VERONESE QUOTIENT MODELS OF $\overline{\mathrm{M}}_{0,n}$ AND CONFORMAL BLOCKS

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Abstract. The moduli space $\overline{\mathrm{M}}_{0,n}$ admits birational maps to what we call Veronese quotients, introduced in the papers [GS11, Gia11, GJM11]. We study divisors on $\overline{\mathrm{M}}_{0,n}$ associated to these morphisms, and show that for particular choices, the maps are given by conformal blocks divisors.

Introduction

The moduli space of Deligne-Mumford stable n-pointed rational curves $\overline{\mathrm{M}}_{0,n}$ is a natural compactification of the moduli space of smooth pointed genus 0 curves, and has figured prominently in the literature. A central motivating question is to describe other compactifications of $\mathrm{M}_{0,n}$ that receive morphisms from $\overline{\mathrm{M}}_{0,n}$. From the perspective of Mori theory, this is tantamount to describing semi-ample divisors on $\overline{\mathrm{M}}_{0,n}$. This work is concerned with two recent constructions that each yield an abundance of such semi-ample divisors on $\overline{\mathrm{M}}_{0,n}$, and the relationship between them. The first comes from Geometric Invariant Theory (GIT), while the second from conformal field theory.

There are birational models of $\overline{\mathrm{M}}_{0,n}$ obtained via GIT which are moduli spaces of pointed rational normal curves of fixed degree d, where the curves and the marked points are weighted by nonnegative rational numbers $(\gamma, A) = (\gamma, (a_1, \cdots, a_n))$ [Gia11, GS11, GJM11]. These so-called *Veronese quotients* $V^d_{\gamma,A}$ are remarkable as they specialize to nearly every known compactification of $\mathrm{M}_{0,n}$ [GJM11]. There are birational morphisms from $\overline{\mathrm{M}}_{0,n}$ to these GIT quotients, and their natural polarization can be pulled back along this morphism, yielding semi-ample divisors $\mathcal{D}_{\gamma,A}$ on $\overline{\mathrm{M}}_{0,n}$.

A second recent development in the birational geometry of $\overline{\mathrm{M}}_{0,n}$ involves divisors that arise from conformal field theory. These divisors are first Chern classes of vector bundles of conformal blocks $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ on the moduli stack $\overline{\mathcal{M}}_{g,n}$. Constructed using the representation theory of affine Lie algebras [TUY89, Fak12], these vector bundles depend on the choice of a simple Lie algebra $\mathfrak{g}$, a nonnegative integer ℓ, and an n-tuple $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ of dominant integral weights in the Weyl alcove for $\mathfrak{g}$ of level ℓ. For the definition of vector bundles of conformal blocks and related representation theoretic notations, see §4.1. Vector bundles of conformal blocks are globally generated when $g = 0$ [Fak12, Lemma 2.5], and their first Chern classes $c_1(\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})) = \mathbb{D}(\mathfrak{g}, \ell, \vec{\lambda})$, the conformal block divisors, are semi-ample.

When $\gamma = 0$, it was shown in [Gia11, GG12] that the divisors $\mathcal{D}_{0,A}$ coincide with conformal block divisors for $\mathfrak{sl}_r$ and level one. Our guiding philosophy is that there is a general correspondence between Veronese quotients and conformal block divisors. After first giving background information about Veronese quotients in Section 1, in support of this we:

(2) derive intersection numbers for all $\mathcal{D}_{\gamma,A}$ with curves on $\overline{\mathrm{M}}_{0,n}$ (Theorem 2.1);
(3) give a new modular interpretation for a particular family of Veronese quotients (§3);
(4) show that the models described in §3 are given by conformal block divisors (Theorem 4.6);
(5) provide several conjectures (and supporting evidence) generalizing these results (§5).

In order to further motivate and put this work in context, we next say a bit more about (2)-(5).

Date: April 14, 2013.

Section 2. The classes of all Veronese quotient divisors $\mathcal{D}_{\gamma,A}$. For each allowable (γ, A), there exists a morphism $\varphi_{\gamma,\mathcal{A}} : \overline{\mathrm{M}}_{0,n} \to V^d_{\gamma,A}$. In §2 we study the divisors $\mathcal{D}_{\gamma,A} = \varphi^*_{\gamma,\mathcal{A}}(L_{\gamma,\mathcal{A}})$, where $L_{\gamma,\mathcal{A}}$ is the canonical ample polarization on $V^d_{\gamma,A}$. In Theorem 2.1, we give a formula for the intersection of $\mathcal{D}_{\gamma,A}$ with F-curves (Definition 1.5), a collection of curves that span the vector space of numerical equivalence classes of 1-cycles on $\overline{\mathrm{M}}_{0,n}$. Theorem 2.1 is a vast generalization of formulas that have appeared for $d \in \{1, 2\}$ and for $\gamma = 0$ (see [AS08, GS11, Gia11, GG12]) and captures a great deal of information about the nef cone $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. For example, since adjacent chambers in the GIT cone correspond to adjacent faces of the nef cone, by combining Theorem 2.1 with the results of [GJM11], we could potentially describe many faces of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. Moreover, Theorem 2.1 is equivalent to giving the class of $\mathcal{D}_{\gamma,A}$ in the Néron-Severi space. To illustrate this, we give the classes of the conformal block divisors with S_n-invariant weights (Corollary 2.12) and the particularly simple formula for the divisors that give rise to the maps to the Veronese quotients $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ (Example 2.13).

Section 3. A new modular interpretation for a particular family of Veronese quotients. Much work has focused on alternative compactifications of $\mathrm{M}_{0,n}$ [Kap93, Kap93b, Bog99, LM00, Has03, Sim08, Smy09, Fed11, GS11, Gia11, GJM11]. As was shown in [GJM11], every choice of allowable weight data for Veronese quotients (Definition 1.1) yields such a compactification, and nearly every previously known compactification arises as such a Veronese quotient. In §3, we study the particular Veronese quotients $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ for $1 \leq \ell \leq g$. In Theorem 3.5, we provide a new modular interpretation for these spaces and we note that, prior to [GJM11], this moduli space had not appeared in the literature (see Remark 3.1). Our main application is to show that the nontrivial conformal block divisors $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ are pullbacks of ample classes from these Veronese quotients (Theorem 4.6). For this, we prove several results concerning morphisms between these Veronese quotients (see Corollary 3.7 and Proposition 3.8).

Section 4. A particular family of conformal block divisors. In [Gia11, GG12] it was shown that the divisors $\mathcal{D}_{0,A}$ coincide with conformal block divisors of $\mathfrak{sl}_r$ and level one, and in [AGS10] it is shown that the divisors $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ and $\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ are proportional for the two special cases $\ell = 1$ and g. In [AGS10] the authors ask whether there is a more general correspondence between $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^n)$ and Veronese quotient divisors. Theorem 4.6 gives a complete, affirmative answer to their question (cf. Remark 4.7). One of the main insights in this work is that, while not proportional for the remaining levels $\ell \in \{2, \ldots, g-1\}$, the divisors $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ and $\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ lie on the same face of the nef cone of $\overline{\mathrm{M}}_{0,n}$. In other words, the two semi-ample divisors define maps to isomorphic birational models of $\overline{\mathrm{M}}_{0,n}$. The corresponding birational models are precisely the spaces described in §3.

Section 5. Generalizations. Evidence suggests that $\mathfrak{sl}_r$ conformal block divisors with nonzero weights give rise to compactifications of $\mathrm{M}_{0,n}$, and that these compactifications coincide with Veronese quotients. This is certainly true for $\ell = 1$, and for the family of higher level $\mathfrak{sl}_2$ divisors considered in this paper, as well as for a large number of cases found using [Swi10], software written for Macaulay 2 by David Swinarski. In §5.1 we provide evidence in support of these ideas in the $\mathfrak{sl}_2$ cases. In §5.2 we describe consequences of and evidence for Conjecture 5.6, which asserts that conformal block divisors (with strictly positive weights) separate points on $\mathrm{M}_{0,n}$.

Acknowledgements. We would like to thank Valery Alexeev, Maksym Fedorchuk, Noah Giansiracusa, Young-Hoon Kiem, Jason Starr, and Michael Thaddeus for many helpful and inspiring discussions. We would also like to thank the referee for many suggestions about an earlier draft of this paper. The first author is supported on NSF DMS-1201268.

1. Background on Veronese quotients

We begin by reviewing general facts about Veronese quotients, including a description of them as moduli spaces of weighted pointed (generalized) Veronese curves (Section 1.1), and the morphisms $\varphi_{\gamma,A} : \overline{\mathrm{M}}_{0,n} \longrightarrow V^d_{\gamma,A}$ (Section 1.2), from [GJM11].

1.1. **The spaces $V^d_{\gamma,A}$.** Following [GJM11], we write $Chow(1, d, \mathbb{P}^d)$ for the irreducible component of the Chow variety parameterizing curves of degree d in $\mathbb{P}^d$ and their limit cycles, and consider the incidence correspondence

$$U_{d,n} := \{(X, p_1, \cdots, p_n) \in Chow(1, d, \mathbb{P}^d) \times (\mathbb{P}^d)^n : p_i \in X \ \forall i\}.$$

There is a natural action of $\mathrm{SL}(d+1)$ on $U_{d,n}$, and one can form the GIT quotients $U_{d,n} /\!/_L \mathrm{SL}(d+1)$ where L is a $\mathrm{SL}(d+1)$-linearized ample line bundle. The Chow variety and each copy of $\mathbb{P}^d$ has a tautological ample line bundle $\mathcal{O}_{Chow}(1)$ and $\mathcal{O}_{\mathbb{P}^d}(1)$, respectively. By taking external tensor products of them, for each sequence of positive rational numbers $(\gamma, (a_1, \cdots, a_n))$, we obtain a $\mathbb{Q}$-linearized ample line bundle $L = \mathcal{O}(\gamma) \otimes \mathcal{O}(a_1) \otimes \cdots \otimes \mathcal{O}(a_n)$.

Definition 1.1. *We say that a linearization L is* **allowable** *if it is an element of the set Δ^0, where*

$$\Delta^0 = \{(\gamma, A) = (\gamma, (a_1, \cdots, a_n)) \in \mathbb{Q}^{n+1}_{\geq 0} : \gamma < 1,\ 0 < a_i < 1,\ \text{and}\ \ (d-1)\gamma + \sum_i a_i = d + 1\}.$$

If $a_1 = \cdots = a_n = a$, then we write a^n for $A = (a_1, \cdots, a_n)$.

Let

$$V^d_{\gamma,A} := U_{d,n} /\!/_{\gamma,A} \mathrm{SL}(d+1).$$

We call GIT quotients of this form Veronese quotients *because they are quotients of a space parametrizing pointed Veronese curves.*

Given $(X, p_1, \cdots, p_n) \in U_{d,n}$, we may think of a choice $L \in \Delta^0$ as an assignment of a rational weight γ to the curve X and another weight a_i to each of the marked points p_i. The conditions $\gamma < 1$ and $0 < a_i < 1$ for all i imply that the quotient $U_{d,n} /\!/_{\gamma,A} \mathrm{SL}(d+1)$ is a compactification of $\mathrm{M}_{0,n}$ [GJM11, Proposition 2.10]. As is reflected in Lemma 1.2 below, the quotients have a modular interpretation parametrizing pointed degenerations of Veronese curves.

By taking $d = 1$, and $\gamma = 0$, one obtains the GIT quotients $(\mathbb{P}^1)^n /\!/_A \mathrm{SL}(2)$ with various weight data A. This quotient, which appears in [MFK94, Chapter 3] under the heading "an elementary example", has been studied by many authors. It was generalized first to $d = 2$ by Simpson in [Sim08] and later Giansiracusa and Simpson in [GS11], and then for arbitrary d, and $\gamma = 0$ by Giansiracusa in [Gia11]. More generally, the quotients for arbitrary d and $\gamma \geq 0$ are defined and studied by Giansiracusa, Jensen and Moon in [GJM11].

The semistable points of $U_{d,n}$ with respect to the linearization (γ, A) have the following nice geometric properties.

Lemma 1.2. [GJM11, Corollary 2.4, Proposition 2.5, Corollary 2.6, Corollary 2.7] *For an allowable choice (γ, A) (Definition 1.1), a semistable point $(X, p_1, \cdots, p_n)$ of $U_{d,n}$ has the following properties.*

(1) X is an arithmetic genus zero curve having at worst multi-nodal singularities.

