14.7.6 Find the local maximum and minimum values and saddle points of $f(x, y) = xy - 2x - 2y - x^2 - y^2$.

\[ \nabla f = (f_x, f_y) = (y - 2 - 2x, \ x - 2 - 2y) \]

\[ \nabla f = 0 \iff y - 2 - 2x = 0, \ x - 2 - 2y = 0 \]

\[ \iff -2x + y = 2, \ x - 2y = 2 \iff x = -2, \ y = -2 \]

Critical point: $(x, y) = (-2, -2)$

\[ f_{xx} = -2, \ f_{xy} = 1, \ f_{yy} = -2 \]

At $(x, y) = (0, 0)$,

\[ D = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 > 0 \]

Also $f_{xx} = -2 < 0$.

Therefore $f(-2, -2) = 4$ is a local maximum.

14.7.10 Find the local maximum and minimum values and saddle points of $f(x, y) = xy(1 - x - y)$.

\[ f(x, y) = xy - x^2y - xy^2 \]

\[ \nabla f = (f_x, f_y) = (y - 2xy - y^2, \ x - x^2 - 2xy) \]

\[ \nabla f = 0 \iff y - 2xy - y^2 = 0, \ x - x^2 - 2xy = 0 \]

\[ \iff y(1 - 2x - y) = 0, \ x(1 - x - 2y) = 0 \]

From $y(1 - 2x - y) = 0$, $y = 0$ or $y = 1 - 2x$.

If $y = 0$, $x(1 - x) = 0 \Rightarrow x = 0$ or $x = 1$. So $(0, 0), (1, 0)$ are critical points.

If $y = 1 - 2x, x(1 - x - 2(1 - 2x)) = 0 \Rightarrow x(-1 + 3x) = 0 \Rightarrow x = 0$ or $x = \frac{1}{3}$.

\[ x = 0 \Rightarrow y = 1, \ x = \frac{1}{3} \Rightarrow y = \frac{1}{3} \]

So $(0, 1), \left( \frac{1}{3}, \frac{1}{3} \right)$ are the other critical points.
In summary, there are four critical points

\[(0, 0), (1, 0), (0, 1), (\frac{1}{3}, \frac{1}{3}).\]

\[f_{xx} = -2y, \quad f_{xy} = 1 - 2x - 2y, \quad f_{yy} = -2x\]

\[D = \begin{vmatrix} -2y & 1 - 2x - 2y \\ 1 - 2x - 2y & -2x \end{vmatrix}\]

At \((0, 0)\),

\[D = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0.\]

So \((0, 0)\) is a saddle point.

At \((1, 0)\),

\[D = \begin{vmatrix} 0 & -1 \\ -1 & -2 \end{vmatrix} = -1 < 0.\]

So \((1, 0)\) is a saddle point.

At \((0, 1)\),

\[D = \begin{vmatrix} -2 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0.\]

So \((0, 1)\) is a saddle point.

At \((\frac{1}{3}, \frac{1}{3})\),

\[D = \begin{vmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -1 & -\frac{1}{3} \end{vmatrix} = \frac{1}{3} > 0\]

Because \(f_{xx}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3} < 0, \ f(\frac{1}{3}, \frac{1}{3}) = \frac{1}{27}\) is a local maximum.

14.7.30 Find the absolute maximum and minimum values of \(f(x, y) = x + y - xy\) on \(D\) where \(D\) is the closed triangular region with vertices \((0, 0), (0, 2)\) and \((4, 0)\).

\[\nabla f = \langle f_x, f_y \rangle = \langle 1 - y, 1 - x \rangle\]

\[\nabla f = 0 \iff 1 - y = 0, \ 1 - x = 0 \iff x = 1, \ y = 1\]

Critical point: \((1, 1)\).

\[f_{xx} = 0, \ f_{xy} = -1, \ f_{yy} = 0\]

\[D = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0\]

So \((1, 1)\) is a saddle point.

