Homework 4 Model Solution

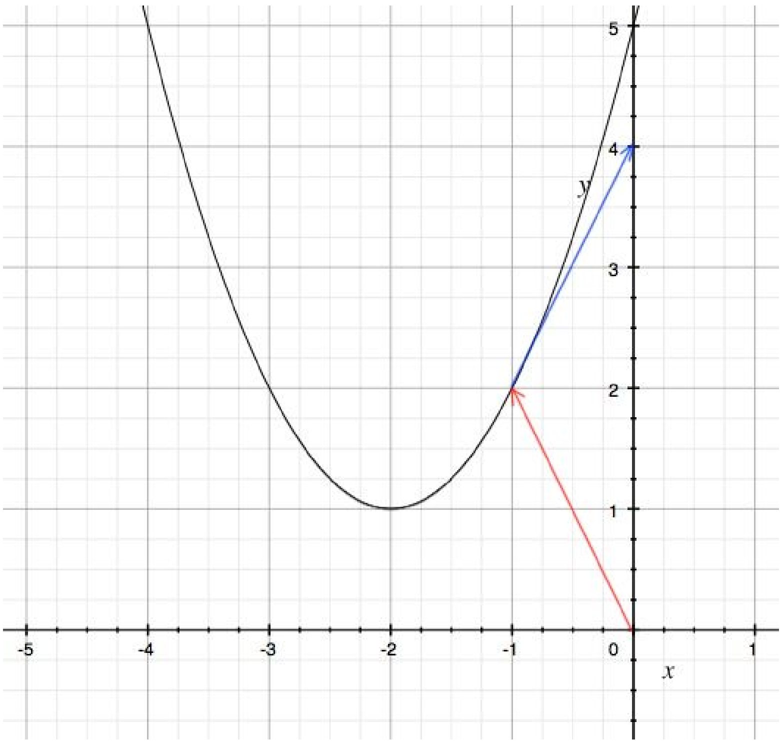
Section 13.2 $\sim$ 13.4.

13.2.3 Let $\mathbf{r}(t) = \langle t-2, t^2+1\rangle$.

(a) Sketch the plane curve with the given vector equation.

$$x = t-2, y = t^2+1 \Rightarrow t = x+2 \Rightarrow y = (x+2)^2+1$$

$$\Rightarrow y = x^2+4x+5$$



(b) Find $\mathbf{r}'(t)$.

$$\mathbf{r}'(t) = \langle 1, 2t\rangle$$

(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for $t = 1$.

$$\mathbf{r}(1) = \langle -1, 2\rangle, \quad \mathbf{r}'(1) = \langle 1, 2\rangle$$

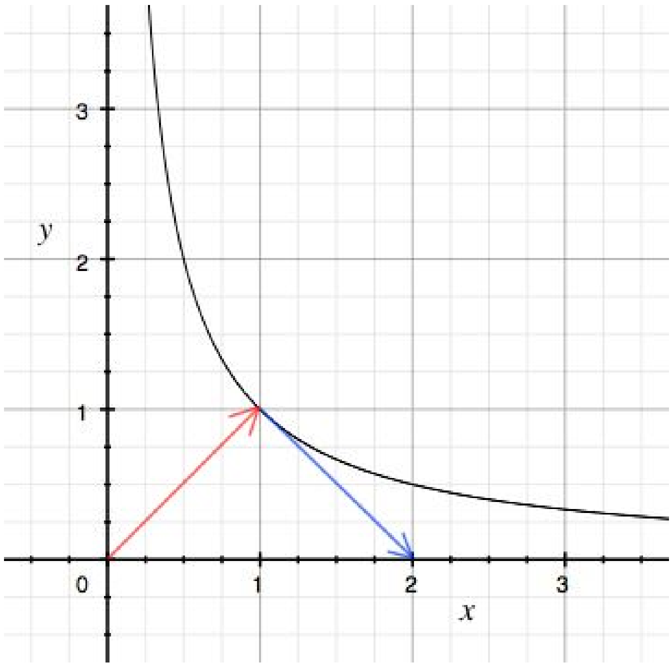
See the above figure. The red vector is $\mathbf{r}(1)$ and the blue one is $\mathbf{r}'(1)$.

13.2.6 Let $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$.

(a) Sketch the plane curve with the given vector equation.

$$x = e^t, y = e^{-t} \Rightarrow xy = e^te^{-t} = 1 \Rightarrow y = \frac{1}{x}$$

Moreover, $x = e^t > 0$.