(2) Given a subset $J \subset \{1, \cdots, n\}$, the marked points $\{p_j : j \in J\}$ can coincide at a point of multiplicity m on X as long as

$$(m-1)\gamma + \sum_{j \in J} a_j \leq 1.$$

In particular, a collection of marked points can coincide at a smooth point of X as long as their total weight is at most 1. Also, a semistable curve cannot have a singularity of multiplicity m unless $\gamma \leq \frac{1}{m-1}$.

(3) X is non-degenerate, i.e., it is not contained in a hyperplane.

1.2. **The morphisms $\varphi_{\gamma,A} : \overline{\mathrm{M}}_{0,n} \longrightarrow V^d_{\gamma,A}$.** In [GJM11] the authors prove the existence and several properties of birational morphisms from $\overline{\mathrm{M}}_{0,n}$ to Veronese quotients.

Proposition 1.3. [GJM11, Theorem 1.2, Proposition 4.7] *For an allowable choice (γ, A), there exists a regular birational map $\varphi_{\gamma,A} : \overline{\mathrm{M}}_{0,n} \to V^d_{\gamma,A}$ preserving the interior $\mathrm{M}_{0,n}$. Moreover, $\varphi_{\gamma,A}$ factors through the contraction maps ρ_A to Hassett's moduli spaces $\overline{\mathrm{M}}_{0,A}$:*

$$\begin{tikzcd}
\overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_A"'] \arrow[dr, "\varphi_{\gamma,A}"] & \\
\overline{\mathrm{M}}_{0,A} \arrow[r, "\phi_\gamma"] & V^d_{\gamma,A}.
\end{tikzcd}$$

By the definition of the projective GIT quotient, there is a natural choice of an ample line bundle on each GIT quotient. By pulling it back to $\overline{\mathrm{M}}_{0,n}$, we obtain a semi-ample divisor.

Definition 1.4. *Let $L = (\gamma, A)$ be an allowable linearization on $U_{d,n}$, and let $\overline{L} = L /\!/_L \mathrm{SL}(d+1)$ be the natural $\mathbb{Q}$-ample line bundle on $V^d_{\gamma,A}$. Define $\mathcal{D}_{\gamma,A}$ to be the semi-ample line bundle $\varphi^*_{\gamma,A}(\overline{L})$.*

Next, following [Has03] and [GJM11], we describe the F-curves contracted by $\varphi_{\gamma,A}$ and ρ_A. To do this, we first define F-curves, which together span the vector space $N_1(\overline{\mathrm{M}}_{0,n})$ of numerical equivalence classes of 1-cycles on $\overline{\mathrm{M}}_{0,n}$.

Definition 1.5. *Let $A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4 = [n] = \{1, \cdots, n\}$ be a partition into nonempty subsets, and set $n_i = |A_i|$. There is an embedding*

$$f_{A_1,A_2,A_3,A_4} : \overline{\mathrm{M}}_{0,4} \longrightarrow \overline{\mathrm{M}}_{0,n}$$

given by attaching four legs $L(A_i)$ to $(X, (p_1, \cdots, p_4)) \in \overline{\mathrm{M}}_{0,4}$ at the marked points. More specifically, to each p_i we attach a stable n_i+1-pointed fixed rational curve $L(A_i) = (X_i, (p^i_1, \cdots, p^i_{n_i}, p^i_a))$ by identifying p^i_a and p_i, while if $n_i = 1$ for some i, we just keep p_i as is. The image is a curve in $\overline{\mathrm{M}}_{0,n}$ whose equivalence class is the F**-curve** *denoted $F(A_1, A_2, A_3, A_4)$. Each member of the* F*-curve consists of a (varying)* **spine** *and 4 (fixed)* **legs***.*

In many parts of this paper, we will focus on symmetric divisors and F*-curves, in which case the equivalence class is determined by the number of marked points on each leg. In this case, we write F_{n_1,n_2,n_3,n_4} for* any F*-curve class $F(A_1, A_2, A_3, A_4)$ with $|A_i| = n_i$.*

The F-curves $F(A_1, A_2, A_3, A_4)$ contracted by the Hassett morphism ρ_A are precisely those for which one of the legs, say $L(A_i)$, has weight $\sum_{j \in A_i} a_j \geq \sum_{j \in [n]} a_j - 1$. We can always order the cells of the partition so that A_4 is the heaviest—that is, $\sum_{j \in A_4} a_j \geq \sum_{j \in A_i} a_j$, for all i. As the morphism $\varphi_{\gamma,A}$ factors through ρ_A, these curves are also contracted by $\varphi_{\gamma,A}$. This morphism may contract additional F-curves as well, which we describe here.

As is proved in [GJM11], the map $\varphi_{\gamma,A}$ contracts those curves $F(A_1, A_2, A_3, A_4)$ for which the sum of the degrees of the four legs is equal to d. We define two functions ϕ and σ below which are useful for computing the degree of the legs of an F-curve.

Definition 1.6. [GJM11, Section 3.1] *Consider the function $\phi : 2^{[n]} \times \Delta^0 \longrightarrow \mathbb{Q}$, given by*

$$\phi(J, \gamma, A) = \frac{a_J - 1}{1 - \gamma}, \text{ where for } J \in 2^{[n]},\ a_J = \sum_{j \in J} a_j.$$

For a fixed allowable linearization $(\gamma, A) = (\gamma, (a_1, \cdots, a_n))$ (cf. Definition 1.1), let

$$\sigma(J) = \begin{cases} \lceil \phi(J, \gamma, A) \rceil & \text{if } 1 < a_J < a_{[n]} - 1, \\ 0 & \text{if } a_J < 1, \\ d & \text{if } a_J > a_{[n]} - 1. \end{cases}$$

Finally, for $(X, p_1, \cdots, p_n) \in U_{d,n}$ and $E \subset X$ a subcurve, define $\sigma(E) = \sigma(\{j \in [n] | p_j \in E\})$.

Proposition 1.7. [GJM11, Proposition 3.5] *For an allowable choice of (γ, A), suppose that $\phi(J, \gamma, A) \notin \mathbb{Z}$ for any nonempty $J \subset [n]$. If X is a GIT-semistable curve and $E \subset X$ a tail (a subcurve such that $E \cap \overline{X - E}$ is one point), then* $\deg(E) = \sigma(E)$.

Corollary 1.8. [GJM11, Corollary 3.7] *Suppose that $\phi(J, \gamma, A) \notin \mathbb{Z}$ for any $\emptyset \neq J \subset [n]$, and let $E \subseteq X$ be a connected subcurve of $(X, p_1, \ldots, p_n) \in U^{ss}_{d,n}$. Then*

$$\deg(E) = d - \sum \sigma(Y)$$

where the sum is over all connected components Y of $\overline{X \backslash E}$.

Given an F-curve $F(A_1, A_2, A_3, A_4)$ as above, Proposition 1.7 says that $\deg(L(A_i)) = \sigma(A_i)$ if $\phi(A_i, \gamma, A)$ is not an integer. It follows that the degree of the spine is zero, and hence the F-curve is contracted, if and only if $\sum_{i=1}^{4} \sigma(A_i) = d$.

Remark 1.9. When $U^{ss}_{d,n}$ has a strictly semistable point, it is possible that $\phi(A_i, \gamma, A) \in \mathbb{Z}$. If $\phi(A_i, \gamma, A) = k$ is an integer, $\deg(L(A_i))$ may be either k or $k+1$. In this case, both curves are identified in the GIT quotient, and it suffices to consider the case where the legs have the maximum possible total degree.

2. The Veronese quotient divisors $\mathcal{D}_{\gamma,A}$

The Veronese quotient divisors $\mathcal{D}_{\gamma,A}$ are semi-ample divisors which give rise to morphisms from $\overline{\mathrm{M}}_{0,n}$ to the Veronese quotients $V^d_{\gamma,A}$. One of the main results of this paper is to give the combinatorial tools necessary to study these divisors as elements of the cone of nef divisors on $\overline{\mathrm{M}}_{0,n}$.

In Theorem 2.1, we give a formula for the intersection of the $\mathcal{D}_{\gamma,A}$ (as given in Definition 1.4) with F-curves on $\overline{\mathrm{M}}_{0,n}$ (as described in Definition 1.5). As a first application of Theorem 2.1, in Section 2.5, we show there is a simple formula for the intersection of the particular divisors $\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ with a basis of F-curves. As a second application of Theorem 2.1, in Corollary 2.12, we write down the class of $\mathcal{D}_{\gamma,A}$ in the case that A is S_n-invariant. While we have already described a criterion for determining when these numbers are zero at the end of §1, to compute these numbers in the non-zero case is substantially more complicated.

We first state Theorem 2.1, and Corollary 2.2, which exhibits the intersection numbers in a particular case. Before proving Theorem 2.1, in Section 2.1, we give an overview of our approach.

Notation. Let $[n] = A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4$ be a partition and let $F(A_1, A_2, A_3, A_4)$ be the corresponding F-curve (cf. Definition 1.5). Recall that $\sigma(A_i)$ is the degree of the leg $L(A_i)$ (Definition 1.6). In this section, we establish the following explicit formula for the intersection of $\mathcal{D}_{\gamma,A}$ and $F(A_1, A_2, A_3, A_4)$.

Theorem 2.1. *Given an allowable linearization (γ, A) with $d \geq 2$, and an* F*-curve $F(A_1, A_2, A_3, A_4)$:*

$$\begin{aligned} F(A_1,A_2,A_3,A_4)\cdot\mathcal{D}_{\gamma,A} \quad &= \quad (\sum_{i=1}^{3} c_{i4}^2)\frac{w}{2d} + (w_{A_4} - \frac{w}{d}\sigma(A_4))b \\ &\quad + \sum_{i=1}^{3}(\frac{w}{d}(\sigma(A_i)+\sigma(A_4)) - w_{A_i} - w_{A_4})c_{i4} \\ &\quad - \frac{1+\gamma}{2d}(\sum_{i=1}^{4}\sigma(A_i)(d-\sigma(A_i)) - \sum_{i=1}^{3}\sigma(A_i\cup A_4)(d-\sigma(A_i\cup A_4))) \end{aligned}$$

where

$$\begin{aligned} c_{ij} &:= d - \sigma(A_i) - \sigma(A_j) - \sigma([n]\backslash(A_i\cup A_j)) \\ &= \sigma(A_i\cup A_j) - \sigma(A_i) - \sigma(A_j), \\ b &= d - \sum_{i=1}^{4}\sigma(A_i), \\ w &= \sum_{i=1}^{n} a_i, \\ w_{A_j} &= \sum_{i\in A_j} a_i. \end{aligned}$$

Note that the case of $d = 1$ was studied previously in [AS08, Section 2]. If there is an A_i such that $\phi(A_i, \gamma, A)$ is an integer, then the σ function does not give a unique degree for each leg (Remark 1.9). But the result of Theorem 2.1 is nevertheless independent of the choice of semistable degree distribution.

As an example for how simple this formula can be, consider the following.

Corollary 2.2. *For $n = 2g+2$, and $1 \leq \ell \leq g$,*

$$F_{n-i-2,i,1,1}\cdot\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}} = \begin{cases} \frac{1}{\ell+1} & \text{if } i \equiv \ell \pmod 2 \text{ and } i \geq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Before delving into the proof of Theorem 2.1 (in Section 2.4), we first explain our approach (in Section 2.1), and develop a tool we will use (in Section 2.2), which is a rational lift to $U_{d,n}$ of the image C in $V^d_{\gamma,A}$ of a given F-curve.

2.1. **Approach to the proof of Theorem 2.1.** Let C be the image of $F(A_1, A_2, A_3, A_4)$ in $V^d_{\gamma,A}$ under the map $\varphi_{\gamma,A}$. Let $L = \mathcal{O}(\gamma, A)$ be an allowable polarization on $U_{d,n}$, and let $\overline{L} = L /\!/_{\gamma,A} \mathrm{SL}(d+1)$ be the associated ample line bundle on $V^d_{\gamma,A}$. By the projection formula,

$$F(A_1, A_2, A_3, A_4)\cdot\mathcal{D}_{\gamma,A} = C\cdot\overline{L}.$$

Therefore, to prove Theorem 2.1 we need to compute $C\cdot\overline{L}$. To do this, we will lift C to an appropriate curve $\widetilde{C}$ on $U_{d,n}$ and do the intersection there.

