Homework 11 Model Solution

Section 15.3.

15.3.1 Evaluate the iterated integral

$$\int_0^4 \int_0^{\sqrt{y}} xy^2 \, dx \, dy.$$

$$\begin{aligned}
\int_0^4 \int_0^{\sqrt{y}} xy^2 \, dx \, dy &= \int_0^4 \left[\frac{x^2y^2}{2}\right]_0^{\sqrt{y}} dy = \int_0^4 \frac{(\sqrt{y})^2y^2}{2} - \frac{0^2y^2}{2} \, dy \\
&= \int_0^4 \frac{y^3}{2} \, dy = \left[\frac{y^4}{8}\right]_0^4 = 32
\end{aligned}$$

15.3.8 Evaluate the double integral

$$\iint_D \frac{y}{x^5+1} \, dA, \quad D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\}.$$

$$\begin{aligned}
\iint_D \frac{y}{x^5+1} \, dA &= \int_0^1 \int_0^{x^2} \frac{y}{x^5+1} \, dy \, dx = \int_0^1 \left[\frac{y^2}{2(x^5+1)}\right]_0^{x^2} dx \\
&= \int_0^1 \frac{(x^2)^2}{2(x^5+1)} - \frac{0^2}{2(x^5+1)} \, dx = \int_0^1 \frac{x^4}{2(x^5+1)} \, dx \\
&= \int_1^2 \frac{1}{10u} \, du \text{ (using substitution } u = x^5+1) \\
&= \left[\frac{1}{10} \ln u\right]_1^2 = \frac{1}{10}\ln 2 - \frac{1}{10}\ln 1 = \frac{\ln 2}{10}
\end{aligned}$$

15.3.18 Evaluate the double integral

$$\iint_D (x^2+2y) \, dA, \quad D \text{ is bounded by } y = x, y = x^3, x \ge 0.$$

$$x = x^3, x \ge 0 \Leftrightarrow x = 0, 1$$

Also if $0 \le x \le 1$, $x \ge x^3$.

$$\begin{aligned}
\iint_D (x^2+2y) \, dA &= \int_0^1 \int_{x^3}^{x} x^2 + 2y \, dy \, dx = \int_0^1 \left[x^2y + y^2\right]_{x^3}^{x} dx \\
&= \int_0^1 \left(x^2 \cdot x + x^2\right) - \left(x^2 \cdot x^3 + (x^3)^2\right) dx \\
&= \int_0^1 x^3 + x^2 - x^6 - x^5 \, dx \\
&= \left[\frac{x^4}{4} + \frac{x^3}{3} - \frac{x^7}{7} - \frac{x^6}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{1}{7} - \frac{1}{6} = \frac{23}{84}
\end{aligned}$$

15.3.19 Evaluate the double integral

$$\iint_D y^2\, dA, \quad D \text{ is the triangular region with vertices } (0,1), (1,2), (4,1).$$

$$(x, y) \in D \Leftrightarrow 1 \le y \le 2,\ y - 1 \le x \le -3y + 7$$

$$\begin{aligned}
\iint_D y^2\, dA &= \int_1^2 \int_{y-1}^{-3y+7} y^2\, dx\, dy = \int_1^2 \left[xy^2\right]_{y-1}^{-3y+7} dy \\
&= \int_1^2 (-3y+7)y^2 - (y-1)y^2\, dy = \int_1^2 -4y^3 + 8y^2\, dy \\
&= \left[-y^4 + \frac{8}{3}y^3\right]_1^2 = \left(-2^4 + \frac{8}{3}\cdot 2^3\right) - \left(-1 + \frac{8}{3}\right) = \frac{11}{3}
\end{aligned}$$

