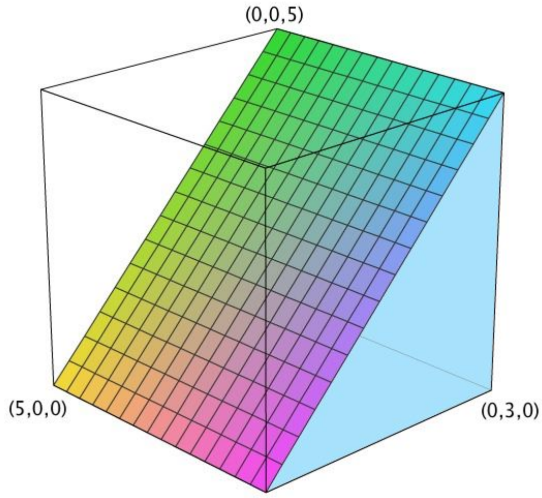
Homework 10 Model Solution

Section 15.1 $\sim$ 15.2.

15.1.12 Evaluate the double integral by first identifying it as the volume of a solid.

$$\iint_R (5-x)\, dA, \quad R = \{(x,y) \mid 0 \le x \le 5, 0 \le y \le 3\}$$



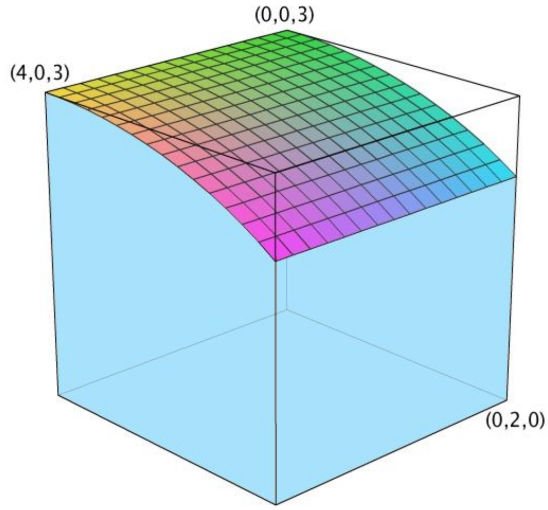
The solid over R bounded above by the graph of $z = 5-x$ is a triangular cylinder, whose base is an isosceles right triangle whose two sides are 5. The height of the cylinder is 3. So

$$\iint_R (5-x)\, dA = \text{ volume } = \text{ area of base } \times \text{ height} = \frac{1}{2}5^2 \cdot 3 = \frac{75}{2}.$$

15.1.14 The integral $\iint_R \sqrt{9-y^2}\, dA$, where $R = [0,4] \times [0,2]$, represents the volume of a solid. Sketch the solid.

$$z = \sqrt{9-y^2} \Rightarrow z^2 + y^2 = 9, z \ge 0$$

So if we fix $x = k$, on the plane the graph is an arc which is a part of a circle whose radius is 3. Therefore the graph of $\sqrt{9-y^2}$ is a part of a circular cylinder whose base is a planar region bounded above by a circle and bounded below by the diameter of the circle.



15.1.17 If f is a constant function, $f(x, y) = k$, and $R = [a, b] \times [c, d]$, show that

$$\iint_R k \, dA = k(b - a)(d - c).$$

If $f(x, y) = k$, the solid over R bounded by xy-plane and $z = f(x, y) = k$ is a rectangular parallelepiped whose three sides are $b - a$, $d - c$, and k. Therefore

$$\iint_R k \, dA = k(b - a)(d - c).$$

15.2.4 Calculate the iterated integral

$$\int_0^1 \int_1^2 (4x^3 - 9x^2y^2) \, dy \, dx.$$

$$\begin{aligned}
\int_0^1 \int_1^2 (4x^3 - 9x^2y^2) \, dy \, dx &= \int_0^1 \left[4x^3y - 3x^2y^3\right]_1^2 \, dx \\
&= \int_0^1 \left(4x^3 \cdot 2 - 3x^2 \cdot 2^3\right) - \left(4x^3 \cdot 1 - 3x^2 \cdot 1^3\right) \, dx \\
&= \int_0^1 4x^3 - 21x^2 \, dx \\
&= \left[x^4 - 7x^3\right]_0^1 = (1^4 - 7 \cdot 1^3) - 0 = -6
\end{aligned}$$