Definition 2.3. *Let C be a curve in $V^d_{\gamma,A}$, and let $\pi : U^{ss}_{d,n} \to V^d_{\gamma,A}$ be the quotient map. A* **rational lift** *of C to $U_{d,n}$ is a curve $\widetilde{C}$ in $U_{d,n}$ such that*

- *a general point of $\widetilde{C}$ lies in $U^{ss}_{d,n}$;*
- *$\overline{\pi(\widetilde{C})} = C$ and $\pi|_{\widetilde{C}} : \widetilde{C} \dashrightarrow C$ is degree 1.*

A section of $\overline{L}$ can be pulled-back to a section of L that vanishes on the unstable locus. It follows that if we have a rational lifting $\widetilde{C}$, then by the projection formula we have

$$C \cdot \overline{L} = \widetilde{C} \cdot (L - \sum t_i E_i)$$

for some rational numbers $t_i > 0$, where the sum is taken over all irreducible unstable divisors. By the proof of [GJM11, Proposition 4.6], there are two types of unstable divisors. One is a divisor of curves with unstable degree distribution and the other is D_{deg}, the divisor of curves contained in a hyperplane. If $\widetilde{C}$ intersects D_{deg} only among unstable divisors, then $C \cdot \overline{L} = \widetilde{C} \cdot (L - tD_{deg})$ for some $t > 0$.

2.2. **An explicit rational lift.** In this section we will construct a rational lift $\widetilde{C}$ to $U_{d,n}$ of the image C in $V^d_{\gamma,A}$ of an F-curve $F(A_1, A_2, A_3, A_4)$ in $\overline{\mathrm{M}}_{0,n}$. This lift $\widetilde{C}$ will be used to prove Theorem 2.1 in Section 2.4.

An F-curve is isomorphic to $\overline{M}_{0,4} \cong \mathbb{P}^1$. Thus the total space of an F-curve is a family of curves over $\mathbb{P}^1$, which is a reducible surface for $n \geq 5$. It consists of five components. One corresponds to a varying spine, which is isomorphic to the universal curve over $\overline{M}_{0,4}$, hence to $\overline{M}_{0,5}$. The other four components are constant families over $\mathbb{P}^1$, which correspond to four fixed legs. We will think of the total space $X \cong \overline{M}_{0,5}$ of spines as the blow-up of 3 points on the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. The points of attachment to the legs $L(A_i)$ labeled by A_1, A_2 and A_3 will correspond to the 3 sections of X through the exceptional divisors, while the point of attachment to the leg $L(A_4)$ will correspond to the diagonal. We denote the classes of the total transforms of two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$ by F (for fiber) and S (for section), and the exceptional divisors by E_i. Then on $X \cong \overline{\mathrm{M}}_{0,5}$, the 10 boundary classes are given by

$$\begin{aligned} &D_{15} = S - E_1, D_{25} = S - E_2, D_{35} = S - E_3, D_{45} = F + S - E_1 - E_2 - E_3,\\ &D_{14} = F - E_1, D_{24} = F - E_2, D_{34} = F - E_3, D_{23} = E_1, D_{13} = E_2, D_{12} = E_3. \end{aligned} \tag{1}$$

Here is an outline of the construction of an explicit rational lift of a curve on $V^d_{\gamma,A}$. For a curve isomorphic to $\mathbb{P}^1$ on $V^d_{\gamma,A}$, we need to construct a family of rational curves in $\mathbb{P}^d$ of degree d over $\mathbb{P}^1$, with n sections. First of all, we will construct a map from X to $\mathbb{P}^d$ by constructing a base-point free sub linear system V of a certain divisor class on X. After that we will attach four fixed legs, to make a family of degree d rational curves. To get a family of curves whose general member is in $U^{ss}_{d,n}$, the general member must satisfy certain degree conditions on each irreducible component and it must be non-degenerate. Let $\sigma(A_i)$ be the degree of the leg containing marked points in A_i.[1] Then the general fiber must have degree

$$b := d - \sum_{i=1}^{4} \sigma(A_i).$$

[1]If $U^{ss}_{d,n} = U^s_{d,n}$, then the degree of the leg is uniquely determined by the σ function in [GJM11], but if there are strictly semi-stable points, then the degree is not determined uniquely. In this case we can take any degree distribution which gives semistable points. See Remark 1.9.

As the cross-ratio of the 4 points on the spine varies, there are 3 points where the spine breaks into two components. The degree of one of these components where A_i and A_j come together is exactly

$$c_{ij} := d - \sigma(A_i) - \sigma(A_j) - \sigma([n]\backslash(A_i \sqcup A_j)) = \sigma(A_i \sqcup A_j) - \sigma(A_i) - \sigma(A_j).$$

We therefore consider the following divisor class on X (which depends on an integer $a \geq 0$):

$$H(a) := aF + bS - \sum_{i=1}^{3} c_{i4}E_i.$$

When $a \gg 0$, it is base-point free (Lemma 2.4), so it defines a map to $\mathbb{P}^d$. Moreover, for $a \gg 0$, we can take a subspace $V \subset H^0(X, H(a))$ of dimension $b+2$ such that its restriction $V|_F$ to every fiber defines a rational normal curve of degree b, thus it is non-degenerate (Lemma 2.6). In Proposition 2.7, we show that the general point of the family obtained by attaching four fixed tails is semistable by showing it satisfies degree conditions.

Lemma 2.4. *For $a \gg 0$, $H(a)$ is base-point free.*

Proof. Since X is a del Pezzo surface, it is well-known that if $H(a)$ is nef, then $H(a)$ is base-point free. On $X \cong \overline{\mathrm{M}}_{0,5}$, the cone of curves is generated by the classes D_{ij}. Thus by using the explicit descriptions of the divisors D_{ij} given in (1) above, it is straightforward to check that $H(a)$ is nef if and only if

$$a \geq c_{i4}, b \geq c_{i4}, a + b \geq \sum_{i=1}^{3} c_{i4}.$$

The second inequality is immediate because $b = c_{12}+c_{34} = c_{13}+c_{24} = c_{14}+c_{23}$. So if a is sufficiently large, then $H(a)$ is nef and base-point free. $\square$

Lemma 2.5. *For $a \gg 0$, the map $H^0(X, H(a)) \to H^0(F, H(a)|_F)$ is surjective.*

Proof. By the exact sequence

$$0 \to H^0(X, H(a) - F) \to H^0(X, H(a)) \to H^0(F, H(a)|_F) \to H^1(X, H(a) - F),$$

it suffices to show that $h^1(X, H(a) - F) = 0$. Since X is a del Pezzo surface, $-K_X$ is ample. Thus $H(a) - K_X$ is ample for $a \gg 0$ by Lemma 2.4 and $h^i(X, H(a)) = h^i(X, H(a) - K_X + K_X) = 0$ for $i > 0$ by the Kodaira vanishing theorem. Since $H(a) - F = H(a-1)$ by definition, $h^1(X, H(a) - F) = 0$ for large a as well. $\square$

By a Riemann-Roch calculation, if $a \gg 0$ then

$$h^0(X, H(a)) = 3ab - \sum_{i=1}^{3} \binom{c_{i4}+1}{2} + 1.$$

For sufficiently large a, $h^0(X, H(a))$ is therefore greater than $d + 1$, so we cannot use the complete linear system $|H(a)|$ to construct a map to $\mathbb{P}^d$. To deal with this problem, we use the following Lemma.

Lemma 2.6. *Let $V \subset H^0(X, H(a))$ be a general linear subspace of dimension $h^0(F, H(a)|_F) + 1 = b + 2$. For $a \gg 0$, the map $V \to H^0(F, H(a)|_F)$ is surjective for every fiber F.*

Proof. For a given fiber F, write K_F for the kernel of the map $H^0(X, H(a)) \to H^0(F, H(a)|_F)$. By Lemma 2.5, K_F is a linear space of dimension $h^0(X, H(a) - F)$. We will show that $\dim V \cap K_F = 1$ for every fiber F. In particular, denote the fiber over a point $y \in \overline{M}_{0,4} \cong \mathbb{P}^1$ by F_y, and consider the variety

$$Z = \{(y, V) \in \mathbb{P}^1 \times Gr(b+2, H^0(X, H(a))) \mid \dim V \cap K_{F_y} \geq 2\}.$$

The fibers of Z over $\mathbb{P}^1$ are Schubert varieties, which are known to be irreducible of codimension 2 in the Grassmannian. It follows that $\dim Z < \dim Gr(b+2, H^0(X, H(a))$, and thus Z does not map onto the Grassmannian. We therefore see that, for the general $V \in Gr(b+2, H^0(X, H(a)))$, $\dim V \cap K_{F_y} < 2$ for every $y \in \mathbb{P}^1$. On the other hand, we see that $\dim V \cap K_{F_y} \geq 1$ trivially for dimension reasons. It follows that the map $V \to H^0(F, H(a)|_F)$ is surjective for every fiber F. $\square$

By Lemma 2.6, if we consider the map $X \to \mathbb{P}^{b+1}$ corresponding to the linear series V, we see that each individual fiber is mapped to $\mathbb{P}^{b+1}$ via a complete linear series. The general fiber therefore maps to a smooth rational normal curve of degree b and the three special fibers map to nodal curves whose two components have the appropriate degrees. Then, as long as $b < d$, one can embed this $\mathbb{P}^{b+1}$ in $\mathbb{P}^d$ and obtain a family of curves in this projective space.

Now consider the case of $b = d$. Because X is a surface and $d \geq 2$, we can take a point $p \in \mathbb{P}^{b+1} \setminus X$. Considering a projection from p, we obtain a family of curves in $\mathbb{P}^d$ with the same degree distribution. We must choose the point p such that a general member of such a family of curves is semistable. Because it has the correct degree distribution, it suffices to check that a general member of the family is not contained in a hyperplane. But the image of a curve under projection is degenerate only if the original curve is degenerate.

To each of the 4 sections we attach a family of curves that does not vary in moduli. Using the same trick as before, we may take 4 copies of $\mathbb{P}^1 \times \mathbb{P}^1$, mapped into $\mathbb{P}^d$ via a linear series $V_i \subset |\mathcal{O}(x_i, y_i)|$, where

$$x_i = \begin{cases} H(a) \cdot (S - E_i) = a - c_{i4}, & i \neq 4, \\ H(a) \cdot (F + S - \sum_{j=1}^{3} E_j) = a + b - \sum_{j=1}^{3} c_{j4}, & i = 4 \end{cases}$$

and $y_i = \sigma(A_i)$ is the degree of the leg. Note that if $b = d$, then $\sigma(A_i) = 0$ so we don't need to worry about the construction of extra components.

Proposition 2.7. *The family we have constructed is a rational lift of $\varphi_{\gamma,A}(F(A_1, A_2, A_3, A_4))$. It does not intersect any GIT-unstable divisor other than D_{deg}.*

Proof. We claim that all of the members of this family satisfy the degree conditions required by semi-stability. Indeed, the general member is a nodal curve with 4 components labeled by the A_i's. The degree of the leg labeled by A_i is $\mathcal{O}(x_i, y_i) \cdot \mathcal{O}(1, 0) = \sigma(A_i)$ and the degree of the spine is $H(a) \cdot F = b = d - \sum_{i=1}^{4} \sigma(A_i)$. As one varies the cross-ratio of the 4 points on the spine, there are 3 points where the spine breaks into two components. The degree of these components are for instance $H(a) \cdot E_1 = c_{14} = d - \sigma(A_4) - \sigma(A_1) - \sigma([n] \backslash (A_4 \cup A_1))$ and $H(a) \cdot (F - E_1) = b - c_{14} = d - \sigma(A_2) - \sigma(A_3) - \sigma([n] \backslash (A_2 \cup A_3))$. $\square$

2.3. **Divisor classes on $U_{d,n}$.** The main result of this section is Lemma 2.9, which gives a numerical relation between several divisor classes on $U_{d,n}$.

Definition 2.8. *Let H be the divisor class on $U_{d,n}$ parameterizes curves which meet a fixed codimension two linear subspace in $\mathbb{P}^d$. Let D_k be the divisor class on $U_{d,n}$ which is the closure of the* *locus parameterizes curves with two irreducible components with degree k and $d-k$ respectively. Finally, let D_{deg} be the divisor of curves contained in a hyperplane.*

Lemma 2.9. *The following numerical relation holds in $\mathrm{N}^1(U_{d,n})$.*

$$D_{deg} = \frac{1}{2d}\Big((d+1)H - \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} k(d-k)D_k\Big).$$

To prove this result, we will use a result of [CHS08] about the moduli space of stable maps. A map $f : (C, p_1, \cdots, p_n) \to \mathbb{P}^r$ from an arithmetic genus 0 curve C with n marked points to $\mathbb{P}^r$ is called **stable** if

- C has at worst nodal singularities,
- p_i are distinct smooth points on C,
- $\omega_C + \sum p_i + f^*\mathcal{O}(3)$ is ample.