15.3.24 Find the volume of the given solid under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$.

$$(x, y) \text{ is in the region } \Leftrightarrow -2 \le y \le 2,\ y^2 \le x \le 4$$

$$\begin{aligned}
\text{volume} &= \int_{-2}^2 \int_{y^2}^4 1 + x^2y^2\, dx\, dy = \int_{-2}^2 \left[x + \frac{x^3}{3}y^2\right]_{y^2}^4 dy \\
&= \int_{-2}^2 \left(4 + \frac{64}{3}y^2\right) - \left(y^2 + \frac{(y^2)^3}{3}y^2\right) dy = \int_{-2}^2 -\frac{y^8}{3} + \frac{61}{3}y^2 + 4\, dy \\
&= \left[-\frac{y^9}{27} + \frac{61}{9}y^3 + 4y\right]_{-2}^2 = \frac{2336}{27}
\end{aligned}$$

15.3.27 Find the volume of the given solid bounded by the coordinate planes and the plane $3x + 2y + z = 6$.

The solid is a tetrahedron over a triangle T bounded by $x = 0$, $y = 0$, and $3x+2y = 6$ under $z = 6 - 3x - 2y$.

$$(x, y) \in T \Leftrightarrow 0 \le x \le 2,\ 0 \le y \le -\frac{3}{2}x + 3$$

$$\begin{aligned}
\text{volume} &= \iint_T 6 - 3x - 2y\, dA \\
&= \int_0^2 \int_0^{-\frac{3}{2}x+3} 6 - 3x - 2y\, dy\, dx = \int_0^2 \left[6y - 3xy - y^2\right]_0^{-\frac{3}{2}x+3} dx \\
&= \int_0^2 6\left(-\frac{3}{2}x + 3\right) - 3x\left(-\frac{3}{2}x + 3\right) - \left(-\frac{3}{2}x + 3\right)^2 dx \\
&= \int_0^2 \frac{9}{4}x^2 - 9x + 9\, dx = \left[\frac{3}{4}x^3 - \frac{9}{2}x^2 + 9x\right]_0^2 = 6
\end{aligned}$$

15.3.36 Find the volume of the solid by subtracting two volumes, where the solid is enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$.

Two planes meet over $3y = 2 + y \Leftrightarrow y = 1$.

D is the planar region that $-1 \le x \le 1,\ x^2 \le y \le 1$.

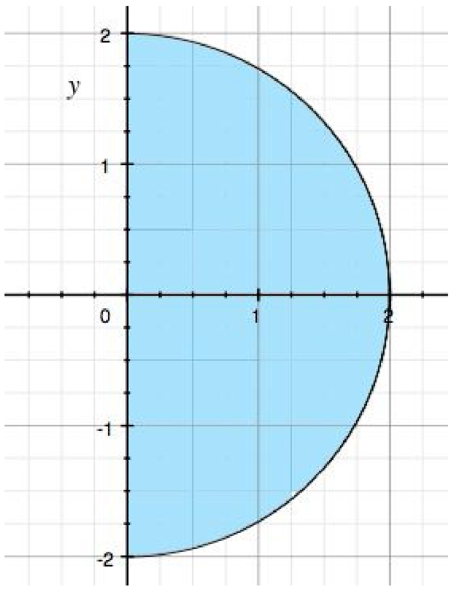
On this region, $2 + y \ge 3y$.

$$\text{volume } = \iint_D 2 + y\ dA - \iint_D 3y\ dA = \iint_D 2 - 2y\ dA$$

$$\begin{aligned}\iint_D 2 - 2y\ dA &= \int_{-1}^{1}\int_{x^2}^{1} 2 - 2y\ dy\ dx = \int_{-1}^{1} \left[2y - y^2\right]_{x^2}^{1}\ dx \\ &= \int_{-1}^{1} 1 - 2x^2 + x^4\ dx = \left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1} = \frac{16}{15}\end{aligned}$$

15.3.46 Sketch the region of integration and change the order of integration.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x, y)\ dx\ dy$$