The boundary consists of 3 line segments, 1) \(y = 0, \ 0 \leq x \leq 4\), 2) \(x = 0, \ 0 \leq y \leq 2\), and 3) \(y = -\frac{1}{2}x + 2, \ 0 \leq x \leq 4\).
On $y = 0$, $0 \leq x \leq 4$,
\[ f(x, y) = x \Rightarrow \text{maximum} = 4, \text{ minimum} = 0. \]

On $x = 0$, $0 \leq y \leq 2$,
\[ f(x, y) = y \Rightarrow \text{maximum} = 2, \text{ minimum} = 0. \]

On $y = -\frac{1}{2}x + 2$, $0 \leq x \leq 4$,
\[ f(x, y) = f(x) = x + \left( -\frac{1}{2}x + 2 \right) - x \left( -\frac{1}{2}x + 2 \right) = \frac{1}{2}x^2 - \frac{3}{2}x + 2. \]
\[
\frac{d}{dx} \left( \frac{1}{2}x^2 - \frac{3}{2}x + 2 \right) = x - \frac{3}{2} = 0 \Leftrightarrow x = \frac{3}{2}
\]
\[ f\left( \frac{3}{2} \right) = \frac{7}{8}, f(0) = 2, f(4) = 4. \]

So the maximum is 4, the minimum is $\frac{7}{8}$.

Therefore, the absolute maximum is $f(4, 0) = 4$ and the absolute minimum is $f(0, 0) = 0$.

14.7.31 Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 + x^2y + 4$
on $D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$.

\[ \nabla f = \langle 2x + 2xy, 2y + x^2 \rangle \]
\[ \nabla f = 0 \Leftrightarrow 2x + 2xy = 0, \quad 2y + x^2 = 0 \Leftrightarrow x(1 + y) = 0, \quad 2y + x^2 = 0 \]

From $x(1 + y) = 0$, $x = 0$ or $y = -1$.
If $x = 0$, then $2y = 0 \Rightarrow y = 0$ and $(x, y) = (0, 0)$.
If $y = -1$, then $-2 + x^2 = 0 \Rightarrow x = \pm \sqrt{2}$. So $(x, y) = (\sqrt{2}, -1)$ or $(-\sqrt{2}, -1)$.
Note that $(\sqrt{2}, -1), (-\sqrt{2}, -1)$ are on the outside of $D$. So on the interior of $D$, there is a unique critical point $(0, 0)$.
\[ f_{xx} = 2 + 2y, \quad f_{xy} = 2x, \quad f_{yy} = 2 \]

At $(0, 0)$,
\[ D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \]

and $f_{xx}(0, 0) = 2 > 0$. So $f(0, 0) = 4$ is a local minimum.
The boundary of $D$ consists of four intervals, 1) $x = 1$, $-1 \leq y \leq 1$, 2) $x = -1$, $-1 \leq y \leq 1$, 3) $y = 1$, $-1 \leq x \leq 1$, and 4) $y = -1$, $-1 \leq x \leq 1$.
On $x = 1$, $f(x, y) = f(y) = y^2 + y + 5$.
\[ f'(y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2} \]
\[ f\left(-\frac{1}{2}\right) = \frac{19}{4}, \quad f(-1) = 5, \quad f(1) = 7 \]

So on \(-1 \leq y \leq 1\), the maximum is 7, the minimum is \(\frac{19}{4}\).

On \(x = -1\), \(f(x, y) = f(y) = y^2 + y + 5\). Therefore the maximum and minimum values are same with the case of \(x = 1\).

On \(y = 1\), \(f(x, y) = f(x) = 2x^2 + 5\).

\[ f'(x) = 4x = 0 \Rightarrow x = 0 \]
\[ f(0) = 5, \quad f(-1) = f(1) = 7 \]

So the maximum is 7, the minimum is 5.

Finally, on \(y = -1\), \(f(x, y) = 5\).

Therefore the absolute maximum is \(f(1, 1) = f(-1, 1) = 7\), the absolute minimum is \(f(0, 0) = 4\).

14.7.41 Find the points on the cone \(z^2 = x^2 + y^2\) that are closest to the point \((4, 2, 0)\).

(Note: You can solve this problem by using Lagrange multiplier. But I will show a solution using the method in Section 14.7.)

In this problem, obviously the closest point exists. So the minimum among values at critical points is the absolute minimum.