We say that f has degree d if $f^*\mathcal{O}(1)$ has degree d on C. A moduli space of stable maps $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is the moduli space of degree d stable maps from genus 0 n-pointed curves to $\mathbb{P}^r$. For more information about moduli space of stable maps, see [FP97].

Here is a list of properties of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$ we will use in this paper.

(1) There is a forgetful map $f : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$, which forgets the n marked points and stabilizes the map.
(2) There are several functorial morphisms. A cycle morphism $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to Chow(1, d, \mathbb{P}^d)$ maps a stable map to its image of fundamental cycle of the domain. There are n evaluation maps $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \mathbb{P}^d$ which send a stable map to the image of i-th marked points on $\mathbb{P}^d$. By taking the product of these maps, we have a cycle map
$$g : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to Chow(1, d, \mathbb{P}^d) \times (\mathbb{P}^d)^n$$
and obviously it factors through $U_{d,n}$.
(3) We can define divisor classes H, D_k, and D_{deg} on $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$ using the descriptions given in Definition 2.8.

Proof of Lemma 2.9. By [CHS08, Lemma 2.1], on the moduli space of stable maps $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$,

$$D_{deg} = \frac{1}{2d}\Big((d+1)H - \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} k(d-k)D_k\Big).$$

If we pull-back D_{deg} by the forgetful map $f : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$, then we obtain the same formula for D_{deg} on $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$. Now for the cycle map $g : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to U_{d,n}$, we have $g_*(H) = H = \mathcal{O}_{Chow}(1)$, $g_*(D_k) = D_k$, and $g_*(D_{deg}) = D_{deg}$. Therefore the same formula holds for $U_{d,n}$. □

2.4. **Proof of Theorem 2.1.** In this section we prove Theorem 2.1, which relies on the curve constructed in Section 2.2.

Proof. As is explained in Section 2.1, to prove Theorem 2.1, we shall compute the intersection of C, the image of $F(A_1, A_2, A_3, A_4)$ in $V^d_{\gamma,A}$, with the natural ample line bundle $\overline{L}$. To do this, it suffices to find a rational lift $\widetilde{C}$ of this curve to $U_{d,n}$ such that a general element of $\widetilde{C}$ is semistable, and compute the intersection in $U_{d,n}$.

By Proposition 2.7, the family constructed in Section 2.2 has this property, so we can use it to carry out these computations. To compute the intersection of $\widetilde{C}$ with $\mathcal{O}_{Chow}(1)$, fix a general codimension 2 linear space in $\mathbb{P}^d$. The intersection number is precisely the number of curves in the family that intersect this linear space. In other words, it is the total degree of our 5 surfaces. Hence

$$\begin{aligned}\widetilde{C}\cdot\mathcal{O}_{Chow}(1) &= H(a)^2+\sum_{i=1}^{4}\mathcal{O}(x_i,y_i)^2\\ &= 2ab-\sum_{i=1}^{3}c_{i4}^2+\sum_{i=1}^{3}2(a-c_{i4})\sigma(A_i)+2(a+b-\sum_{j=1}^{3}c_{j4})\sigma(A_4)\\ &= 2ad+2\sigma(A_4)b-\sum_{i=1}^{3}c_{i4}^2-\sum_{i=1}^{3}2(\sigma(A_i)+\sigma(A_4))c_{i4}.\end{aligned}$$

Similarly, to compute the intersection of $\widetilde{C}$ with $\mathcal{O}_{\mathbb{P}^d_j}(1)$, fix a general hyperplane in $\mathbb{P}^d$. The intersection number is precisely the number of points at which the j-th section meets this hyperplane. In other words, it is the degree of the j-th section. If A_i is the part of the partition containing j, then we see that

$$\widetilde{C}\cdot\mathcal{O}_{\mathbb{P}^d_j}(1)=\mathcal{O}(x_i,y_i)\cdot\mathcal{O}(0,1)=x_i=\begin{cases}a-c_{i4}, & i\neq 4,\\ a+b-\sum_{k=1}^{3}c_{k4}, & i=4.\end{cases}$$

One can then easily compute the intersection with $L=\bigotimes_{j=1}^{n}\mathcal{O}_{\mathbb{P}^d_j}(a_j)\otimes\mathcal{O}_{Chow}(\gamma)$ by linearity. If we denote $\sum_{i\in A_j}a_i$ by w_{A_j} and $w=\sum_{i=1}^{n}a_i$, then

$$\begin{aligned}\widetilde{C}\cdot L &= \gamma(2ad+2\sigma(A_4)b-\sum_{i=1}^{3}c_{i4}^2-\sum_{i=1}^{3}2(\sigma(A_i+A_4))c_{i4})\\ &\quad+\sum_{i=1}^{3}w_{A_i}(a-c_{i4})+w_{A_4}(a+b-\sum_{i=1}^{3}c_{i4})\\ &= (2d\gamma+w)a-\sum_{i=1}^{3}c_{i4}^2\gamma+(2\sigma(A_4)\gamma+w_{A_4})b\\ &\quad-\sum_{i=1}^{3}(2\gamma(\sigma(A_1)+\sigma(A_4))+w_{A_i}+w_{A_4})c_{i4}.\end{aligned}$$

Recall that $C\cdot\overline{L}=\widetilde{C}\cdot(L-tD_{deg})$ for some positive rational number t (Section 2.1). It remains to determine the value of t. By Lemma 2.9,

$$\begin{aligned}\widetilde{C}\cdot D_{deg} &= \frac{d+1}{2d}\Big(2ad+2\sigma(A_4)b-\sum_{i=1}^{3}c_{i4}^2-\sum_{i=1}^{3}2(\sigma(A_i)+\sigma(A_4))c_{i4}\Big)\\ &\quad+\frac{1}{2d}\Big(\sum_{i=1}^{4}\sigma(A_i)(d-\sigma(A_i))-\sum_{i=1}^{3}(\sigma(A_i\cup A_4))(d-\sigma(A_i\cup A_4))\Big).\end{aligned}$$

Note that the rational lift depends on the choice of a. To obtain an intersection number $C\cdot\overline{L}=\widetilde{C}\cdot(L-tD_{deg})$ that is independent of the choice of a, the coefficient of a must be 0. Thus

$$2d\gamma+w-t\frac{(d+1)}{2d}2d=0$$

and $t = \frac{2d\gamma+w}{1+d} = 1+\gamma$.

Therefore,

$$\begin{aligned} C\cdot\overline{L} &= \widetilde{C}\cdot(L-(1+\gamma)D_{deg}) \\ &= (\sum_{i=1}^{3} c_{i4}^2)\frac{w}{2d} + (w_{A_4} - \frac{w}{d}\sigma(A_4))b + \sum_{i=1}^{3}(\frac{w}{d}(\sigma(A_i)+\sigma(A_4)) - w_{A_i} - w_{A_4})c_{i4} \\ & \quad -\frac{1+\gamma}{2d}(\sum_{i=1}^{4}\sigma(A_i)(d-\sigma(A_i)) - \sum_{i=1}^{3}\sigma(A_i\cup A_4)(d-\sigma(A_i\cup A_4))). \end{aligned}$$

□

2.5. **Example and application of Theorem 2.1.** As the F-curves span the vector space of 1-cycles, Theorem 2.1 gives, in principal, the class of $\mathcal{D}_{\gamma,A}$ in the Nerón Severi space. Using a particular basis (described in Definition 2.10), we explicitly write down the class of $\mathcal{D}_{\gamma,A}$ for S_n-invariant weights A. The classes depend on the intersection numbers, which as we see below in Example 2.13, are particularly simple for $\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$.

Definition 2.10. [AGSS12, Section 2.2.2, Proposition 4.1] *For $1 \le j \le g := \lfloor \frac{n}{2} - 1\rfloor$, let F_j be the S_n-invariant* F*-curve $F_{1,1,j,n-j-2}$. The set $\{\mathrm{F}_j : 1 \le j \le g\}$ forms a basis for the group of* 1*-cycles $\mathrm{N}_1(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$.*

Definition 2.11. [KM96, Section 3] *For $2 \le j \le \lfloor \frac{n}{2} \rfloor$, let B_j be the S_n-invariant divisor given by the sum of boundary divisors indexed by sets of size j:*

$$B_j = \sum_{J\subset[n],|J|=j} \delta_J.$$

*The set $\{\mathrm{B}_j : 2 \le j \le g+1\}$ forms a basis for the group of codimension-*1*-cycles $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$.*

Corollary 2.12. *Fix $n = 2g+2$ or $n = 2g+3$ and $j \in \{1,\cdots,g\}$, and write $a(\gamma,A)_j = \mathcal{D}_{\gamma,A}\cdot F_j$. If A is an S_n-invariant choice of weights, then $\mathcal{D}_{\gamma,A} \equiv \sum_{r=1}^{g} b(\gamma,A)_r B_{r+1}$, where*

$$b(\gamma,A)_r = \sum_{j=1}^{r-1}\left(\frac{r(r+1)}{n-1} - (r-j)\right)a(\gamma,A)_j + \frac{r(r+1)}{n-1}\sum_{j=r}^{g} a(\gamma,A)_j,$$

when $n = 2g+3$ is odd, and

$$b(\gamma,A)_r = \sum_{j=1}^{r-1}\left(\frac{r(r+1)}{n-1} - (r-j)\right)a(\gamma,A)_j + \frac{r(r+1)}{n-1}\sum_{j=r}^{g-1} a(\gamma,A)_j + \frac{r(r+1)}{2(n-1)}a(\gamma,A)_g$$

when $n = 2g+2$ is even.

Proof. This follows from the formula given in [AGSS12, Theorem 5.1]. □

Example 2.13.

$$\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}} = \frac{1}{\ell+1}\sum_{r=1}^{g}\left(\frac{r(r+1)}{n-1}\left(\frac{g-\ell+1}{2}\right) - \left\lceil\frac{r-\ell+1}{2}\right\rceil_+ \left\lfloor\frac{r-\ell+1}{2}\right\rfloor_+\right)B_{r+1},$$

where

$$\lceil x\rceil_+ = \max\{\lceil x\rceil, 0\}, \qquad \lfloor x\rfloor_+ = \max\{\lfloor x\rfloor, 0\}.$$

Proof. Indeed, by the previous results, we have

$$\begin{aligned} b(\gamma, A)_r &= \sum_{j=1}^{r-1}\left(\frac{r(r+1)}{n-1} - (r-j)\right)a(\gamma,A)_j + \frac{r(r+1)}{n-1}\sum_{j=r}^{g-1} a(\gamma,A)_j + \frac{r(r+1)}{2(n-1)}a(\gamma,A)_g \\ &= \frac{r(r+1)}{n-1}\sum_{j=1}^{g} a(\gamma,A)_j - \frac{r(r+1)}{2(n-1)}a(\gamma,A)_g - \sum_{j=1}^{r-1}(r-j)a(\gamma,A)_j. \end{aligned}$$

By Corollary 2.2,

$$\sum_{j=1}^{g} a(\gamma,A)_j = \begin{cases} \frac{1}{\ell+1}\left(\frac{g-\ell}{2}+1\right), & g \equiv \ell \pmod 2, \\ \frac{1}{\ell+1}\left(\frac{g-\ell+1}{2}\right), & g \not\equiv \ell \pmod 2. \end{cases}$$

Also $a(\gamma,A)_g = \frac{1}{\ell+1}$ if $g \equiv \ell \pmod 2$ and zero if $g \not\equiv \ell \pmod 2$, so we can write

$$b(\gamma,A)_r = \frac{1}{\ell+1}\frac{r(r+1)}{n-1}\frac{g-\ell+1}{2} - \sum_{j=1}^{r-1}(r-j)a(\gamma,A)_j.$$

By a similar case-by-case computation, one obtains

$$\begin{aligned} \sum_{j=1}^{r-1}(r-j)a(\gamma,A)_j &= \begin{cases} \left(\frac{r-\ell+1}{2}\right)^2, & r \not\equiv \ell \pmod 2 \text{ and } \ell \le r-1, \\ \frac{(r-\ell)(r-\ell+2)}{4}, & r \equiv \ell \pmod 2 \text{ and } \ell \le r-1, \\ 0, & \ell > r-1 \end{cases} \\ &= \lceil \frac{r-\ell+1}{2} \rceil_+ \lfloor \frac{r-\ell+1}{2} \rfloor_+. \end{aligned}$$

□

3. A new modular interpretation for a particular family of Veronese quotients

In this section, we study the family $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ of birational models for $\overline{\mathrm{M}}_{0,n}$, where $n = 2(g+1)$ and $1 \le \ell \le g$. In Theorem 3.5 we give a new modular interpretation of them as certain contractions of Hassett spaces

$$\tau_\ell : \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^n} \longrightarrow V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}},$$

where so called *even chains*, described in Definition 3.4, are replaced by particular curves. In order to see that the morphisms τ_ℓ exist, we first prove Proposition 3.2, which identifies the Veronese quotient associated to a nearby linearization with the Hassett space $\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^n}$. The results in this section allow us in §4 to prove that nontrivial conformal block divisors $\mathbb{D}(\mathfrak{sl}_2,\ell,\omega_1^{2g+2})$ are pullbacks of ample classes from $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$.