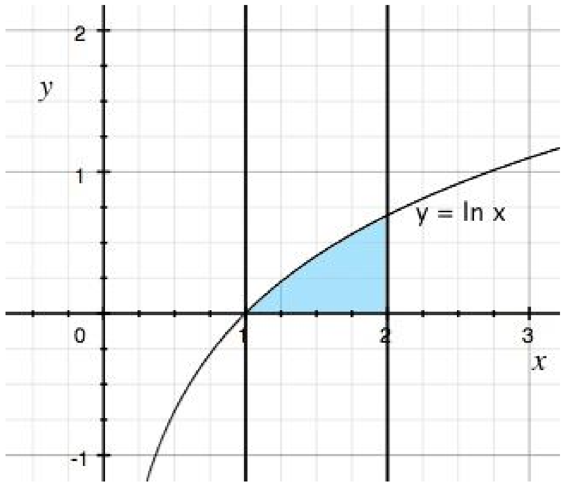
$$x = \sqrt{4 - y^2} \Rightarrow x^2 = 4 - y^2 \Rightarrow x^2 + y^2 = 4$$

$$-2 \le y \le 2,\ 0 \le x \le \sqrt{4 - y^2} \Leftrightarrow 0 \le x \le 2,\ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$$

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x, y)\ dx\ dy = \int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\ dy\ dx$$

15.3.47 Sketch the region of integration and change the order of integration.

$$\int_{1}^{2}\int_{0}^{\ln x} f(x, y)\ dy\ dx$$



$$1 \leq x \leq 2,\ 0 \leq y \leq \ln x \Leftrightarrow 0 \leq y \leq \ln 2,\ e^y \leq x \leq 2$$

$$\int_1^2 \int_0^{\ln x} f(x,y)\ dy\ dx = \int_0^{\ln 2} \int_{e^y}^2 f(x,y)\ dx\ dy$$

15.3.49 Evaluate

$$\int_0^1 \int_{3y}^3 e^{x^2}\ dx\ dy$$

by reversing the order of integration.

$$0 \leq y \leq 1,\ 3y \leq x \leq 3 \Leftrightarrow 0 \leq x \leq 3,\ 0 \leq y \leq \frac{x}{3}$$

$$\begin{aligned} \int_0^1 \int_{3y}^3 e^{x^2}\ dx\ dy &= \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\ dy\ dx = \int_0^3 \left[ye^{x^2}\right]_0^{\frac{x}{3}}\ dx \\ &= \int_0^3 \frac{1}{3}xe^{x^2}\ dx = \left[\frac{1}{6}e^{x^2}\right]_0^3 = \frac{1}{6}e^9 - \frac{1}{6} \end{aligned}$$

15.3.52 Evaluate

$$\int_0^1 \int_x^1 e^{\frac{x}{y}}\ dy\ dx$$

by reversing the order of integration.

$$0 \leq x \leq 1,\ x \leq y \leq 1 \Leftrightarrow 0 \leq y \leq 1,\ 0 \leq x \leq y$$

$$\begin{aligned} \int_0^1 \int_x^1 e^{\frac{x}{y}}\ dy\ dx &= \int_0^1 \int_0^y e^{\frac{x}{y}}\ dx\ dy = \int_0^1 \left[ye^{\frac{x}{y}}\right]_0^y\ dy = \int_0^1 ye - y\ dy \\ &= \left[\frac{y^2}{2}(e-1)\right]_0^1 = \frac{e-1}{2} \end{aligned}$$

15.3.60 Find the average value of $f(x,y) = x \sin y$ over the region D where D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$.

$$(x,y) \in D \Leftrightarrow 0 \le x \le 1,\ 0 \le y \le x^2$$

$$\begin{aligned}
\iint_D x \sin y \, dA &= \int_0^1 \int_0^{x^2} x \sin y \, dy \, dx = \int_0^1 [-x \cos y]_0^{x^2} \, dx \\
&= \int_0^1 -x\cos(x^2) - (-x \cos 0) \, dx = \int_0^1 x - x\cos(x^2) \, dx \\
&= \left[\frac{x^2}{2} - \frac{1}{2}\sin(x^2) \right]_0^1 = \frac{1}{2} - \frac{1}{2}\sin 1
\end{aligned}$$

$$\text{area}(D) = \int_0^1 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\text{average} = \frac{1}{\text{area}(D)} \iint_D x \sin y \, dA = \frac{3}{2} - \frac{3}{2}\sin 1$$