Distance from \((x, y, z)\) to \((4, 2, 0)\)

\[
= d = \sqrt{(x - 4)^2 + (y - 2)^2 + z^2} = \sqrt{x^2 - 8x + y^2 - 4y + 20 + z^2}
= \sqrt{x^2 - 8x + y^2 - 4y + 20 + x^2 + y^2} = \sqrt{2x^2 - 8x + 2y^2 - 4y + 20}
\]

Let \(f(x, y) = d^2 = 2x^2 - 8x + 2y^2 - 4y + 20\).

\[
\nabla f = (f_x, f_y) = (4x - 8, 4y - 4) \\
\n\nabla f = 0 \Rightarrow 4x - 8 = 0, \quad 4y - 4 = 0 \Rightarrow x = 2, \quad y = 1 
\]

So \((x, y, z) = (2, 1, \pm \sqrt{2^2 + 1^2}) = (2, 1, \pm \sqrt{5})\) are the critical points. At these two points, the distances to \((4, 2, 0)\) are same. So both of them are local (and the absolute) minimum.

14.7.55 Suppose that a scientist has reason to believe that two quantities \(x\) and \(y\) are related linearly, that is \(y = mx + b\), at least approximately, for some values of \(m\) and \(b\). The scientist performs an experiment and collects data in the form of points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), and then plots these points. The points don’t lie exactly on a straight line, so the scientist wants to find constants \(m\) and \(b\) so that the line \(y = mx + b\) “fits” the points as well as possible.

Let \(d_i = y_i - (mx_i + b)\) be the vertical deviation of the point \((x_i, y_i)\) from the line.

The **method of least squares** determines \(m\) and \(b\) so as to minimize \(\sum_{i=1}^{n} d_i^2\), the
sum of the squares of these deviations. Show that, according to this method, the line of best fit is obtained when

\[ m \sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i, \]

\[ m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i. \]

Thus the line is found by solving these two equations in the two unknowns \( m \) and \( b \).

\[ f(m, b) = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - mx_i - b)^2 \]

At \((m, b)\) minimize \( f(m, b) \), \( \nabla f = 0 \).

\[ \nabla f = \langle f_m, f_b \rangle = \left\langle \sum_{i=1}^{n} 2(y_i - mx_i - b)(-x_i), \sum_{i=1}^{n} 2(y_i - mx_i - b)(-1) \right\rangle \]

\[ = \left\langle -2 \sum_{i=1}^{n} (y_i - mx_i - b)x_i, -2 \sum_{i=1}^{n} (y_i - mx_i - b) \right\rangle \]

\[ \nabla f = 0 \Rightarrow \sum_{i=1}^{n} (y_i - mx_i - b)x_i = 0, \sum_{i=1}^{n} (y_i - mx_i - b) = 0 \]

\[ \sum_{i=1}^{n} (y_i - mx_i - b)x_i = 0 \Rightarrow \sum_{i=1}^{n} y_ix_i - m \sum_{i=1}^{n} x_i^2 - b \sum_{i=1}^{n} x_i = 0 \]

\[ \Rightarrow m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \]

\[ \sum_{i=1}^{n} (y_i - mx_i - b) = 0 \Rightarrow \sum_{i=1}^{n} y_i - m \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} b = 0 \]

\[ \Rightarrow m \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} b = \sum_{i=1}^{n} y_i \Rightarrow m \sum_{i=1}^{n} x_i + nb = \sum_{i=1}^{n} y_i \]

14.8.3 Use Lagrange multipliers to find the maximum and minimum values of \( f(x, y) = x^2 + y^2 \) subject to the given constraint \( xy = 1 \).

\[ g(x, y) = xy \]

\[ \nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle \]

\[ \nabla g = \langle g_x, g_y \rangle = \langle y, x \rangle \]

\[ \nabla f = \lambda \nabla g, \ g = 1 \iff 2x = \lambda y, \ 2y = \lambda x, \ xy = 1 \]

\[ x = \frac{\lambda y}{2} \Rightarrow 2y = \lambda \cdot \frac{\lambda y}{2} = \lambda^2 y \Rightarrow 4y = \lambda^2 y \Rightarrow (4 - \lambda^2)y = 0 \]
⇒ \( y = 0 \) or \( \lambda = \pm 2 \)
\( y = 0 \) ⇒ \( xy = 0 \) so it is impossible.

\[ \lambda = 2 \Rightarrow x = y \Rightarrow x = y = 1 \text{ or } x = y = -1 \]

⇒ \((x, y) = (1, 1), (-1, -1)\)

\[ \lambda = -2 \Rightarrow x = -y \Rightarrow xy = -x^2 < 0 \]

So it is impossible, too.