(b) Find $\mathbf{r}'(t)$.

$$\mathbf{r}'(t) = \langle e^t, -e^{-t} \rangle$$

(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for $t = 0$.

$$\mathbf{r}(0) = \langle 1, 1 \rangle, \quad \mathbf{r}'(0) = \langle 1, -1 \rangle$$

See the above figure. The red vector is $\mathbf{r}(0)$ and the blue one is $\mathbf{r}'(0)$.

13.2.13 Find the derivative of the vector function $\mathbf{r}(t) = e^{t^2}\mathbf{i} - \mathbf{j} + \ln(1+3t)\mathbf{k}$.

$$\mathbf{r}(t) = \langle e^{t^2}, -1, \ln(1+3t) \rangle$$

$$\mathbf{r}'(t) = \langle 2te^{t^2}, 0, \frac{3}{1+3t} \rangle$$

13.2.18 Find the unit tangent vector $\mathbf{T}(t)$ at the point with $t = 1$ for $\mathbf{r}(t) = \langle t^3 + 3t, t^2 + 1, 3t + 4 \rangle$.

$$\mathbf{r}'(t) = \langle 3t^2 + 3, 2t, 3 \rangle$$

$$\mathbf{r}'(1) = \langle 6, 2, 3 \rangle$$

$$|\mathbf{r}'(1)| = \sqrt{6^2 + 2^2 + 3^2} = 7$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \langle \frac{6}{7}, \frac{2}{7}, \frac{3}{7} \rangle$$

13.2.21 If $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\mathbf{r}'(1) = \langle 1, 2, 3 \rangle, \quad |\mathbf{r}'(1)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \rangle$$
$$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle$$
$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle$$

13.2.24 Find parametric equations for the tangent line to the curve with

$$x = e^t, \quad y = te^t, \quad z = te^{t^2}$$

at $(1, 0, 0)$.

$$\mathbf{r}(t) = \langle e^t, te^t, te^{t^2} \rangle$$

A point:

$$(1, 0, 0) = \mathbf{r}(0)$$
$$\mathbf{r}'(t) = \langle e^t, e^t + te^t, e^{t^2} + 2t^2e^{t^2} \rangle$$

Direction vector:

$$\mathbf{r}'(0) = \langle 1, 1, 1 \rangle$$

Tangent line:

$$x = 1 + t, \quad y = t, \quad z = t$$

13.2.27 Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point $(3, 4, 2)$.

$$x^2 + y^2 = 25 \Rightarrow x^2 = 25 - y^2 \Rightarrow x = \sqrt{25 - y^2}$$
$$y^2 + z^2 = 20 \Rightarrow z^2 = 20 - y^2 \Rightarrow z = \sqrt{20 - y^2}$$

Note that we are interested in a point $(3, 4, 2)$ whose three coordinates are positive. So we can take the positive square root to represent a parametric curve near $(3, 4, 2)$.

A parameterization of the intersection curve near $(3, 4, 2)$:

$$\mathbf{r}(t) = \langle \sqrt{25 - t^2}, t, \sqrt{20 - t^2} \rangle$$

$$\mathbf{r}(4) = \langle 3, 4, 2 \rangle$$
$$\mathbf{r}'(t) = \langle \frac{-t}{\sqrt{25 - t^2}}, 1, \frac{-t}{\sqrt{20 - t^2}} \rangle$$
$$\mathbf{r}'(4) = \langle -\frac{4}{3}, 1, -2 \rangle$$

A vector equation of the tangent line:

$$\mathbf{s}(t) = \langle 3 - \frac{4}{3}t, 4 + t, 2 - 2t \rangle$$

13.2.36 Evaluate the integral

$$\int_0^1 \left(\frac{4}{1+t^2}\mathbf{j} + \frac{2t}{1+t^2}\mathbf{k} \right)\, dt.$$

$$\int_0^1 \frac{4}{1+t^2}\, dt = [4\arctan t]_0^1 = 4 \cdot \frac{\pi}{4} - 0 = \pi$$

$$\int_0^1 \frac{2t}{1+t^2}\, dt = \left[\ln(1+t^2)\right]_0^1 = \ln 2 - \ln 1 = \ln 2$$

$$\int_0^1 \left(\frac{4}{1+t^2}\mathbf{j} + \frac{2t}{1+t^2}\mathbf{k} \right)\, dt = \pi\mathbf{j} + \ln 2\mathbf{k}$$

