Homework 6 Model Solution

Section 14.2 $\sim$ 14.3.

14.2.6 Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(1,-1)} e^{xy}\cos(x+y)$$

e^{xy}, $\cos(x+y)$, $e^{xy}\cos(x+y)$ are all continuous functions.

$$\lim_{(x,y)\to(1,-1)} e^{xy}\cos(x+y) = e^{1+(-1)}\cos(1+(-1)) = e^0\cos 0 = 1$$

14.2.8 Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$$

$\ln t$ is a continuous function.

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \ln\left(\frac{1+0^2}{1^2+1\cdot 0}\right) = \ln 1 = 0$$

14.2.9 Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$$

If $(x,y)\to(0,0)$ along x-axis so $y=0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2} = \lim_{x\to 0}\frac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0$$

If $(x,y)\to(0,0)$ along y-axis so $x=0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2} = \lim_{y\to 0}\frac{-4y^2}{2y^2} = \lim_{y\to 0} -2 = -2$$

So the limit does not exist.

14.2.13 Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} &= \lim_{r\to 0}\frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}} = \lim_{r\to 0}\frac{r^2\cos\theta\sin\theta}{r}\\ &= \lim_{r\to 0} r\cos\theta\sin\theta = 0\end{aligned}$$

14.2.20 Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2}$$

If $(x, y, z) \to (0, 0, 0)$ along x-axis (so $y = z = 0$), then

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

If $(x, y, z) \to (0, 0, 0)$ along a line $x = y = z$, then

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2} = \lim_{x\to 0} \frac{x^2+x^2}{x^2+x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{3x^2} = \lim_{x\to 0} \frac{2}{3} = \frac{2}{3}$$

So the limit does not exist.

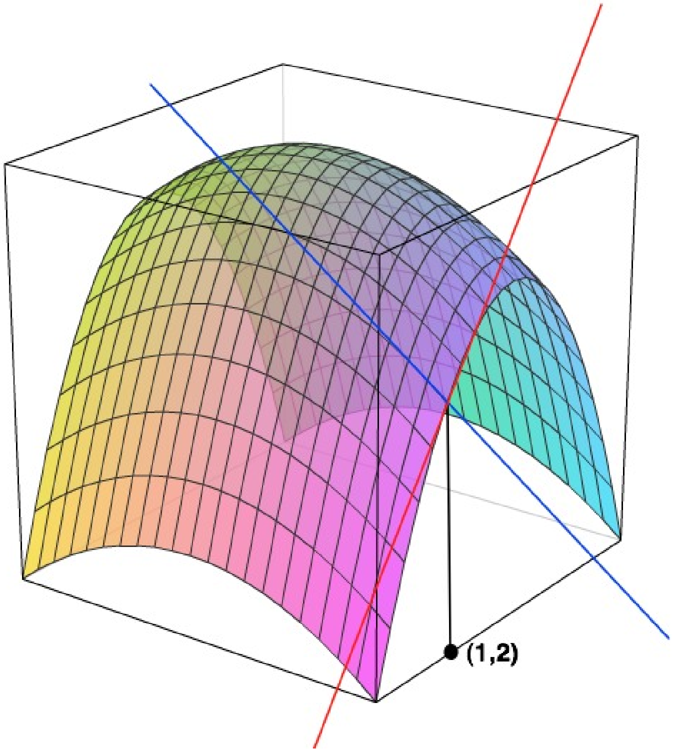
14.2.39 Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geq 0$, note that $r \to 0$ as $(x, y) \to (0, 0)$.]

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$$

14.3.11 If $f(x, y) = 16 - 4x^2 - y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes. Illustrate with either hand-drawn sketches or computer plots.

$$f_x = -8x, \quad f_y = -2y$$
$$f_x(1, 2) = -8, \quad f_y(1, 2) = -4$$



$f_x(1,2) = -8$ is the slope of the tangent line which is the intersection of the tangent plane at $(1, 2, f(1,2))$ and a vertical plane containing x-axis. On the figure above, it is red line.

$f_y(1,2) = -4$ is the slope of blue tangent line which is the intersection of the tangent plane at $(1, 2, f(1,2))$ and a vertical plane containing y-axis.

14.3.16 Find the first partial derivatives of $f(x,y) = x^4y^3 + 8x^2y$.

$$f_x = 4x^3y^3 + 16xy$$
$$f_y = 3x^4y^2 + 8x^2$$

14.3.32 Find the first partial derivatives of $f(x,y,z) = x\sin(y-z)$.

$$f_x = \sin(y-z)$$
$$f_y = x\cos(y-z)$$
$$f_z = x\cos(y-z)\cdot(-1) = -x\cos(y-z)$$

14.3.43 Find $f_y(2,1,-1)$ of $f(x,y,z) = \dfrac{y}{x+y+z}$.

$$f_y = \frac{1\cdot(x+y+z) - y\cdot 1}{(x+y+z)^2} = \frac{x+z}{(x+y+z)^2}$$
$$f_y(2,1,-1) = \frac{2+(-1)}{(2+1+(-1))^2} = \frac{1}{4}$$

14.3.48 Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ of $x^2 - y^2 + z^2 - 2z = 4$.

$$\begin{aligned}
\frac{\partial}{\partial x}\left(x^2 - y^2 + z^2 - 2z\right) &= \frac{\partial}{\partial x}4 \\
2x + 0 + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} &= 0 \\
(2z-2)\frac{\partial z}{\partial x} &= -2x \\
\frac{\partial z}{\partial x} = \frac{-2x}{2z-2} &= -\frac{x}{z-1}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial y}\left(x^2 - y^2 + z^2 - 2z\right) &= \frac{\partial}{\partial y}4 \\
0 - 2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} &= 0 \\
(2z-2)\frac{\partial z}{\partial y} &= 2y \\
\frac{\partial z}{\partial y} = \frac{2y}{2z-2} &= \frac{y}{z-1}
\end{aligned}$$

14.3.53 Find all the second partial derivatives of $f(x, y) = x^3y^5 + 2x^4y$.

$$\begin{aligned}
f_x &= 3x^2y^5 + 8x^3y \\
f_y &= 5x^3y^4 + 2x^4 \\
f_{xx} &= 6xy^5 + 24x^2y \\
f_{xy} &= 15x^2y^4 + 8x^3 \\
f_{yx} = f_{xy} &= 15x^2y^4 + 8x^3 \\
f_{yy} &= 20x^3y^3
\end{aligned}$$