15.2.8 Calculate the iterated integral

$$\int_1^3 \int_1^5 \frac{\ln y}{xy} \, dy \, dx.$$

$$\begin{aligned}
\int_1^3\int_1^5 \frac{\ln y}{xy}\,dy\,dx &= \int_1^3 \left[\frac{1}{2x}(\ln y)^2\right]_1^5 dx\\
&= \int_1^3 \left(\frac{1}{2x}(\ln 5)^2\right) - \left(\frac{1}{2x}(\ln 1)^2\right) dx\\
&= \int_1^3 \frac{(\ln 5)^2}{2x}\,dx = \left[\frac{(\ln 5)^2}{2}\ln x\right]_1^3\\
&= \left(\frac{(\ln 5)^2}{2}\ln 3\right) - \left(\frac{(\ln 5)^2}{2}\ln 1\right) = \frac{(\ln 5)^2\ln 3}{2}
\end{aligned}$$

15.2.17 Calculate the double integral

$$\iint_R \frac{xy^2}{x^2+1}\,dA, \quad R = \{(x,y) | 0 \le x \le 1, -3 \le y \le 3\}.$$

$$\begin{aligned}
\iint_R \frac{xy^2}{x^2+1}\,dA &= \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \left[\frac{xy^3}{3(x^2+1)}\right]_{-3}^3 dx\\
&= \int_0^1 \frac{3^3x}{3(x^2+1)} - \frac{(-3)^3x}{3(x^2+1)}\,dx = \int_0^1 \frac{18x}{x^2+1}\,dx\\
&= \left[9\ln(x^2+1)\right]_0^1 = 9\ln 2 - 9\ln 1 = 9\ln 2
\end{aligned}$$

15.2.20 Calculate the double integral

$$\iint_R \frac{x}{1+xy}\,dA, \quad R = [0,1]\times[0,1].$$

$$\begin{aligned}
\iint_R \frac{x}{1+xy}\,dA &= \int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 [\ln(1+xy)]_0^1\,dx\\
&= \int_0^1 \ln(1+x) - \ln 1\,dx = \int_0^1 \ln(1+x)\,dx\\
&= \int_1^2 \ln u\,du \text{ (Use } u = 1+x \text{ substitution)}\\
&= [(\ln u)u]_1^2 - \int_1^2 \frac{1}{u}\cdot u\,du \text{ (integration by parts with } f = \ln u,\ g' = 1)\\
&= 2\ln 2 - \ln 1 - \int_1^2 1\,du = 2\ln 2 - [u]_1^2\\
&= 2\ln 2 - (2-1) = 2\ln 2 - 1
\end{aligned}$$

15.2.21 Calculate the double integral

$$\iint_R ye^{-xy}\,dA, \quad R = [0,2]\times[0,3].$$

$$\begin{aligned}
\iint_R ye^{-xy}\, dA &= \int_0^3 \int_0^2 ye^{-xy}\, dx\, dy = \int_0^3 \left[-e^{-xy}\right]_0^2\, dy \\
&= \int_0^3 -e^{-2y} - \left(-e^0\right)\, dy = \int_0^3 -e^{-2y} + 1\, dy \\
&= \left[\frac{1}{2}e^{-2y} + y\right]_0^3 \\
&= \left(\frac{1}{2}e^{-6} + 3\right) - \left(\frac{1}{2}e^0 + 0\right) = \frac{1}{2}e^{-6} + \frac{5}{2}
\end{aligned}$$