13.2.53 If $\mathbf{r}(t) \neq 0$, show that

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

(Hint: $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$.)

$$\frac{d}{dt}|\mathbf{r}(t)|^2 = 2|\mathbf{r}(t)|\frac{d}{dt}|\mathbf{r}(t)|$$

On the other hand,

$$\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t)) = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t)$$

Therefore

$$2|\mathbf{r}(t)|\frac{d}{dt}|\mathbf{r}(t)| = 2\mathbf{r}(t) \cdot \mathbf{r}'(t)$$

and

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{2|\mathbf{r}(t)|}2\mathbf{r}(t) \cdot \mathbf{r}'(t) = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

13.2.55 If $\mathbf{u}(t) = \mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t))$, show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot \left(\mathbf{r}'(t) \times \mathbf{r}'''(t)\right).$$

$$\begin{aligned}
\mathbf{u}'(t) = \frac{d}{dt}\left(\mathbf{r}(t) \cdot \left(\mathbf{r}'(t) \times \mathbf{r}''(t)\right)\right) &= \mathbf{r}'(t) \cdot \left(\mathbf{r}'(t) \times \mathbf{r}''(t)\right) + \mathbf{r}(t) \cdot \frac{d}{dt}\left(\mathbf{r}'(t) \times \mathbf{r}''(t)\right) \\
&= \mathbf{r}'(t) \cdot \left(\mathbf{r}'(t) \times \mathbf{r}''(t)\right) + \mathbf{r}(t) \cdot \left(\mathbf{r}''(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'''(t)\right) \\
&= \mathbf{r}(t) \cdot \left(\mathbf{r}'(t) \times \mathbf{r}'''(t)\right),
\end{aligned}$$

because 1) $\mathbf{r}'(t) \times \mathbf{r}''(t)$ is perpendicular to $\mathbf{r}'(t)$ so $\mathbf{r}'(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t)) = 0$ and 2) $\mathbf{r}''(t) \times \mathbf{r}''(t) = 0$.

13.3.2 Find the length of the curve

$$\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle, \quad 0 \le t \le 1.$$

$$\mathbf{r}'(t) = \langle 2, 2t, t^2 \rangle$$
$$|\mathbf{r}'(t)| = \sqrt{2^2 + (2t)^2 + (t^2)^2} = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2+2)^2} = t^2 + 2$$
$$\text{length } = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 t^2 + 2\, dt = \left[\frac{1}{3}t^3 + 2t\right]_0^1 = \frac{1}{3} + 2 = \frac{7}{2}$$

13.3.7 Find the length of the curve

$$\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle, \quad 0 \le t \le 2$$

correct to four decimal places. (Use your calculator to approximate the integral.)

$$\mathbf{r}'(t) = \langle 2t, 3t^2, 4t^3 \rangle$$
$$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (4t^3)^2} = \sqrt{4t^2 + 9t^4 + 16t^6}$$
$$\text{length } = \int_0^2 |\mathbf{r}'(t)|\, dt = \int_0^2 \sqrt{4t^2 + 9t^4 + 16t^6}\, dt \risingdotseq 18.6833$$

In this problem, it is extremely hard to find the antiderivative. It is enough to compute an approximation of the integral by using your calculator. Or you may use an on-line integral calculator - click here.

13.3.11 Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$.

$$x^2 = 2y \Rightarrow y = \frac{x^2}{2}$$
$$3z = xy \Rightarrow z = \frac{xy}{3} = \frac{x^3}{6}$$

Curve of intersection:

$$\mathbf{r}(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{6} \rangle$$
$$(0,0,0) = \mathbf{r}(0)$$
$$(6,18,36) = \mathbf{r}(6)$$
$$\mathbf{r}'(t) = \langle 1, t, \frac{t^2}{2} \rangle$$
$$|\mathbf{r}'(t)| = \sqrt{1^2 + t^2 + \left(\frac{t^2}{2}\right)^2} = \sqrt{1 + t^2 + \frac{t^4}{4}} = \sqrt{\frac{4 + 4t^2 + t^4}{4}}$$
$$= \sqrt{\frac{(t^2+2)^2}{4}} = \frac{t^2+2}{2} = \frac{t^2}{2} + 1$$
$$\text{length } = \int_0^6 |\mathbf{r}'(t)|\, dt = \int_0^6 \frac{t^2}{2} + 1\, dt = \left[\frac{t^3}{6} + t\right]_0^6 = 36 + 6 = 42$$