Remark 3.1. Because their defining linearizations $(\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2})$ lie on GIT walls, these Veronese quotients $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ admit strictly semistable points, and thus their corresponding moduli functors are not in fact separated. The quotient described in Theorem 3.5 is not isomorphic to a modular compactification in the sense of [Smy09] (cf. Remark 3.6). In fact, the only known method for constructing this compactification is via GIT. This highlights the strength of the Veronese quotient construction, as we show here that you can use them to construct "new" compactifications of $\mathrm{M}_{0,n}$ — compactifications that have not been described and cannot be described through any of the previously developed techniques.

3.1. **Defining the maps τ_ℓ.** In this section we define the morphism

$$\tau_\ell : \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}} \to V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}.$$

obtained by variation of GIT. As mentioned above, each of the linearizations $(\frac{\ell-1}{\ell+1}, (\frac{1}{\ell+1})^{2g+2})$ lies on a wall. To show that τ_ℓ exists, we will use the general variation of GIT fact that any quotient corresponding to a GIT chamber admits a morphism to a quotient corresponding to a wall of that chamber. Namely, in Proposition 3.2 below, we identify the Veronese quotient corresponding to a GIT chamber that borders the GIT wall that contains the linearization $(\frac{\ell-1}{\ell+1}, (\frac{1}{\ell+1})^{2g+2})$. We then use this to describe the morphism to the Veronese quotient $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$. We note that the Veronese quotient discussed in Proposition 3.2 is in general not a modular compactification of $\mathrm{M}_{0,n}$ but it is isomorphic to one (unlike the quotients described in Theorem 3.5).

Proposition 3.2. *For $2 \le \ell \le g-1$, and $\epsilon > 0$ sufficiently small, the Hassett space $\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}}$ is isomorphic to the normalization of the Veronese quotient*

$$V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\epsilon',(\frac{1}{\ell+1}-\epsilon)^{2g+2}}.$$

Here ϵ' is a positive number that is uniquely determined by the data $d = g+1-\ell$ and $A = (\frac{1}{\ell+1}-\epsilon)^{2g+2}$ (cf. Definition 1.1).

Proof. By Proposition 1.3, there is a morphism $\phi_\gamma : \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}} \to V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\epsilon',(\frac{1}{\ell+1}-\epsilon)^{2g+2}}$, which fits into the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_{(\frac{1}{\ell+1}-\epsilon)^{2g+2}}"'] \arrow[dr, "\varphi_{\gamma,(\frac{1}{\ell+1}-\epsilon)^{2g+2}}"] & \\
\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}} \arrow[r, "\phi_\gamma"] & V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\epsilon',(\frac{1}{\ell+1}-\epsilon)^{2g+2}}.
\end{tikzcd}$$

Thus to prove the result, it suffices to show that ϕ_γ is bijective. Since $(g-\ell)\gamma + (2g+2)(\frac{1}{\ell+1}-\epsilon) = g+2-\ell$,

$$\gamma = 1 - \frac{2}{\ell+1} + \frac{2(g+1)}{g-\ell}\epsilon.$$

If $\ell \ge 3$, we have $\gamma > \frac{1}{2}$ and a curve in the semistable locus $U^{ss}_{g+1-\ell,2g+2}$ does not have multinodal singularities by Lemma 1.2. Similarly, the sum of the weights at a node cannot exceed $1-\gamma = \frac{2}{\ell+1} - \frac{2(g+1)}{g-\ell}\epsilon < 2(\frac{1}{\ell+1}-\epsilon)$. So at a node, there is at most one marked point.

If $\ell = 2$, $\gamma > \frac{1}{3}$, so a curve in $U^{ss}_{g+1-\ell,2g+2}$ has at worst a multinodal point of multiplicity 3. Note that $1-(3-1)\gamma = \frac{1}{3} - \frac{4(g+1)}{g-2}\epsilon < \frac{1}{3}-\epsilon$, so by Lemma 1.2, there can be no marked point at a triple point. Similarly, since $1-(2-1)\gamma = \frac{2}{3} - \frac{2(g+1)}{g-2}\epsilon < 2(\frac{1}{3}-\epsilon)$, there can be at most one marked point at a node.

To summarize, there is no positive dimensional moduli of curves contracted to the same curve. In other words, ϕ_γ is an injective map. The surjectivity comes directly from the properness of both sides. ∎

Remark 3.3. (1) In [GJM11, Theorem 7.1, Corollary 7.2], the authors show that for certain values of γ and A, the corresponding Veronese quotient is $\overline{\mathrm{M}}_{0,A}$. Proposition 3.2 indicates precise values of γ and A when A is symmetric.

(2) The normalization map of a Veronese quotient is always bijective ([GJM11, Remark 6.2]). Thus at least on the level of topological spaces, the normalization is equal to the Veronese quotient itself.

(3) If $g \equiv \ell \bmod 2$, there are strictly semi-stable points on $U_{g+1-\ell,2g+2}$ for the linearization $(\frac{\ell-1}{\ell+1}+\epsilon', (\frac{1}{\ell+1}-\epsilon)^{2g+2})$. Indeed, for a set J of $g+1$ marked points, the weight function

$$\phi(J,\gamma,A) = \frac{(g+1)(\frac{1}{\ell+1}-\epsilon)-1}{\frac{2}{\ell+1}-\frac{2(g+1)}{g-\ell}\epsilon} = \frac{(g-\ell)(g-\ell)-(g-\ell)(g+1)(\ell+1)\epsilon}{2(g-\ell)-2(g+1)(\ell+1)\epsilon} = \frac{g-\ell}{2}$$

is an integer. So the quotient stack $[U^{ss}_{g+1-\ell,2g+2}/\operatorname{SL}(g+2-\ell)]$ is not modular in the sense of [Smy09].

(4) Even if the GIT quotient is modular, the moduli theoretic meaning of

$$\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}} \text{ and } V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\epsilon',(\frac{1}{\ell+1}-\epsilon)^{2g+2}}$$

may be different in general because on the GIT quotient, multinodal singularities and a marked point on a node are allowed. But the moduli spaces are nevertheless isomorphic.

(5) The condition on ℓ is necessary. Indeed, if $\ell = 1$ or g, the GIT quotient is not isomorphic to a Hassett space.

3.2. **The new modular interpretation.** In this section we will prove Theorem 3.5, which describes the Veronese quotients $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ as images of contractions where the so-called *even chains* in the Hassett spaces $\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^n}$ are replaced by other curves, described below.

Definition 3.4. *A curve* $(C, x_1, \ldots, x_{2g+2}) \in \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}}$ *is an* **odd chain** *(resp.* **even chain***) if* C *contains a connected chain* $C_1 \cup \cdots \cup C_k$ *of rational curves such that:*

(1) each C_i *contains exactly two marked points;*

(2) each interior component C_i *for* $2 \le i \le k-1$ *contains exactly two nodes* $C_i \cap C_{i-1}$ *and* $C_i \cap C_{i+1}$*;*

(3) aside from the two marked points, each of the two end components C_1 *and* C_k *contains two "special" points, where a special point is either a node or a point at which* $\ell+1$ *marked points coincide. In the first case, we will refer to the connected components of* $\overline{C \setminus \cup_{i=1}^k C_i}$ *as "tails". We will regard the second type of special point as a "tail" of degree* 0*;*

(4) the number of marked points on each of tails is odd (resp. even).

Figure 1 shows two examples of odd chains, when ℓ is even.

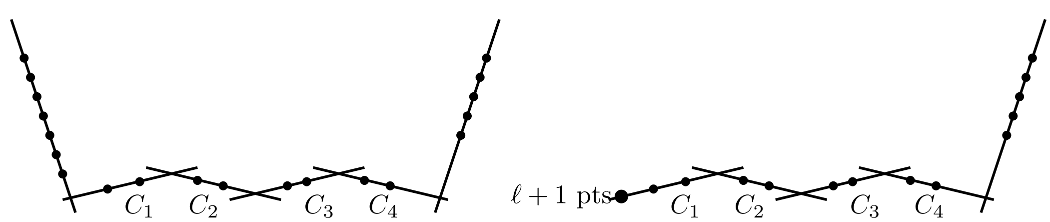

FIGURE 1. Examples of odd chains

Theorem 3.5. *If* $3 \le \ell \le g-1$ *and* ℓ *is even (resp. odd), then the map* τ_ℓ *restricts to an isomorphism away from the locus of odd chains (resp. even chains). If* $(C, x_1, \ldots, x_{2g+2})$ *is an* *odd chain (resp. even chain), then* $\tau_\ell(C, x_1, \ldots, x_{2g+2})$ *is strictly semistable, and its orbit closure contains a curve where the chain* $C_1 \cup \cdots \cup C_k$ *has been replaced by a chain* $D_1 \cup \cdots \cup D_{k+1}$ *with two marked points at each node* $D_i \cap D_{i+1}$ *(see Figure 2).*

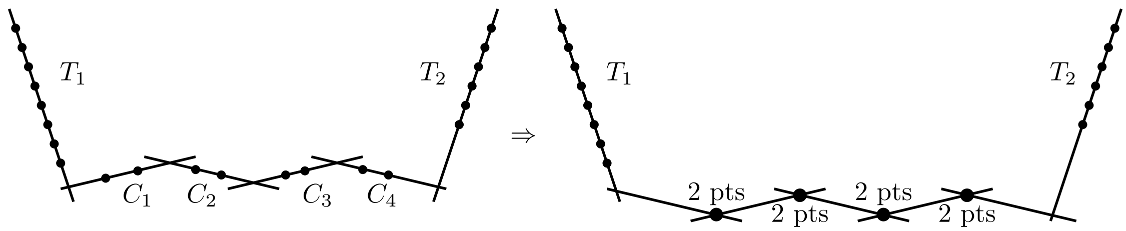

FIGURE 2. The contraction

Proof. Note that both the Hassett space and the GIT quotient are stratified by the topological types of parametrized curves. Furthermore, τ_ℓ is compatible with these stratifications, so τ_ℓ contracts a curve B if and only if

(1) B is in the closure of a stratum,
(2) a general point $(C, x_1, \cdots, x_{2g+2})$ of B has irreducible components $C_1, C_2, \cdots, C_k$ with positive dimensional moduli,
(3) C_i is contracted by the map from C to $\tau_\ell(C, x_1, \cdots, x_{2g+2})$ and
(4) the configurations of points on the irreducible components other than the C_i's are fixed.

A component $C_i \subset C \in \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}}$ has positive dimensional moduli if it has four or more distinct special points. If a tail with k points is contracted, then $k(\frac{1}{\ell+1}) \leq 1$. But then $k(\frac{1}{\ell+1}-\epsilon) < 1$, so such a tail is impossible on $\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}}$. Thus no tail is contracted. Now $\gamma = \frac{\ell-1}{\ell+1} \geq \frac{1}{2}$. By Lemma 1.2, a curve $(D, y_1, \cdots, y_{2g+2}) \in U^{ss}_{g+1-\ell,2g+2}$ has at worst triplenodes if $\ell = 3$ and nodes if $\ell \geq 4$. Moreover, the sum of the weights on triplenodes cannot exceed $1 - 2\gamma = \frac{3-\ell}{\ell+1} \leq 0$, so there are no marked points at a triplenode. Since $1 - \gamma = \frac{2}{\ell+1}$, there are at most two marked points at a node. Therefore, the only possible contracted component is an interior component C_i with two points of attachment and two marked points.