Therefore there are at most two local extrema at \((1, 1)\) and \((-1, -1)\). And \( f(1, 1) = f(-1, -1) = 2 \).

Note that if \( x \) is very large positive number, \( f(x, y) = x^2 + y^2 \) becomes very large without any bound. So it has no absolute maximum. But its absolute minimum is \( f(1, 1) = f(-1, -1) = 2 \). (Note that \( xy = 1 \) is not bounded so there may not be an absolute maximum.)

14.8.5 Use Lagrange multipliers to find the maximum and minimum values of \( f(x, y) = y^2 - x^2 \) subject to the given constraint \( \frac{1}{4}x^2 + y^2 = 1 \).

\[ g(x, y) = \frac{1}{4}x^2 + y^2 \]
\[ \nabla f = \langle f_x, f_y \rangle = \langle -2x, 2y \rangle \]
\[ \nabla g = \langle g_x, g_y \rangle = \langle x, 2y \rangle \]

\[ \nabla f = \lambda \nabla g, \quad g = 1 \Rightarrow -2x = \lambda \frac{x}{2}, 2y = \lambda 2y, \frac{1}{4}x^2 + y^2 = 1 \]

⇒ \( y = \lambda y \Rightarrow \lambda = 1 \text{ or } y = 0 \)

\[ \lambda = 1 \Rightarrow -2x = \frac{x}{2} \Rightarrow x = 0 \Rightarrow \frac{1}{4}x^2 + y^2 = 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \]

⇒ \((x, y) = (0, 1) \text{ or } (0, -1)\)

\[ y = 0 \Rightarrow \frac{1}{4}x^2 + y^2 = 1 \Rightarrow \frac{1}{4}x^2 = 1 \Rightarrow x = \pm 2 \]

⇒ \((x, y) = (2, 0) \text{ or } (-2, 0)\)

\[ f(0, 1) = 1, \quad f(0, -1) = 1, \quad f(2, 0) = -4, \quad f(-2, 0) = -4 \]

So the absolute maximum is \( f(0, 1) = f(0, -1) = 1 \), the absolute minimum is \( f(2, 0) = f(-2, 0) = -4 \).

14.8.10 Use Lagrange multipliers to find the maximum and minimum values of \( f(x, y, z) = x^2 y^2 z^2 \) subject to the given constraint \( x^2 + y^2 + z^2 = 1 \).

First of all, because \( f(x, y, z) = x^2 y^2 z^2 \geq 0 \), it is easy to see that the absolute minimum is 0 at \((x, y, z)\) with at least one of \( x, y, \text{ and } z \) is zero. So let’s find a local maximum, minimum with \( x, y, z \neq 0 \).

\[ g(x, y, z) = x^2 + y^2 + z^2 \]
\[ \nabla f = \langle f_x, f_y, f_z \rangle = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle \]
\[ \nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, 2y, 2z \rangle \]
\[ \nabla f = \lambda \nabla g, \quad g = 1 \Rightarrow 2xy^2z^2 = \lambda 2x, \quad 2yx^2z^2 = \lambda 2y, \quad 2zx^2y^2 = \lambda 2z, \quad x^2 + y^2 + z^2 = 1 \]
\[ 2xy^2z^2 = \lambda 2x \Rightarrow x = 0 \text{ or } y^2z^2 = \lambda \]

Now suppose that \( x, y, z \neq 0 \). Then by a similar computation,
\[ \lambda = y^2z^2 = x^2z^2 = x^2y^2 \Rightarrow x^2 = y^2 = z^2 \]
\[ \Rightarrow x^2 + y^2 + z^2 = 1 \Rightarrow x^2 = y^2 = z^2 = \frac{1}{3} \]
\[ f(x, y, z) = x^2y^2z^2 = \frac{1}{27} \]

Therefore the absolute maximum is \( \frac{1}{27} \), the absolute minimum is 0.