13.3.19 Let $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$.

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

$$\mathbf{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{e^t}{e^t + e^{-t}}, -\frac{e^{-t}}{e^t + e^{-t}} \rangle = \langle \frac{\sqrt{2}e^t}{e^{2t} + 1}, \frac{e^{2t}}{e^{2t} + 1}, -\frac{1}{e^{2t} + 1} \rangle$$

$$\mathbf{T}'(t) = \langle \frac{\sqrt{2}e^t(e^{2t} + 1) - \sqrt{2}e^t 2e^{2t}}{(e^{2t} + 1)^2}, \frac{2e^{2t}(e^{2t} + 1) - e^{2t}2e^{2t}}{(e^{2t} + 1)^2}, \frac{0(e^{2t} + 1) - (-1) \cdot 2e^{2t}}{(e^{2t} + 1)^2} \rangle$$

$$= \langle \frac{\sqrt{2}e^t(1 - e^{2t})}{(e^{2t} + 1)^2}, \frac{2e^{2t}}{(e^{2t} + 1)^2}, \frac{2e^{2t}}{(e^{2t} + 1)^2} \rangle$$

$$|\mathbf{T}'(t)| = \sqrt{\left(\frac{\sqrt{2}e^t(1 - e^{2t})}{(e^{2t} + 1)^2}\right)^2 + \left(\frac{2e^{2t}}{(e^{2t} + 1)^2}\right)^2 + \left(-\frac{2e^{2t}}{(e^{2t} + 1)^2}\right)^2}$$

$$= \sqrt{\frac{2e^{2t}(1 - e^{2t})^2 + 4e^{4t} + 4e^{4t}}{(e^{2t} + 1)^4}} = \sqrt{\frac{2e^{2t} - 4e^{4t} + 2e^{8t} + 8e^{4t}}{(e^{2t} + 1)^4}}$$

$$= \sqrt{\frac{2e^{2t}(1 + 4e^{2t} + e^{4t})}{(e^{2t} + 1)^4}} = \sqrt{\frac{2e^{2t}(e^{2t} + 1)^2}{(e^{2t} + 1)^4}} = \frac{\sqrt{2}e^t}{e^{2t} + 1}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle \frac{1 - e^{2t}}{e^{2t} + 1}, \frac{\sqrt{2}e^t}{e^{2t} + 1}, -\frac{\sqrt{2}e^t}{e^{2t} + 1} \rangle$$

(b) Use formula 9 to find the curvature.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{\sqrt{2}e^t}{e^{2t}+1}}{e^t + e^{-t}} = \frac{\frac{\sqrt{2}e^t}{e^{2t}+1}}{\frac{e^{2t}+1}{e^t}} = \frac{\sqrt{2}e^{2t}}{(e^{2t} + 1)^2}$$

13.3.22 Use Theorem 10 to find the curvature for $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k}$.

$$\mathbf{r}'(t) = \langle 1, 2t, e^t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + (e^t)^2} = \sqrt{1 + 4t^2 + e^{2t}}$$

$$\mathbf{r}''(t) = \langle 0, 2, e^t \rangle$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & e^t \\ 0 & 2 & e^t \end{vmatrix} = \langle 2(t - 1)e^t, -e^t, 2 \rangle$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(2(t - 1)e^t)^2 + (-e^t)^2 + 2^2} = \sqrt{4(t - 1)^2e^{2t} + e^{2t} + 4}$$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{4(t - 1)^2e^{2t} + e^{2t} + 4}}{(1 + 4t^2 + e^{2t})^{\frac{3}{2}}}$$

13.3.30 At what point does $y = \ln x$ have maximum curvature? What happens to the curvature as $x \to \infty$?

$$\frac{dy}{dx} = \frac{1}{x}$$
$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$$
$$\kappa(x) = \frac{|\frac{d^2y}{dx^2}|}{(1+(\frac{dy}{dx})^2)^{\frac{3}{2}}} = \frac{\frac{1}{x^2}}{(1+\frac{1}{x^2})^{\frac{3}{2}}} = \frac{\frac{1}{x^2}}{\frac{(x^2+1)^{\frac{3}{2}}}{x^3}} = \frac{x}{(x^2+1)^{\frac{3}{2}}}$$
$$\kappa'(x) = \frac{1\cdot(x^2+1)^{\frac{3}{2}} - x\frac{3}{2}(x^2+1)^{\frac{1}{2}}2x}{(x^2+1)^3} = \frac{((x^2+1)-3x^2)(x^2+1)^{\frac{1}{2}}}{(x^2+1)^3} = \frac{1-2x^2}{(x^2+1)^{\frac{5}{2}}}$$