Now, let C_i be such a component. Connected to C_i there are two tails T_1 and T_2 (not necessarily irreducible), with i and $2g - i$ marked points respectively. (Here we will regard a point with $\ell + 1$ marked points, or equivalently, total weight $1 - (\ell + 1)\epsilon$ as a 'tail' of degree 0.) If $i \equiv \ell \bmod 2$, then $\phi(T_1) = \frac{i-\ell-1}{2}$ and $\phi(T_2) = \frac{2g-i-\ell-1}{2}$ (see Definition 1.6), so neither is an integer. Hence the degree of the component C_i is

$$d - (\sigma(T_1) + \sigma(T_2)) \;=\; g + 1 - \ell - (\lceil \frac{i-\ell-1}{2} \rceil + \lceil \frac{2g-i-\ell-1}{2} \rceil) = 1,$$

and thus C_i is not contracted by the map to $\tau_\ell(C, x_1, \cdots, x_{2g+2})$. On the other hand, if $i \equiv \ell + 1 \bmod 2$, then both $\phi(T_1) = \frac{i-\ell-1}{2}$ and $\phi(T_2) = \frac{2g-i-\ell-1}{2}$ are integers. Therefore, this curve lies in the strictly semi-stable locus and the image $\tau_\ell(C, x_1, \cdots, x_{2g+2})$ can be represented by several possible topological types. By [GJM11, Proposition 6.7], the orbit closure of $\tau_\ell(C, x_1, \cdots, x_{2g+2})$ contains a curve in which C_i is replaced by the union of two lines $D_1 \cup D_2$, with two marked points at the node $D_1 \cap D_2$. $\square$

Remark 3.6. Note that τ_ℓ restricts to an isomorphism on the (non-closed) locus of curves consisting of two tails connected by an irreducible bridge with 4 marked points. But, on the locus of curves consisting of two tails connected by a chain of two bridges with two marked points each, τ_ℓ forgets the data of the chain. The map τ_ℓ therefore fails to satisfy axiom (3) of [Smy09, Definition 1.5]. In particular, the Veronese quotient described in Theorem 3.5 is not isomorphic to a modular compactification in the sense of [Smy09].

3.3. **Morphisms between the moduli spaces we have described.**

Corollary 3.7. *If $1 \le \ell \le g-2$, then there is a morphism*

$$\psi_{\ell,\ell+2} : \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}} \to V^{g-1-\ell}_{\frac{\ell+1}{\ell+3},(\frac{1}{\ell+3})^{2g+2}}$$

preserving the interior.

Proof. For $1 \le \ell \le g-3$, we consider the composition

$$\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}} \to \overline{\mathrm{M}}_{0,(\frac{1}{\ell+3}-\epsilon)^{2g+2}} \to V^{g-1-\ell}_{\frac{\ell+1}{\ell+3},(\frac{1}{\ell+3})^{2g+2}},$$

where the first morphism is Hassett's reduction morphism [Has03, Theorem 4.1] and the last morphism is $\tau_{\ell+2}$.

If $\ell = g-2$, then $V^{g-1-\ell}_{\frac{\ell+1}{\ell+3}+\epsilon',(\frac{1}{\ell+3}-\epsilon)^{2g+2}} = (\mathbb{P}^1)^{2g+2} /\!/ \mathrm{SL}(2)$ with symmetric weight datum. Because there is a morphism $\overline{\mathrm{M}}_{0,A} \to (\mathbb{P}^1)^{2g+2} /\!/ \mathrm{SL}(2)$ for any symmetric weight datum A ([Has03, Theorem 8.3]), we obtain $\psi_{g-2,g}$. □

To obtain morphisms between the moduli spaces described in Theorem 3.5, we consider the following diagram.

$$\begin{tikzcd}
 & \overline{\mathrm{M}}_{0,n} \arrow[dl] \arrow[dr] & \\
\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}} \arrow[rr] \arrow[d, "\cong"] \arrow[dd, bend right=60, "\tau_\ell"'] \arrow[ddrr, "\psi_{\ell,\ell+2}"] & & \overline{\mathrm{M}}_{0,(\frac{1}{\ell+3}-\epsilon)^{2g+2}} \arrow[d, "\cong"'] \arrow[dd, bend left=60, "\tau_{\ell+2}"] \\
V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\epsilon',(\frac{1}{\ell+1}-\epsilon)^{2g+2}} \arrow[d] & & V^{g-1-\ell}_{\frac{\ell+1}{\ell+3}+\epsilon',(\frac{1}{\ell+3}-\epsilon)^{2g+2}} \arrow[d] \\
V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}} \arrow[rr, dashed] & & V^{g-1-\ell}_{\frac{\ell+1}{\ell+3},(\frac{1}{\ell+3})^{2g+2}}
\end{tikzcd}$$

Proposition 3.8. *If $3 \le \ell \le g-2$, then the morphism $\psi_{\ell,\ell+2}$ factors through τ_ℓ.*

Proof. By the rigidity lemma ([Kee99, Definition-Lemma 1.0]), it suffices to show that for any curve $B \subset \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\epsilon)^{2g+2}}$ contracted by τ_ℓ, the morphism $\psi_{\ell,\ell+2}$ is constant. We have already described the curves contracted by τ_ℓ in the proof of Theorem 3.5, so it suffices to show that the same curves B are contracted by $\psi_{\ell,\ell+2}$.

When $\ell < g-2$, $i \equiv \ell+1 \bmod 2$ if and only if $i \equiv (\ell+2)+1 \bmod 2$. So the image of B is contracted by

$$\tau_{\ell+2} : \overline{\mathrm{M}}_{0,(\frac{1}{\ell+3}-\epsilon)^{2g+2}} \to V^{g-1-\ell}_{\frac{\ell+1}{\ell+3},(\frac{1}{\ell+3})^{2g+2}}.$$

If $\ell = g-2$, then $\psi_{g-2,g}$ is Hassett's reduction morphism

$$\overline{\mathrm{M}}_{0,(\frac{1}{g-1}-\epsilon)^{2g+2}} \to (\mathbb{P}^1)^{2g+2} /\!/ SL(2).$$

In this case there are two types of odd/even chains (of length 1 or 2). It is straightforward to check that these curves contracted to an isolated singular point of $(\mathbb{P}^1)^{2g+2} /\!/ SL(2)$ parameterizing strictly semi-stable curves. □

4. Higher level conformal block divisors and Veronese quotients

The main goal of this section is to prove Theorem 4.6, which says that when $n = 2g+2$ the divisors $\mathcal{D}_{\gamma,A} = \mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ and $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ determine the same birational models. To prove this, we will show that $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ and $\mathcal{D}_{\gamma,A}$ lie on the same face of the semi-ample cone.

To carry this out, we use a set of S_n-invariant F-curves, given in Definition 2.10, which were shown in [AGSS12, Proposition 4.1] to form a basis for $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$. Using Theorem 2.1, we obtained a simple formula for the intersection of these curves with $\mathcal{D}_{\gamma,A}$ in Corollary 2.2. We then show that $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ is equivalent to a nonnegative combination of the divisors $\{\mathcal{D}_{\frac{\ell+2k-1}{\ell+2k+1},(\frac{1}{\ell+2k+1})^{2g+2}} : k \in \mathbb{Z}_{\geq 0}, \ell+2k \leq g\}$ (Corollary. 4.5). This follows from Proposition 4.4 which shows that the nonzero intersection numbers $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_i$ are nondecreasing.

4.1. Definition of vector bundles of conformal blocks and related notation.

In this section we briefly give the definition of conformal block divisors and explain our notation. The readers can find the details in [Uen08, Chapter 3, 4]. For representation theoretic terminologies and definitions, consult [Hum78].

Let $\mathfrak{g}$ be a simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and positive roots Δ^+. Let $\theta \in \Delta^+$ be the highest root, and let $(\ ,\ )$ be the Killing form normalized so that $(\theta, \theta) = 2$. For a nonnegative integer ℓ, the Weyl alcove P_ℓ is the set of dominant integral weights λ satisfying $(\lambda, \theta) \leq \ell$.

For a collection of data $(\mathfrak{g}, \ell, \vec{\lambda} = (\lambda_1, \cdots, \lambda_n))$, where $\mathfrak{g}$ is a simple Lie algebra, ℓ is a nonnegative integer, and $\vec{\lambda}$ is a collection of weights in P_ℓ, we can construct a conformal block vector bundle $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ as follows.

For a simple Lie algebra $\mathfrak{g}$, we can construct an affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c$$

where c is a central element, with bracket operation

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + c(X, Y)\mathrm{Res}_{z=0}\, g df.$$

For each ℓ and $\lambda \in P_\ell$, there is a unique integrable highest weight $\hat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ where c acts as multiplication by ℓ. Let $\mathcal{H}_{\vec{\lambda}} = \bigotimes_{i=1}^n \mathcal{H}_{\lambda_i}$. There is a natural $\hat{\mathfrak{g}}_n$-action on $\mathcal{H}_{\vec{\lambda}}$ where

$$\hat{\mathfrak{g}}_n = \bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((z_i)) \oplus \mathbb{C}c.$$

Now fix a stable curve $X = (C, p_1, \cdots, p_n) \in \overline{\mathcal{M}}_{g,n}$. Set $U = C - \{p_1, \cdots, p_n\}$. There is a natural map $\mathcal{O}_C(U) \hookrightarrow \bigoplus_{i=1}^n \mathbb{C}((z_i))$. Thus we have a map (indeed it is a Lie algebra homomorphism)

$$\mathfrak{g}(X) = \mathfrak{g} \otimes \mathcal{O}_C(U) \hookrightarrow \bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((z_i)) \oplus \mathbb{C}c = \hat{\mathfrak{g}}_n.$$

The vector space $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})|_X$ of conformal blocks is defined by $\mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(X)\mathcal{H}_{\vec{\lambda}}$. In [Uen08], it is proved that this construction can be sheafified ([Uen08, Theorem 4.4]), and these vector spaces form a vector bundle $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ of finite rank over the moduli stack $\overline{\mathcal{M}}_{g,n}$ ([Uen08, Theorem 4.19]). Finally, a **conformal block divisor** $\mathbb{D}(\mathfrak{g}, \ell, \vec{\lambda})$ is the first Chern class of $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$.

In this paper, we focus on $\mathfrak{g} = \mathfrak{sl}_2$ cases.

4.2. **Intersections of F_i with $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ are nondecreasing.** In this section we prove that the nonzero intersection numbers of $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ with the S_n invariant F-curves F_i are nondecreasing.

We recall some notation from [AGS10].

Definition 4.1. *We define*

$$r_\ell(a_1, \cdots, a_n) := \operatorname{rank} \mathbb{V}(\mathfrak{sl}_2, \ell, (a_1\omega_1, \cdots, a_n\omega_1))$$

and as a special case,

$$r_\ell(k^j, t) := \operatorname{rank} \mathbb{V}(\mathfrak{sl}_2, \ell, (\underbrace{k\omega_1, \cdots, k\omega_1}_{j \text{ times}}, t\omega_1)).$$

For the basic numerical properties of $r_\ell(a_1, \cdots, a_n)$, see [AGS10, Section 3].

Proposition 4.2. *The ranks $r_\ell(1^j, t)$ are determined by the system of recurrences*

$$r_\ell(1^j, t) = r_\ell(1^{j-1}, t-1) + r_\ell(1^{j-1}, t+1), \qquad t = 1, \cdots, \ell. \tag{2}$$

together with seeds

$$r_\ell(1^j, j) = 1, \quad \text{if } j \le \ell, \quad \text{and} \quad r_\ell(1^j, j) = 0, \quad \text{if } j > \ell.$$

Remark. The recurrence (2) is somewhat reminiscent of the recurrence for Pascal's triangle.