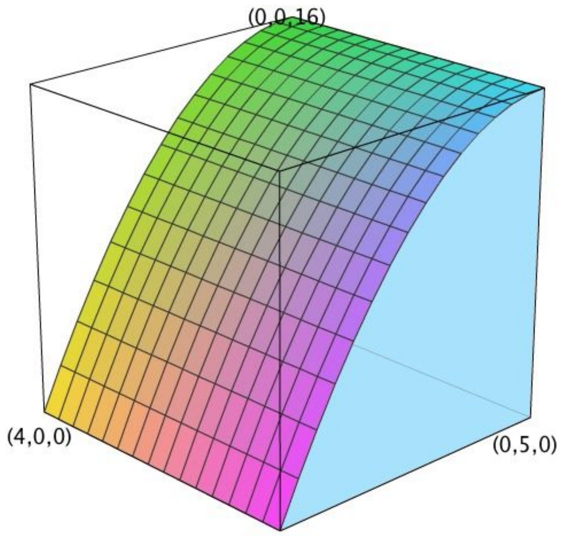
15.2.26 Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

The volume is the integral of $f(x, y) = 3y^2 - x^2 + 2$ over R.

$$\begin{aligned}
\text{volume} &= \iint_R 3y^2 - x^2 + 2\, dA = \int_{-1}^1 \int_1^2 3y^2 - x^2 + 2\, dy\, dx \\
&= \int_{-1}^1 \left[y^3 - x^2y + 2y\right]_1^2\, dx \\
&= \int_{-1}^1 \left(2^3 - x^2 \cdot 2 + 2 \cdot 2\right) - \left(1^3 - x^2 \cdot 1 + 2 \cdot 1\right)\, dx \\
&= \int_{-1}^1 9 - x^2\, dx = \left[9x - \frac{x^3}{3}\right]_{-1}^1 \\
&= \left(9 \cdot 1 - \frac{1^3}{3}\right) - \left(9 \cdot (-1) - \frac{(-1)^3}{3}\right) \\
&= 18 - \frac{2}{3} = \frac{52}{3}
\end{aligned}$$

15.2.30 Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

The first octant is bounded by three coordinate planes $x = 0$, $y = 0$, and $z = 0$. The plane $z = 0$ intersects $z = 16 - x^2$ along $x^2 - 16 = 0$, which is $x = \pm 4$. Because the solid is on the first octant $x \geq 0$ and $z = 16 - x^2$ intersects $z = 0$ at $x = 4$. The solid is over $R = [0, 4] \times [0, 5]$ and under $z = 16 - x^2$.



$$\begin{aligned}
\text{volume} &= \iint_R 16 - x^2\, dA = \int_0^4 \int_0^5 16 - x^2\, dy\, dx \\
&= \int_0^4 \left[16y - x^2 y\right]_0^5\, dx \\
&= \int_0^4 80 - 5x^2\, dx = \left[80x - \frac{5}{3}x^3\right]_0^4 \\
&= 320 - \frac{5}{3}\cdot 4^3 - 0 = \frac{640}{3}
\end{aligned}$$

15.2.35 Find the average value of $f(x, y) = x^2 y$ over the rectangle R whose vertices are $(-1, 0)$, $(-1, 5)$, $(1, 5)$, and $(1, 0)$.

$R = [-1, 1] \times [0, 5]$

$$\begin{aligned}
\iint_R x^2 y\, dA &= \int_{-1}^1 \int_0^5 x^2 y\, dy\, dx = \int_{-1}^1 \left[x^2 \frac{y^2}{2}\right]_0^5\, dx \\
&= \int_{-1}^1 \frac{25}{2}x^2\, dx = \left[\frac{25}{6}x^3\right]_{-1}^1 \\
&= \frac{25}{6}\cdot 1^3 - \frac{25}{6}(-1)^3 = \frac{25}{3}
\end{aligned}$$

$$\text{area of } R = (1 - (-1)) \cdot (5 - 0) = 10$$

$$\text{average} = \frac{1}{\text{area of } R} \iint_R x^2 y\, dA = \frac{1}{10} \cdot \frac{25}{3} = \frac{25}{30} = \frac{5}{6}$$