So $\kappa'(x) = 0$ if $x = \frac{1}{\sqrt{2}}$. (Note that x is positive.) Therefore $\kappa(x)$ is maximum if $x = \frac{1}{\sqrt{2}}$.

$$\lim_{x\to\infty} \frac{x}{(x^2+1)^{\frac{3}{2}}} = \lim_{x\to\infty} \frac{1}{\frac{3}{2}(x^2+1)^{\frac{1}{2}}2x} = \lim_{x\to\infty} \frac{1}{3x(x^2+1)^{\frac{1}{2}}} = 0$$

So as x grows, the curvature approaches 0.

13.3.65 The DNA molecule has the shape of a double helix (see Figure 3 on page 866). The radius of each helix is about 10 angstroms (1 Å = 10^{-8} cm). Each helix rises about 34 Å during each complete turn, and there are about 2.9×10^8 complete turns. Estimate the length of each helix.

It is a helix of radius 10 Å:

$$\mathbf{r}(t) = \langle 10\cos t, 10\sin t, ct\rangle$$

When $t = 2\pi$ (a complete turn), it rises 34 Å:

$$c \cdot 2\pi = 34 \Rightarrow c = \frac{17}{\pi}$$
$$\mathbf{r}(t) = \langle 10\cos t, 10\sin t, \frac{17}{\pi}t\rangle, \quad 0 \le t \le 2.9\times 10^8 \cdot 2\pi$$
$$\mathbf{r}'(t) = \langle -10\sin t, 10\cos t, \frac{17}{\pi}\rangle$$
$$|\mathbf{r}'(t)| = \sqrt{(-10\sin t)^2 + (10\cos t)^2 + \left(\frac{17}{\pi}\right)^2} = \sqrt{100 + \frac{17^2}{\pi^2}} \fallingdotseq 11.370216$$
$$\text{length } = \int_0^{2.9\times10^8\cdot2\pi} |\mathbf{r}'(t)|\, dt \fallingdotseq \int_0^{2.9\times10^8\cdot2\pi} 11.370216\, dt$$
$$\fallingdotseq 207.179413 \times 10^8 \text{ Å} = 207.179413 \text{ cm}$$

13.3.66 Let's consider the problem of designing a railroad track to make a smooth transition between sections of straight track. Existing track along the negative x-axis is to be joined smoothly to a track along the line $y = 1$ for $x \ge 1$.

(a) Find a polynomial $P = P(x)$ of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

is continuous and has continuous slope and continuous curvature.

Let $P(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5$. Because $\lim_{x\to 0^+} P(x) = F(0) = 0$ and $\lim_{x\to 1^-} P(x) = F(1) = 1$,

$$\lim_{x\to 0^+} P(x) = P(0) = a = 0, \quad \lim_{x\to 1^-} P(x) = P(1) = a + b + c + d + e + f = 1.$$

$$P'(x) = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4$$

Since $F(x)$ has continuous slope,

$$\lim_{x\to 0^+} P'(x) = \lim_{x\to 0^+} F'(x) = \lim_{x\to 0^-} F'(x) = 0$$

and

$$\lim_{x\to 1^-} P'(x) = \lim_{x\to 1^-} F'(x) = \lim_{x\to 1^+} F'(x) = 0.$$

So

$$0 = \lim_{x\to 0^+} P'(x) = P'(0) = b, \quad 0 = \lim_{x\to 1^-} P'(x) = P'(1) = b+2c+3d+4e+5f.$$

$$P''(x) = 2c + 6dx + 12ex^2 + 20fx^3$$

On $0 < x < 1$,

$$\kappa(x) = \frac{|P''(x)|}{(1 + (P'(x))^2)^{\frac{3}{2}}}$$

The curvature of a line segment is zero, so to obtain a continuous curvature at $x = 0$ and $x = 1$, $\lim_{x\to 0^+} \kappa(x) = \lim_{x\to 1^-} \kappa(x) = 1$.