Proof. Partition the weight vector $(1, \cdots, 1, t) = 1^j t$ as $1^{j-1} \cup (1, t)$. If $j+t$ is odd, then by the odd sum rule ([AGS10, Proposition 3.5]) $r_\ell(1^j, t) = 0$. So assume $j+t$ is even. Then the factorization formula ([AGS10, Proposition 3.3]) states

$$r_\ell(1^j, t) = \sum_{\mu=0}^{\ell} r_\ell(1^{j-1}, \mu) r_\ell(1, t, \mu). \tag{3}$$

We can simplify this expression. Recall that by the $\mathfrak{sl}_2$ fusion rules ([AGS10, Proposition 3.4]), $r_\ell(1, t, \mu)$ is 0 if $\mu > t+1$ or if $\mu < t-1$. Thus the only possibly nonzero summands in (3) are when $\mu = t-1$, t, or $t+1$. But when $\mu = t$, by the odd sum rule ([AGS10, Proposition 3.5]), we have $r_\ell(1, t, t) = 0$. Thus (3) simplifies to the following:

$$\begin{aligned} r_\ell(1^j, t) &= r_\ell(1^{j-1}, t-1) + r_\ell(1^{j-1}, t+1) \qquad & t = 1, \cdots, \ell-1; \\ r_\ell(1^j, \ell) &= r_\ell(1^{j-1}, \ell-1). \end{aligned}$$

Since $r_\ell(1^{j-1}, \ell+1) = 0$, we can unify the two lines above, yielding (2). □

Lemma 4.3. *Let $i_1 < i_2$ and $j_1 < j_2$. Suppose $i_1 \equiv i_2 \equiv j_1 \equiv j_2 \pmod 2$. Then $r_\ell(1^{i_1}, j_1) r_\ell(1^{i_2}, j_2) - r_\ell(1^{i_1}, j_2) r_\ell(1^{i_2}, j_1) \ge 0$.*

Proof. We prove the result by induction on i_2. For the base case, we can check $(i_1, i_2) = (0, 2)$ and $(i_1, i_2) = (1, 3)$. If $(i_1, i_2) = (0, 2)$ the result is true since $r_\ell(t) = 0$ if $t > 0$. Similarly if $(i_1, i_2) = (1, 3)$ the result is true since $r_\ell(1, t) = 0$ if $t > 1$.

So suppose the result has been established for all quadruples (i_1, i_2, j_1, j_2) with $i_2 \leq k - 1$. We apply the recurrence (2):

$$\begin{aligned}
&r_\ell(1^{i_1}, j_1) r_\ell(1^{i_2}, j_2) - r_\ell(1^{i_1}, j_2) r_\ell(1^{i_2}, j_1) \\
&= \left(r_\ell(1^{i_1-1}, j_1 - 1) + r_\ell(1^{i_1-1}, j_1 + 1)\right)\left(r_\ell(1^{i_2-1}, j_2 - 1) + r_\ell(1^{i_2-1}, j_2 + 1)\right) \\
&\qquad - \left(r_\ell(1^{i_1-1}, j_2 - 1) + r_\ell(1^{i_1-1}, j_2 + 1)\right)\left(r_\ell(1^{i_2-1}, j_1 - 1) + r_\ell(1^{i_2-1}, j_1 + 1)\right) \\
&= r_\ell(1^{i_1-1}, j_1 - 1) r_\ell(1^{i_2-1}, j_2 - 1) - r_\ell(1^{i_1-1}, j_2 - 1) r_\ell(1^{i_2-1}, j_1 - 1) \\
&\qquad + r_\ell(1^{i_1-1}, j_1 - 1) r_\ell(1^{i_2-1}, j_2 + 1) - r_\ell(1^{i_1-1}, j_2 + 1) r_\ell(1^{i_2-1}, j_1 - 1) \\
&\qquad + r_\ell(1^{i_1-1}, j_1 + 1) r_\ell(1^{i_2-1}, j_2 - 1) - r_\ell(1^{i_1-1}, j_2 - 1) r_\ell(1^{i_2-1}, j_1 + 1) \\
&\qquad + r_\ell(1^{i_1-1}, j_1 + 1) r_\ell(1^{i_2-1}, j_2 + 1) - r_\ell(1^{i_1-1}, j_2 + 1) r_\ell(1^{i_2-1}, j_1 + 1).
\end{aligned}$$

By the induction hypothesis, each of the last four lines is nonnegative. □

Proposition 4.4. *Suppose* $\ell \leq i \leq g - 2$ *and* $i \equiv \ell \pmod 2$. *Then* $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_i \leq \mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_{i+2}$. *If, on the other hand,* $i \equiv \ell + 1 \pmod 2$, *then* $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_i = 0$.

Proof. By [AGS10, Theorem 4.2] we have $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_i = r_\ell(1^i, \ell) r_\ell(1^{n-i-2}, \ell)$. By the odd sum rule ([AGS10, Proposition 3.5]) we see that $r_\ell(1^i, \ell) = 0$ if $i \equiv \ell + 1 \pmod 2$.

In the remaining cases, we seek to show that

$$r_\ell(1^{i+2}, \ell) r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell) r_\ell(1^{n-i-2}, \ell) \geq 0.$$

We apply the recurrence (2) and use $r_\ell(1^j, t) = 0$ if $t > \ell$ to obtain

$$\begin{aligned}
&r_\ell(1^{i+2}, \ell) r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell) r_\ell(1^{n-i-2}, \ell) \\
&= \left(r_\ell(1^{i+1}, \ell - 1) + r_\ell(1^{i+1}, \ell + 1)\right) r_\ell(1^{n-i-4}, \ell) \\
&\qquad - r_\ell(1^i, \ell)\left(r_\ell(1^{n-i-3}, \ell - 1) + r_\ell(1^{n-i-3}, \ell + 1)\right) \\
&= r_\ell(1^{i+1}, \ell - 1) r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell) r_\ell(1^{n-i-3}, \ell - 1) \\
&= \left(r_\ell(1^i, \ell - 2) + r_\ell(1^i, \ell)\right) r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell)\left(r_\ell(1^{n-i-4}, \ell - 2) + r_\ell(1^{n-i-2}, \ell)\right) \\
&= r_\ell(1^i, \ell - 2) r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell) r_\ell(1^{n-i-4}, \ell - 2).
\end{aligned}$$

By Lemma 4.3, we have $r_\ell(1^i, \ell - 2) r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell) r_\ell(1^{n-i-4}, \ell - 2) \geq 0$. □

Corollary 4.5. *The divisor* $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ *is a nonnegative linear combination of the divisors* $\{\mathcal{D}_{\frac{\ell+2k-1}{\ell+2k+1}, (\frac{1}{\ell+2k+1})^{2g+2}} : k \in \mathbb{Z}_{\geq 0}, \ell + 2k \leq g\}$. *Moreover, the coefficient of* $\mathcal{D}_{\frac{\ell-1}{\ell+1}, (\frac{1}{\ell+1})^{2g+2}}$ *in this expression is strictly positive.*

Proof. This follows from Proposition 4.4 and the intersection numbers computed in Corollary 2.2. □

4.3. **Morphisms associated to conformal block divisors.** We are now in a position where we can prove that the divisors $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ give maps to Veronese quotients.

Theorem 4.6. *The conformal block divisor $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ on $\overline{\mathrm{M}}_{0,2g+2}$ for $1 \le \ell \le g$ is the pullback of an ample class via the morphism*

$$\varphi_{\frac{\ell-1}{\ell+1},A} = \phi_{\frac{\ell-1}{\ell+1}} \circ \rho_A : \overline{\mathrm{M}}_{0,n} \xrightarrow{\rho_A} \overline{\mathrm{M}}_{0,A} \xrightarrow{\phi_{\frac{\ell-1}{\ell+1}}} V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},A},$$

where $A = (\frac{1}{\ell+1})^{2g+2}$, and ρ_A is the contraction to Hassett's moduli space $\overline{\mathrm{M}}_{0,A}$ of stable weighted pointed rational curves.

Proof. By [AGS10, Corollary 4.7 and 4.9] and Corollary 2.2, $\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}} \equiv \mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ if $\ell = 1, 2$. When $\ell \ge 3$, by Corollary 4.5, $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ is a non-negative linear combination of $\mathcal{D}_{\frac{\ell+2k-1}{\ell+2k+1},(\frac{1}{\ell+2k+1})^{2g+2}}$ where $k \in \mathbb{Z}_{\ge 0}$ and $\ell + 2k \le g$. In the latter case, by Proposition 3.8, we see that all of the divisors in this non-negative linear combination are pullbacks of semi-ample divisors from $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},A}$. Moreover, one of them is ample, and it appears with strictly positive coefficient. The result follows. $\square$

Remark 4.7. If n is odd, then all $\mathbb{D}(\mathfrak{sl}_2, \ell, (\omega_1, \cdots, \omega_1))$ is trivial ([Fak12, Lemma 4.1]). So it suffices to consider $n = 2g + 2$ cases.

We note that, for a sequence of dominant integral weights $(k_1\omega_1, \cdots, k_n\omega_1)$ of $\mathfrak{sl}_2$, the integer $(\sum_{i=1}^n \frac{k_i}{2}) - 1$ is called the **critical level** $c\ell$. By [Fak12, Lemma 4.1], if ℓ is strictly greater than the critical level, then $\mathbb{D}(\mathfrak{sl}_2, \ell, (k_1\omega_1, \cdots, k_n\omega_1)) \equiv 0$, so the corresponding morphism is a constant map.

When $k_1 = \cdots = k_n = 1$, the critical level is equal to g. Thus it is sufficient to study the cases $1 \le \ell \le g$. Therefore Theorem 4.6 is a complete answer for the Lie algebra $\mathfrak{sl}_2$ and weight data ω_1^n cases.

5. Conjectural generalizations

Numerical evidence suggests that the connection between Veronese quotients and $\mathfrak{sl}_r$ conformal block divisors holds in a more general setting. In this section, we provide some of this evidence and make a few conjectures.

5.1. **$\mathfrak{sl}_2$ cases.** In this section, we consider $\mathfrak{sl}_2$ symmetric weight cases, i.e, $\mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$ for $1 \le k \le \ell$. Theorem 4.6 tells us that when $k = 1$, the associated birational models are Veronese quotients. Before we can predict the birational models associated to other conformal block divisors, we need the following useful lemma.

Lemma 5.1. [RW02, Formula (17)] *The rank $r_\ell(a_1, \cdots, a_n) \ne 0$ if and only if $\Lambda = \sum_{i=1}^n a_i$ is even and, for any subset $I \subset \{1, \dots n\}$ with $n - |I|$ odd, we have*

$$\Lambda - (n-1)\ell \le \sum_{i \in I} (2a_i - \ell).$$

Note that for a given weight datum, the left-hand side of this expression is fixed, while the right-hand side is minimized by summing over all weights such that $2a_i < \ell$.

The next result shows that when $k = \ell$, we get the same birational model as in the case of $k = 1$.

Proposition 5.2. *We have the following equalities between conformal block divisors:*

$$\mathbb{D}(\mathfrak{sl}_2, \ell, \ell\omega_1^n) = \ell\mathbb{D}(\mathfrak{sl}_2, 1, \omega_1^n) = \frac{\ell}{k}\mathbb{D}(\mathfrak{sl}_{2k}, 1, \omega_k^n).$$

Proof. The second assertion is a direct application of [GG12, Proposition 5.1], which says that

$$\mathbb{D}(\mathfrak{sl}_r, 1, (\omega_{z_1}, \cdots, \omega_{z_n})) = \frac{1}{k}\mathbb{D}(\mathfrak{sl}_{rk}, 1, (\omega_{kz_1}, \cdots, \omega_{kz_n})).$$

For the first assertion, let $\mathbb{D} = \mathbb{D}(\mathfrak{sl}_2, \ell, \ell\omega_1^n)$. It suffices to consider intersection numbers of $\mathbb{D}$ with F-curves of the form $F_i = F_{n-i-2,i,1,1}$. Then

$$\mathbb{D} \cdot F_{i_1,i_2,i_3,i_4} = \sum_{\vec{u} \in P_\ell^4} \deg(\mathbb{V}(\mathfrak{sl}_2, \ell, (u_1\omega_1, u_2\omega_1, u_3\omega_1, u_4\omega_1)) \prod_{k=1}^{4} r_\ell(\ell^{i_k}, t),$$

where $P_\ell = \{0, 1, \cdots, \ell\}$. In the case where $i_3 = i_4 = 1$, we may use the two-point fusion rule for $\mathfrak{sl}_2$ to obtain:

$$\mathbb{D} \cdot F_i = \sum_{0 \le u_1, u_2 \le \ell} \deg(\mathbb{V}(\mathfrak{sl}_2, \ell, (u_1\omega_1, u_2\omega_1, \ell\omega_1, \ell\omega_1)) r_\ell(\ell^{n-i-2}, u_1) r_\ell(\ell^i, u_2).$$

By the case $I = \{n\}$ if n is even and $I = \emptyset$ if n is odd in Lemma 5.1, we see that $r_\ell(\ell^j, t) = 0$ if $0 < t < \ell$. Hence

$$\mathbb{D} \cdot F_i = \sum_{u_1, u_2 = 0, \ell} \deg(\mathbb{V}(\mathfrak{sl}_2, \ell, (u_1\omega_1, u_2\omega_1, \ell\omega_1, \ell\omega_1)) r_\ell(\ell^{n-i-2}, u_1) r_\ell(\ell^i, u_2).$$

But by [Fak12, Proposition 4.2], $\deg(\mathbb{V}(\mathfrak{sl}_2, \ell, (0, 0, \ell\omega_1, \ell\omega_1))) = \deg(\mathbb{V}(\mathfrak{sl}_2, \ell, (0, \ell\omega_1, \ell\omega_1, \ell\omega_1))) = 0$ and $\deg(\mathbb{V}(\mathfrak{sl}_2, \ell, (\ell\omega_1, \ell\omega_1, \ell\omega_1, \ell\omega_1))) = \ell$. Thus

$$\mathbb{D} \cdot F_i = \ell r_\ell(\ell^{n-i-2}, \ell) r_\ell(\ell^i, \ell).$$

It therefore suffices to show that $r_\ell(\ell^t, \ell) = r_\ell(1^t, 1)$, but this follows by induction via the factorization rules and the propagation of vacua. □

For the majority of values of k such that $1 < k < \ell$, the divisor $\mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$ appears to give a map to a Hassett space. To establish our evidence for this, we first start with a lemma.

