pointed rational curve is stable. So $M_{0,n} \subset \overline{M}_{0,n}$.
- $\overline{M}_{0,n} - M_{0,n}$ is a simple normal crossing divisor.
- An irreducible component of $\overline{M}_{0,n} - M_{0,n}$ is isomorphic to $\overline{M}_{0,i} \times \overline{M}_{0,j}$.
- Many topological invariants such as cohomology ring, Hodge numbers are already known.

Mori's program - Birational geometry of moduli space

We are interested in birational geometry of moduli spaces.

The Mori's program for a moduli space M consists of

1. Compute the cone of effective (nef, movable) divisors of M.
2. For an effective $\mathbb{Q}$-divisor D, find the projective model
$$M(D) := \mathrm{Proj} \bigoplus_{m \geq 0} H^0(M, \mathcal{O}(mD)).$$
3. Study the moduli theoretic meaning of $M(D)$ and its relation with M.

Example: Hassett-Keel program $=$ Mori's program for $\overline{M}_g$ with divisors of the form $K_{\overline{M}_g} + \alpha\Delta$, $0 < \alpha < 1$.

Birational geometry of $\overline{M}_{0,n}$

For $\overline{M}_{0,n}$, none of them was fully understood. (Cone of divisors of $\overline{M}_{0,n}$ is extremely complicate.)

- Cone of nef divisors ($\Leftrightarrow$ cone of curves). F-conjecture: The cone of curves is generated by 1-dimensional intersections of boundary divisors.
- Cone of effective divisors. Castravet-Tevelev conjecture: The cone of effective divisors is generated by a finite set of divisors which can be described in terms of combinatorics.
- Mori dream space conjecture. $\overline{M}_{0,n}$ is a Mori dream space (nef $=$ semi-ample, finite generation of cone of effective divisors, $\cdots$).

Alternative approach: Construct (modular) birational models of $\overline{M}_{0,n}$ and describe them in terms of $\overline{M}_{0,n}(D)$ for some D.

Example - Hassett's space

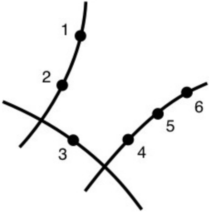
Define new moduli problems:

$\mathcal{A} := (a_1, \cdots, a_n)$, $0 < a_i \leq 1$. $(C, p_1, \cdots, p_n)$ is $\mathcal{A}$-stable if

- $(C, \sum a_i p_i)$ is an slc pair (C has at worst nodal singularities, all p_i's are smooth points, if $p_{i_1} = \cdots = p_{i_k}$, then $\sum_{j=1}^{k} a_{i_j} \leq 1$);
- $\omega_C + \sum a_i p_i$ is ample (For each component C', $\sum_{p_i \in C'} a_i + \#|\text{singular pts}| > 2$).

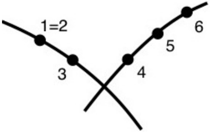
$\overline{M}_{0,\mathcal{A}}$ $\cdots$ moduli space of genus 0 $\mathcal{A}$-stable curves.

Example: $\mathcal{A} = (\frac{1}{2}, \cdots, \frac{1}{2})$



$\in \overline{M}_{0,n}$
$\notin \overline{M}_{0,\mathcal{A}}$

$\Rightarrow$



$\notin \overline{M}_{0,n}$
$\in \overline{M}_{0,\mathcal{A}}$

Example - Hassett's space

Q. Can we describe $\overline{M}_{0,\mathcal{A}}$ in terms of Mori's program?

For $(C, p_1, \cdots, p_n) \in \overline{M}_{0,n}$, take the cotangent space of C at p_i.
It forms a rank 1 bundle $\mathbb{L}_i$ on $\overline{M}_{0,n}$.
$\psi_i := c_1(\mathbb{L}_i)$

Theorem (M)

Let $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ be an allowable weight datum.

$$\overline{M}_{0,n}(K_{\overline{M}_{0,n}} + \sum_{i=1}^{n} a_i \psi_i) \cong \overline{M}_{0,\mathcal{A}}.$$

GIT compactifications

Goal: Construct many modular birational models of $\overline{M}_{0,n}$.

A typical way to construct (projective) moduli spaces: GIT quotient!

$\mathrm{Chow}_{1,d}(\mathbb{P}^d) :=$ an irreducible component of the Chow variety of dimension 1, degree d algebraic cycles in $\mathbb{P}^d$ which contains rational normal curves of degree d

$U_{d,n} := \{(C, x_1, \cdots, x_n) \mid x_i \in C\} \subset \mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n$

SL_{d+1} acts on $U_{d,n}$ via $SL_{d+1} \to PGL_{d+1} = \mathrm{Aut}(\mathbb{P}^d)$

$U_{d,n}//SL_{d+1}$ is a candidate of a modular birational model of $\overline{M}_{0,n}$.

GIT compactifications

The GIT quotient depends on the choice of a linearization.

$\Rightarrow$ Can obtain various compactifications.