$$\lim_{x\to 0^+} \kappa(x) = \lim_{x\to 0^+} \frac{|P''(x)|}{(1 + (P'(x))^2)^{\frac{3}{2}}} = \frac{|P''(0)|}{(1 + (P'(0))^2)^{\frac{3}{2}}} = 2c$$

$$\lim_{x\to 1^-} \kappa(x) = \lim_{x\to 1^-} \frac{|P''(x)|}{(1 + (P'(x))^2)^{\frac{3}{2}}} = \frac{|P''(1)|}{(1 + (P'(1))^2)^{\frac{3}{2}}} = |2c + 6d + 12e + 20f|$$

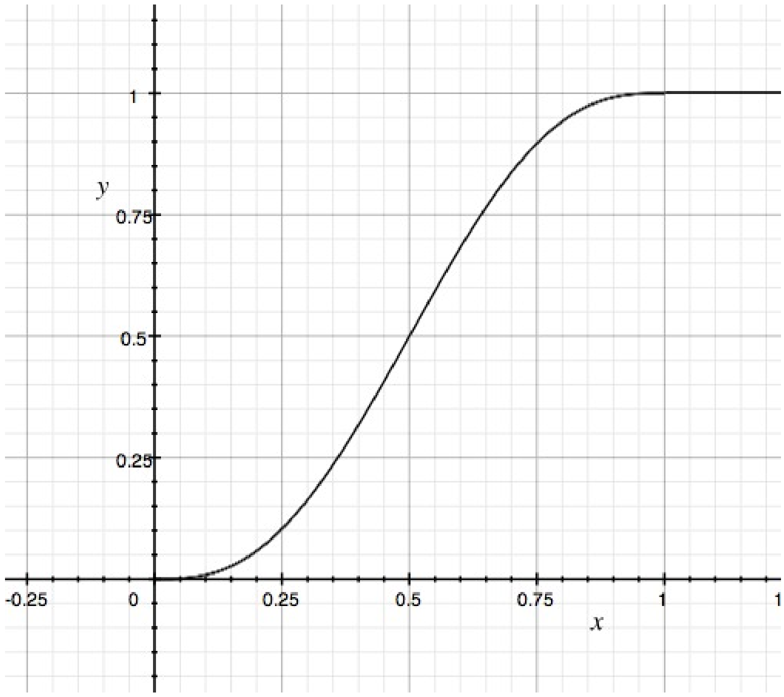
So $2c = 0$ and $2c + 6d + 12e + 20f = 0$ as well.

In summary, we have a system of linear equations.

$$\begin{aligned} a &= 0 \\ b &= 0 \\ 2c &= 0 \\ a + b + c + d + e + f &= 1 \\ b + 2c + 3d + 4e + 5f &= 0 \\ 2c + 6d + 12e + 20f &= 0 \end{aligned}$$

The solution of this system of linear equation is $a = b = c = 0, d = 10, e = -15$, and $f = 6$. Therefore $P(x) = 10x^3 - 15x^4 + 6x^5$.

(b) Use a graphing calculator or computer to draw the graph of F.



13.4.6 Find the velocity, acceleration, and speed of a particle with the position function $\mathbf{r}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j}$. Sketch the path of the particle and draw the velocity and acceleration vectors for $t = 0$.

Velocity:

$$\mathbf{r}'(t) = \langle e^t, 2e^{2t} \rangle$$

Speed:

$$|\mathbf{r}'(t)| = \sqrt{(e^t)^2 + (2e^{2t})^2} = \sqrt{e^{2t} + 4e^{4t}}$$

Acceleration:

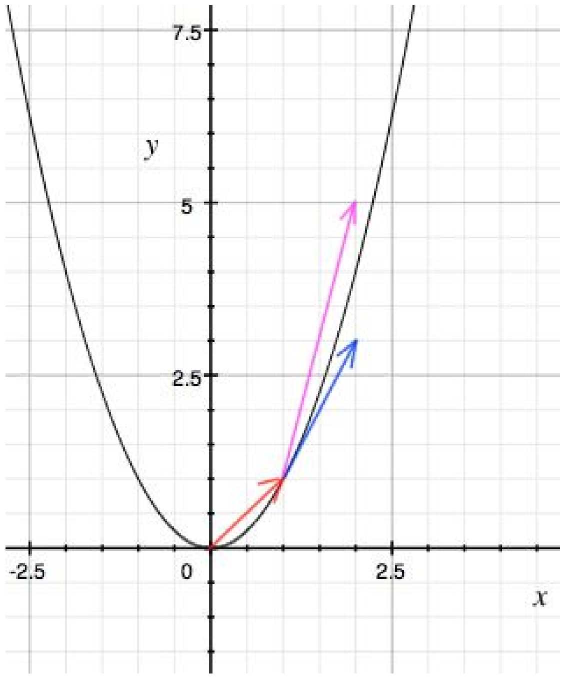
$$\mathbf{r}''(t) = \langle e^t, 4e^{2t} \rangle$$