14.8.20 Find the extreme values of \( f(x, y) = 2x^2 + 3y^2 - 4x - 5 \) on the region described by the inequality \( x^2 + y^2 \leq 16 \).

\[ \nabla f = \langle f_x, f_y \rangle = \langle 4x - 4, 6y \rangle \]
\[ \nabla f = 0 \Rightarrow 4x - 4 = 0, \quad 6y = 0 \Rightarrow (x, y) = (1, 0) \]
\[ f_{xx} = 4, \quad f_{xy} = 0, \quad f_{yy} = 6 \]

At \((1, 0)\),
\[ D = \begin{vmatrix} 4 & 0 \\ 0 & 6 \end{vmatrix} = 24 > 0, \quad f_{xx}(1, 0) = 4 > 0. \]

So \( f(1, 0) = -7 \) is a local minimum.

Now the boundary is \( g(x, y) = x^2 + y^2 = 16 \).
\[ \nabla g = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle \]
\[ \nabla f = \lambda \nabla g, \quad g = 16 \Rightarrow 4x - 4 = \lambda 2x, \quad 6y = \lambda 2y, \quad x^2 + y^2 = 16 \]
\[ 6y = \lambda 2y \Rightarrow \lambda = 3 \text{ or } y = 0 \]

If \( \lambda = 3 \),
\[ 4x - 4 = 6x \Rightarrow x = -2 \Rightarrow x^2 + y^2 = 16 \Rightarrow y = \pm \sqrt{12} \]
\[ (x, y) = (-2, \sqrt{12}) \text{ or } (-2, -\sqrt{12}) \]

If \( y = 0 \),
\[ x^2 + y^2 = 16 \Rightarrow x = \pm 4 \]
\[ (x, y) = (4, 0) \text{ or } (-4, 0) \]
\[ f(-2, \pm \sqrt{12}) = 47, \quad f(4, 0) = 11, \quad f(-4, 0) = 43 \]

Therefore the absolute maximum is \( f(-2, \pm \sqrt{12}) = 47 \), the absolute minimum is \( f(1, 0) = -7 \).
14.8.27 Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter \( p \) is a square.

Let \( x \) be the width and \( y \) be the height of the rectangle. Then \( g(x, y) = 2x + 2y = p \) and the area is \( A = xy \).

\[
\nabla A = \langle A_x, A_y \rangle = \langle y, x \rangle
\]

\[
\nabla g = \langle g_x, g_y \rangle = \langle 2, 2 \rangle
\]

At the maximum,

\[
\nabla A = \lambda \nabla g, \quad g = p \Rightarrow y = \lambda 2, \quad x = \lambda 2, \quad 2x + 2y = p.
\]

\[
\Rightarrow x = 2\lambda = y
\]

So the rectangle should be a square.

14.8.32 Find the points on the surface \( y^2 = 9 + xz \) that are closest to the origin.

Note that in this problem obviously there exists an absolute minimum.

Let \( f(x, y, z) \) be the distance square from the origin to \((x, y, z)\). Then

\[
f(x, y, z) = d^2 = x^2 + y^2 + z^2.
\]

\[
g(x, y, z) = y^2 - 9 - xz = 0
\]

We are finding the point \((x, y, z)\) giving the minimum of \( f(x, y, z) \) on \( g(x, y, z) = 0 \).

\[
\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x, 2y, 2z \rangle
\]

\[
\nabla g = \langle g_x, g_y, g_z \rangle = \langle -z, 2y, -x \rangle
\]

\[
\nabla f = \lambda \nabla g, \quad g = 0 \Rightarrow 2x = -\lambda z, \quad 2y = \lambda 2y, \quad 2z = -\lambda x, \quad y^2 - 9 - xz = 0
\]

\[
2y = \lambda 2y \Rightarrow y = 0 \text{ or } \lambda = 1
\]

If \( y = 0, \)

\[
2x = -\lambda z \Rightarrow x = \frac{-\lambda}{2} z \Rightarrow 2z = -\lambda x = \frac{\lambda^2}{2} z \Rightarrow z = 0 \text{ or } \lambda = \pm 2
\]

\( z = 0 \) is impossible because \( y^2 - 9 - xz = 0, \quad y = 0 \Rightarrow xz = -9. \)

\[
\lambda = 2 \Rightarrow x = -z \Rightarrow -9 + x^2 = 0 \Rightarrow x = \pm 3
\]

\[(x, y, z) = (3, 0, -3) \text{ or } (-3, 0, 3)\]

\[
\lambda = -2 \Rightarrow x = z \Rightarrow -9 - x^2 = 0
\]

This is impossible because the left hand side is negative.