$(\gamma, \vec{c}) = (\gamma, c_1, \cdots, c_n) \in \mathbb{Q}_{>0}^{n+1}$

$\Rightarrow L_{\gamma,\vec{c}} := \mathcal{O}_{\mathrm{Chow}_{1,d}(\mathbb{P}^d)}(\gamma) \otimes \mathcal{O}_{\mathbb{P}^d}(c_1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^d}(c_n)$ $\cdots$ a linearization

$\Rightarrow U_{d,n} /\!/_{L_{\gamma,\vec{c}}} SL_{d+1}$

$\Delta_{d,n}^0 := \{(\gamma, c_1, \cdots, c_n) \in \mathbb{Q}_{>0}^{n+1} \mid (d-1)\gamma + \sum c_i = d+1,\ 0 < \gamma, c_i < 1\}$
$\cdots$ generalized hypersimplex

GIT compactifications

$\Delta^0_{d,n} := \{(\gamma, c_1, \cdots, c_n) \in \mathbb{Q}^{n+1}_{>0} \mid (d-1)\gamma + \sum c_i = d+1, 0 < \gamma, c_i < 1\}$

Theorem (Giansiracusa, Jensen, M)

For every $(\gamma, \vec{c}) \in \Delta^0_{d,n}$,

1. $U^{ss}_{d,n}(L_{\gamma,\vec{c}})$ *is nonempty. So* $U_{d,n}//_{L_{\gamma,\vec{c}}}SL_{d+1}$ *is nonempty.*
2. *There is an explicit combinatorial way to describe (semi-)stable locus of* $U_{d,n}$ *for each linearization.*
3. $U_{d,n}//_{L_{\gamma,\vec{c}}}SL_{d+1}$ *has an open dense subset which is isomorphic to* $M_{0,n}$*, i.e., it is a compactification of* $M_{0,n}$*.*
4. *If* $U^{ss}_{d,n}(L_{\gamma,\vec{c}}) = U^{s}_{d,n}(L_{\gamma,\vec{c}})$*, then* $U_{d,n}//_{L_{\gamma,\vec{c}}}SL_{d+1}$ *is a fine moduli space.*

GIT compactifications

Description of stability criterion.

Fix $(\gamma, \vec{c}) = (\gamma, c_1, \cdots, c_n) \in \Delta^0_{d,n}$.

Suppose that $U^{ss}_{d,n}(L_{\gamma,\vec{c}}) = U^{s}_{d,n}(L_{\gamma,\vec{c}})$.

A curve $(C, x_1, \cdots, x_n) \in U_{d,n}$ is stable if and only if

- C is non-degenerated;
- For any (possibly reducible) tail $T \subset C$,

$$\deg T = \lceil \frac{\sum_{x_i \in T} c_i - 1}{1 - \gamma} \rceil.$$

GIT compactifications

- All of previously constructed projective modular birational models of $\overline{M}_{0,n}$ can be obtained in this way. $\overline{M}_{0,n}$, $\overline{M}_{0,\mathcal{A}}$, $\overline{M}_{0,n}^{Bog}$, $\cdots$.
- They admit a birational morphism

$$\overline{M}_{0,n} \to U_{d,n}//_{L_{\gamma,\vec{c}}}SL_{d+1}.$$

  Thus they give information about nef cone of $\overline{M}_{0,n}$.
- So far, there is no known modular flip of $\overline{M}_{0,n}$, i.e., a moduli space which is a flip of $\overline{M}_{0,n}$.

Applications - 1. Projectivity of modular birational models

Theorem (Smyth)

As algebraic stacks, there are many moduli spaces $\overline{M}_{0,n}(Z)$ which are birational to $\overline{M}_{0,n}$. They obtained by allowing worse singularities and collisions of some points. They depend on certain combinatorial data Z.

- By definition, it has a modular meaning.
- Hard to obtain good geometric properties, for example projectivity.

We prove the projectivity of a lot of $\overline{M}_{0,n}(Z)$ by identifying them with $U_{d,n}/\!/_{L_{\gamma,\vec{c}}} SL_{d+1}$.

Applications - 2. Birational models for conformal block divisors

There is a machinery

$$\left\{ \begin{array}{c} \text{simple Lie algebra } \mathfrak{g},\ \ell \in \mathbb{Z}_{\geq 0} \\ \text{dominant weights } \lambda_1, \cdots, \lambda_n \\ \text{such that } (\theta, \lambda_i) \leq \ell \end{array} \right\} \to \mathbb{D}(\mathfrak{g}, \ell, (\lambda_1, \cdots, \lambda_n))$$

to construct a semi-ample divisor $\mathbb{D}(\mathfrak{g}, \ell, (\lambda_1, \cdots, \lambda_n))$ so called a conformal block divisor, originated from the conformal field theory and representations of affine Lie algebras.

Q. What is the birational model corresponding to conformal block divisors?

Applications - 2. Birational models for conformal block divisors

Theorem (Gibney, Jensen, M, Swinarski)

For any non-trivial symmetric $\mathfrak{sl}_2$ weight 1 conformal block divisors,

$$\overline{M}_{0,n}(\mathbb{D}(\mathfrak{sl}_2, \ell, (\omega_1, \cdots, \omega_1))) \cong U_{g+1-\ell,n} /\!/_L SL_{g+2-\ell}$$

where $g = \lfloor \frac{n}{2} \rfloor - 1$ *and* $L = (\frac{\ell-1}{\ell+1}, \frac{1}{\ell+1}, \cdots, \frac{1}{\ell+1})$.

Thank you!