At $t = 0$,

$$\mathbf{r}(0) = \langle 1, 1 \rangle, \quad \mathbf{r}'(0) = \langle 1, 2 \rangle, \quad \mathbf{r}''(0) = \langle 1, 4 \rangle$$

$$y = e^{2t} = (e^t)^2 = x^2$$

So the graph is



The red vector is $\mathbf{r}(0)$, the blue one is the velocity $\mathbf{r}'(0)$, and the purple vector is $\mathbf{r}''(0)$.

13.4.12 Find the velocity, acceleration, and speed of a particle with the position function $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \ln t\mathbf{k}$.

Velocity:

$$\mathbf{r}'(t) = \langle 2t, 2, \frac{1}{t}\rangle$$

Acceleration:

$$\mathbf{r}''(t) = \langle 2, 0, -\frac{1}{t^2}\rangle$$

Speed:

$$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 2^2 + \left(\frac{1}{t}\right)^2} = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\left(2t + \frac{1}{t}\right)^2} = \left|2t + \frac{1}{t}\right|$$

13.4.16 Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position

$$\mathbf{a}(t) = 2\mathbf{i} + 6t\mathbf{j} + 12t^2\mathbf{k}, \quad \mathbf{v}(0) = \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}.$$

$$\mathbf{a}(t) = 2\mathbf{i} + 6t\mathbf{j} + 12t^2\mathbf{k} \Rightarrow \mathbf{v}(t) = \int \mathbf{a}(t)\, dt = 2t\mathbf{i} + 3t^2\mathbf{j} + 4t^3\mathbf{k} + \overrightarrow{C}$$

$$\mathbf{k} = \mathbf{v}(0) = \overrightarrow{C} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + 3t^2\mathbf{j} + (4t^3 + 1)\mathbf{k}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt \Rightarrow \mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + (t^4 + t)\mathbf{k} + \overrightarrow{D}$$

$$\mathbf{j} + \mathbf{k} = \mathbf{r}(0) = \overrightarrow{D} \Rightarrow \mathbf{r}(t) = t^2\mathbf{i} + (t^3 + 1)\mathbf{j} + (t^4 + t + 1)\mathbf{k}$$

13.4.26 A gun is fired with angle of elevation $30°$. What is the muzzle speed if the maximum height of the shell is $500\ m$?

Let v be the muzzle speed. Set $g = 9.8\ (m/s^2)$.

$$\Rightarrow \mathbf{v}(0) = \langle v\cos 30°, v\sin 30°\rangle = \langle \frac{\sqrt{3}v}{2}, \frac{v}{2}\rangle$$

$$\mathbf{a}(t) = \langle 0, -g\rangle \Rightarrow \mathbf{v}(t) = \langle 0, -gt\rangle + \overrightarrow{C}$$

$$\langle \frac{\sqrt{3}v}{2}, \frac{v}{2}\rangle = \mathbf{v}(0) = \overrightarrow{C}$$

$$\Rightarrow \mathbf{v}(t) = \langle 0, -gt\rangle + \langle \frac{\sqrt{3}v}{2}, \frac{v}{2}\rangle = \langle \frac{\sqrt{3}v}{2}, -gt + \frac{v}{2}\rangle$$

$$\mathbf{r}(t) = \langle \frac{\sqrt{3}v}{2}t, -\frac{g}{2}t^2 + \frac{v}{2}t\rangle + \overrightarrow{D}$$

$$\langle 0, 0\rangle = \mathbf{r}(0) = \overrightarrow{D}$$

$$\Rightarrow \mathbf{r}(t) = \langle \frac{\sqrt{3}v}{2}t, -\frac{g}{2}t^2 + \frac{v}{2}t\rangle$$

At maximum height, $\frac{dy}{dt} = 0$.