If \( \lambda = 1, \)

\[
2x = -z, \quad 2z = -x \Rightarrow x = z = 0 \Rightarrow y^2 - 9 = 0 \Rightarrow y = \pm 3
\]
\[(x, y, z) = (0, 3, 0) \text{ or } (0, -3, 0)\]

\[f(3, 0, -3) = f(-3, 0, 3) = 18, \, f(0, 3, 0) = f(0, -3, 0) = 9\]

Therefore the absolute minimum is 9.

14.8.36 Find the dimensions of the box with volume 1000 cm\(^3\) that has minimal surface area.

Let \(x, y, z\) be the length, width, and height of the box respectively. Then \(V = xyz = 1000\). The surface area is \(A = 2xy + 2yz + 2zx\).

\[
\nabla A = (A_x, A_y, A_z) = (2y + 2z, 2x + 2z, 2x + 2y)
\]

\[
\nabla V = (V_x, V_y, V_z) = (yz, xz, xy)
\]

From \(xyz = 1000\), we know that none of \(x, y, z\) is zero.

\[
\nabla A = \lambda \nabla V, \, V = 1000 \Rightarrow 2y + 2z = \lambda yz, \, 2x + 2z = \lambda xz, \, 2x + 2y = \lambda xy, \, xyz = 1000
\]

\[
\Rightarrow \frac{2}{y} + \frac{2}{z} = \lambda, \, \frac{2}{z} + \frac{2}{x} = \lambda, \, \frac{2}{y} + \frac{2}{x} = \lambda
\]

\[
\Rightarrow \frac{2}{y} + \frac{2}{z} = \frac{2}{z} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y} \Rightarrow \frac{2}{y} = \frac{2}{z} = \frac{2}{x} \Rightarrow x = y = z
\]

\[xyz = 1000 \Rightarrow x = y = z = \sqrt[3]{1000}\]

The minimum surface area is \(A(\sqrt[3]{1000}, \sqrt[3]{1000}, \sqrt[3]{1000}) = 6(1000)^{\frac{2}{3}} = 600\) (cm\(^2\)).

14.8.41 If the length of the diagonal of a rectangular box must be \(L\), what is the largest possible volume?

Let \(x, y, z\) be the length, width, and height of the box, respectively. Then \(L = \sqrt{x^2 + y^2 + z^2}\). Let \(g(x, y, z) = x^2 + y^2 + z^2 = L^2\). The volume is \(V = xyz\).

\[
\nabla V = (V_x, V_y, V_z) = (yz, xz, xy)
\]
\[
\nabla g = (g_x, g_y, g_z) = (2x, 2y, 2z)
\]

\[
\nabla V = \lambda \nabla g, \, g = L^2 \Rightarrow yz = \lambda 2x, \, xz = \lambda 2y, \, xy = \lambda 2z, \, x^2 + y^2 + z^2 = L^2
\]

\[
\Rightarrow xyz = 2\lambda x^2, \, xyz = 2\lambda y^2, \, xyz = 2\lambda z^2
\]

\[
\Rightarrow 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2 \Rightarrow \lambda = 0 \text{ or } x^2 = y^2 = z^2
\]

If \(\lambda = 0\),

\[
yz = 0 \Rightarrow y = 0 \text{ or } z = 0 \Rightarrow V = xyz = 0
\]

So obviously it does not give the absolute maximum.

If \(x^2 = y^2 = z^2\),

\[
3x^2 = L^2 \Rightarrow x^2 = y^2 = z^2 = \frac{L^2}{3}
\]

In this case,

\[
V = xyz = \sqrt{x^2y^2z^2} = \sqrt{\left(\frac{L^2}{3}\right)^3} = \sqrt{\frac{L^6}{27}} = \frac{L^3}{\sqrt{27}}.
\]