Lemma 5.3. *Suppose that $1 < k < \ell$. Then $r_\ell(k^i, t) = 0$ if and only if either $ki + t$ is odd or one of the following holds:*

(1) $2k \le \ell$ and $i < \max\{\frac{t}{k}, 2 - \frac{t}{k}\}$;
(2) $2k > \ell$, i is even, and $i < \max\{\frac{t}{\ell-k}, 2 - \frac{t}{\ell-k}\}$;
(3) $2k > \ell$, i is odd, and $i < \max\{\frac{\ell-t}{\ell-k}, 2 - \frac{\ell-t}{\ell-k}\}$.

Proof. Each of these follows from case by case analysis of Lemma 5.1 above and the remark that follows it. □

We now consider which S_n-invariant F-curves have trivial intersection with the divisors in question.

Proposition 5.4. *Suppose that $1 < k < \frac{3}{4}\ell$ and let $\mathbb{D} = \mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$. Assume that n is even and $\ell \le \frac{kn}{2} - 1$. (Recall that this is necessary for the non-triviality of $\mathbb{D}$ by remark 4.7.) If $a \le b \le c \le d$, then $\mathbb{D} \cdot F_{a,b,c,,d} = 0$ if and only if $a + b + c \le \frac{\ell+1}{k}$.*

Proof. By [Fak12, Proposition 4.7], the map associated to $\mathbb{D}$ factors through the map $\overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,(\frac{k}{\ell+1})^n}$. It follows that, if $a+b+c \le \frac{\ell+1}{k}$, then $\mathbb{D}\cdot F_{a,b,c,d} = 0$. It therefore suffices to show the converse. We assume throughout that $a+b+c > \frac{\ell+1}{k}$.

By [Fak12, Proposition 2.7], we have

$$\mathbb{D}\cdot F_{a,b,c,d} = \sum_{\vec{u}\in P_\ell^4} \deg(\mathbb{V}(\mathfrak{sl}_2,\ell,(u_1\omega_1,u_2\omega_1,u_3\omega_1,u_4\omega_1))r_\ell(k^a,u_1)r_\ell(k^b,u_2)r_\ell(k^c,u_3)r_\ell(k^d,u_4).$$

Since each term in the sum above is nonnegative, it suffices to show that a single term is nonzero.

We first consider the case that $2k \le \ell$. We set

$$w_a = \begin{cases} \min\{ka,\ell\} & \text{if } ka \equiv \ell \pmod 2, \\ \min\{ka,\ell-1\} & \text{if } ka \not\equiv \ell \pmod 2. \end{cases}$$

Note that by assumption, both $k(a+b+c+d) = kn$ and $k(a+b+c)+\ell$ are strictly greater than $2\ell+1$. So it is straightforward to check that $w_a+w_b+w_c+w_d > 2\ell$ and $\ell+1 > w_d$. Note further that $2\ell+2+2w_a > 2w_a+w_c+w_d$ and $2w_a \ge 4$. It follows that there is an integer w_b' such that $w_b' \equiv w_b \pmod 2$, $2\ell < w_a+w_b'+w_c+w_d < 2\ell+2+2w_a$. Then $w_a+w_b'+w_c+w_d \equiv w_a+w_b+w_c+w_d \equiv k(a+b+c+d) \equiv 0 \pmod 2$. Thus $\deg(\mathbb{V}(\mathfrak{sl}_2,\ell,(w_a\omega_1,w_b'\omega_1,w_c\omega_1,w_d\omega_1))) \neq 0$ by [Fak12, Proposition 4.2]. It therefore suffices to show that $r_\ell(k^a,w_a) \neq 0$. But in this case, Lemma 5.3 tells us that $r_\ell(k^a,w_a)=0$ only if $a < \max\{\frac{w_a}{k}, 2-\frac{w_a}{k}\} \le \max\{a,2\}$, which is possible only if $a=1$. But then $w_a = k$ so $r_\ell(k^a,w_a) = r_\ell(k,k) = 1 \neq 0$ by the two point fusion rule. Therefore it is always nonzero.

We next consider the case that $2k > \ell$. Note that in this case, $\frac{\ell+1}{k} < 3$, so we must show that no F-curves are contracted. We set

$$w_a = \begin{cases} k & \text{if } a \text{ is odd}, \\ 2(\ell-k) & \text{if } a \text{ is even}. \end{cases}$$

Again we have that $\ell+1 > \max\{w_a,w_b,w_c,w_d\}$, $2\ell < w_a+w_b+w_c+w_d < 2\ell+2+2\min\{w_a,w_b,w_c,w_d\}$, and $w_a+w_b+w_c+w_d \equiv 0 \pmod 2$. Thus $\deg(\mathbb{V}(\mathfrak{sl}_2,\ell,(w_a\omega_1,w_b\omega_1,w_c\omega_1,w_d\omega_1))) \neq 0$ by [Fak12, Proposition 4.2]. If a is odd, we see that $r_\ell(k^a,w_a)=0$ if and only if $a<1$. If a is even, we see that $r_\ell(k^a,w_a) = 0$ only if $a < 2$. It follows that no F-curves are contracted. $\square$

By Proposition 5.4, we see that the F-curves that have trivial intersection with $\mathbb{D}(\mathfrak{sl}_2,\ell,k\omega_1^n)$ are precisely those that are contracted by the morphism $\rho_{(\frac{k}{\ell+1})^n} : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,(\frac{k}{\ell+1})^n}$. At the present time, this is not sufficient to conclude that $\mathbb{D}(\mathfrak{sl}_2,\ell,k\omega_1^n)$ is in fact the pullback of an ample divisor from this Hassett space, although this would follow from a well-known conjecture (see [KM96, Question 1.1]).

Theorem 5.5. *Assume that n is even, $1 < k < \frac{3}{4}\ell$, and $\ell \le \frac{kn}{2}-1$. If the* F*-Conjecture holds (see* [KM96, Question 1.1]*), then the divisor $\mathbb{D}(\mathfrak{sl}_2,\ell,k\omega_1^n)$ is the pullback of an ample class via the morphism $\rho_{(\frac{k}{\ell+1})^n} : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,(\frac{k}{\ell+1})^n}$. In particular, if $\frac{\ell+1}{3} < k < \frac{3}{4}\ell$, then $\mathbb{D}$ is ample.*

We note further that Proposition 5.4 does not cover all of the possible cases of symmetric-weight $\mathfrak{sl}_2$ conformal block divisors. In particular, if $k \ge \frac{3}{4}\ell$, then $\mathbb{D}(\mathfrak{sl}_2,\ell,k\omega_1^n)$ may in fact have trivial intersection with an F-curve if all of the legs contain an even number of marked points. Such is the case, for example, of the divisor $\mathbb{D}(\mathfrak{sl}_2,4,3\omega_1^8)$. This divisor has zero intersection with the F-curve $F(2,2,2,2)$ and positive intersection with every other F-curve. It is not difficult to see that the associated birational model is the Kontsevich-Boggi compactification of $\mathrm{M}_{0,8}$ (see [GJM11, Section 7.2] for details on this moduli space).

5.2. **Birational properties of $\mathfrak{sl}_r$ conformal blocks.** We note that in every known case the birational model associated to conformal block divisors is in fact a compactification of $\mathrm{M}_{0,n}$. In other words, the associated morphism restricts to an isomorphism on the interior. We pose this as a conjecture.

Conjecture 5.6. *Let $\mathbb{D}$ be a non-trivial conformal block divisor of $\mathfrak{sl}_r$ with strictly positive weights. Then $\mathbb{D}$ separates all points on $\mathrm{M}_{0,n}$. More precisely, for any two distinct points $x_1, x_2 \in \mathrm{M}_{0,n}$, the morphism*

$$H^0(\overline{\mathrm{M}}_{0,n}, \mathbb{D}) \to \mathbb{D}|_{x_1} \oplus \mathbb{D}|_{x_2}$$

is surjective.

If true, this conjecture would have several interesting consequences. Among them is the following simple description of the maps associated to conformal block divisors. Let $\mathbb{D}$ be a conformal block divisor of $\mathfrak{sl}_r$ and $\rho_{\mathbb{D}} : \overline{\mathrm{M}}_{0,n} \to X$ the associated morphism. Consider a boundary stratum

$$\prod_{i=1}^{m} \mathrm{M}_{0,k_i} \hookrightarrow \overline{\mathrm{M}}_{0,n}.$$

By factorization [Fak12, Proposition 2.4], the pullback of a $\mathfrak{sl}_r$ conformal block divisor to $\overline{\mathrm{M}}_{0,k_m}$ and its interior M_{0,k_m} is an effective sum of $\mathfrak{sl}_r$ conformal block divisors. If all of the divisors in this sum are trivial, then the restriction of $\rho_{\mathbb{D}}$ to this boundary stratum forgets a component of the curve:

$$\begin{tikzcd} \prod_{i=1}^{m} \mathrm{M}_{0,k_i} \arrow[r] \arrow[d] & \overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_{\mathbb{D}}"] \\ \prod_{i=1}^{m-1} \mathrm{M}_{0,k_i} \arrow[r] & X. \end{tikzcd}$$

If the only non-trivial divisors in this sum have weight zero on some subset of the attaching points, then these divisors are pullbacks of non-trivial conformal block divisors via the map that forgets these points. Hence, the restriction of $\rho_{\mathbb{D}}$ to this boundary stratum forgets these attaching points:

$$\begin{tikzcd} \mathrm{M}_{0,k_i} \arrow[r] \arrow[d] & \overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_{\mathbb{D}}"] \\ \mathrm{M}_{0,k_i-j} \arrow[r] & X. \end{tikzcd}$$

Finally, if any of the non-trivial conformal block divisors in this sum has strictly positive weights, then by Conjecture 5.6, the restriction of $\rho_{\mathbb{D}}$ to the interior of this stratum is an isomorphism:

$$\begin{tikzcd} \mathrm{M}_{0,k_i} \arrow[r] \arrow[d, "\cong"] & \overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_{\mathbb{D}}"] \\ \mathrm{M}_{0,k_i} \arrow[r] & X. \end{tikzcd}$$

In summary, Conjecture 5.6 implies that the image of a boundary stratum $\prod_{i=1}^{m} \mathrm{M}_{0,k_i}$ in X is isomorphic to

$$\prod_{i=1}^{a} \mathrm{M}_{0,k_i} \times \prod_{i=a+1}^{b} \mathrm{M}_{0,k_i-j_i}$$

for some $1 \leq a \leq b \leq n$ and $1 \leq j_i \leq k_i - 3$.

In this way, the morphisms associated to $\mathfrak{sl}_r$ conformal blocks are somewhat reminiscent of Smyth's modular compactifications (see [Smy09]). Each of Smyth's compactifications can be described by assigning, to each boundary stratum, a collection of "forgotten" components. In a similar way, the morphism $\rho_{\mathbb{D}}$ appears to assign to each boundary stratum a collection of forgotten components and forgotten points of attachment. It follows that, if Conjecture 5.6 holds, one can understand the morphism $\rho_{\mathbb{D}}$ completely from such combinatorial data.

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