$$\mathbf{v}(t) = \langle c, 0\rangle \Rightarrow -gt + \frac{v}{2} = 0 \Rightarrow t = \frac{v}{2g}$$

At this time, the height is $500\ m$.

$$-\frac{g}{2}\left(\frac{v}{2g}\right)^2 + \frac{v}{2}\left(\frac{v}{2g}\right) = 500$$

$$\Rightarrow -\frac{v^2}{8g} + \frac{v^2}{4g} = 500 \Rightarrow \frac{v^2}{8g} = 500$$

$$v = \sqrt{500 \cdot 8g} = \sqrt{500 \cdot 8 \cdot 9.8} \fallingdotseq 197.99\ m/s$$

13.4.27 A gun has muzzle speed $150\ m/s$. Find two angles of elevation that can be used to hit a target $800\ m$ away.

Let θ be the angle of elevation. Then $\mathbf{v}(0) = \langle 150\cos\theta, 150\sin\theta\rangle$.

$$\mathbf{a}(t) = \langle 0, -9.8\rangle \Rightarrow \mathbf{v}(t) = \langle 0, -9.8t\rangle + \overrightarrow{C}$$

$$\langle 150\cos\theta, 150\sin\theta\rangle = \mathbf{v}(0) = \overrightarrow{C} \Rightarrow \mathbf{v}(t) = \langle 150\cos\theta, -9.8t + 150\sin\theta\rangle$$

$$\mathbf{r}(t) = \langle 150(\cos\theta)t, -4.9t^2 + 150(\sin\theta)t\rangle + \overrightarrow{D}$$

$$\langle 0, 0\rangle = \mathbf{r}(0) = \overrightarrow{D} \Rightarrow \mathbf{r}(t) = \langle 150(\cos\theta)t, -4.9t^2 + 150(\sin\theta)t\rangle$$

The bullet hit the ground at a target $800\ m$ away. That means when y- coordinate is zero, x-coordinate is 800.

$$-4.9t^2 + 150(\sin\theta)t = 0 \Rightarrow t = 0 \text{ or } t = \frac{150\sin\theta}{4.9}$$

$$x\text{-coordinate } = 150(\cos\theta)\frac{150\sin\theta}{4.9} = 800 \Rightarrow 22500\cos\theta\sin\theta = 800\cdot 4.9 = 3920$$
$$\Rightarrow 11250\sin 2\theta = 3920 \Rightarrow \sin 2\theta = \frac{3920}{11250} \fallingdotseq 0.348444$$
$$2\theta \fallingdotseq \arcsin 0.348444 \fallingdotseq 0.3559 \text{ or } \pi - 0.3559 \fallingdotseq 2.7857 \Rightarrow \theta \fallingdotseq 0.1774 \text{ or } 1.3928$$

In degree,
$$\theta \fallingdotseq 10.16^\circ \text{ or } 79.80^\circ$$

13.4.28 A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/s at an angle 50° above the horizontal. Is it a home run? (In other words, does the ball clear the fence?)

From the conditions, we know

$$\mathbf{r}(0) = \langle 0, 3\rangle, \quad \mathbf{v}(0) = \langle 115\cos 50^\circ, 115\sin 50^\circ\rangle \fallingdotseq \langle 73.92058, 88.09511\rangle$$
$$\mathbf{a}(t) = \langle 0, -32.174\rangle \Rightarrow \mathbf{v}(t) = \langle 0, -32.174t\rangle + \overrightarrow{C}$$
$$\langle 73.92058, 88.09511\rangle = \mathbf{v}(0) = \overrightarrow{C}$$
$$\mathbf{v}(t) = \langle 73.92058, -32.174t + 88.09511\rangle$$
$$\mathbf{r}(t) = \langle 73.92058t, -16.087t^2 + 88.09511t\rangle + \overrightarrow{D}$$
$$\langle 0, 3\rangle = \mathbf{r}(0) = \overrightarrow{D} \Rightarrow \mathbf{r}(t) = \langle 73.92058t, -16.087t^2 + 88.09511t + 3\rangle$$

If $x = 400$,
$$73.92058t = 400 \Rightarrow t = \frac{400}{73.92058} \fallingdotseq 5.41121.$$

At this time, y-coordinate is

$$-16.087(5.41121)^2 + 88.09511\cdot 5.41121 + 3 \fallingdotseq 8.655 < 10.$$

Therefore it is not a home run.