Classical Invariant Theory and Birational Geometry of Moduli Spaces

How not to prove the S_n-invariant F-conjecture

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Part I

Invariant theory

An old question

Question

Let A be a finitely generated algebra (over a field k or $\mathbb{Z}$). Let G be a subgroup of $\mathrm{Aut}(A)$. Calculate the invariant subring

$$A^G := \{x \in A \mid g(x) = x,\ \forall g \in G\}.$$

Mainly, we will work on a special situation:

V: G-respresentation

$k[V]$: ring of polynomial functions on V

There is an induced G-action on $k[V]$.

Question

Calculate $k[V]^G$.

First example - Gauss

Question (Gauss, 1801)

How is a binary quadratic form with integer coefficients affected by a linear transformation of its variables?

$$ax^2 + 2bxy + cy^2, \quad a, b, c \in \mathbb{Z}$$

$$\begin{array}{rcl} x & = & \alpha x' + \beta y' \\ y & = & \gamma x' + \delta y', \end{array} \ \alpha, \beta, \gamma, \delta \in \mathbb{Z} \Rightarrow a'(x')^2 + 2b'x'y' + c'(y')^2$$

$$(b')^2 - a'c' = (b^2 - ac)(\alpha\delta - \beta\gamma)^2$$

If $\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \in \mathrm{SL}_2$, $(b')^2 - a'c' = b^2 - ac \cdots$ discovery of an invariant

tried to classify binary quadratic forms

Example - Symmetric group

Consider the polynomial ring $k[X_1, X_2, \cdots, X_n]$.

There is a natural S_n-action permuting the variables.

Examples of S_n-invariants:

$$
\begin{aligned}
E_1 &:= X_1 + X_2 + \cdots + X_n, \\
E_2 &:= \sum_{i<j} X_i X_j, \\
E_3 &:= \sum_{i<j<k} X_i X_j X_k, \\
&\vdots \\
E_n &:= X_1 X_2 \cdot \cdots \cdot X_n
\end{aligned}
$$

Theorem (Fundamental theorem of symmetric polynomials, Gauss, 1815)

$k[X_1, X_2, \cdots, X_n]^{S_n} \cong k[E_1, E_2, \cdots, E_n]$.

Example - Linear algebra

$V = \mathrm{M}_{n\times n}$: set of $n \times n$ matrices, $G = \mathrm{GL}_n$

There is a conjugate action on V defined by $\sigma \cdot A := \sigma^{-1} A \sigma$ (a basis change).

$k[V] = k[X_{11}, X_{12}, \cdots, X_{nn}]$

Examples of invariants:

$$\text{trace} : X_{11} + X_{22} + \cdots + X_{nn},$$

$$\text{determinant} : \sum_{\tau \in S_n} \mathrm{sgn}(\tau) \prod_{i=1}^{n} X_{i\tau(i)}.$$

More generally, coefficients of the characteristic polynomial are invariants.

Theorem

As a k-algebra, $k[V]^G$ is generated by coefficients of the characteristic polynomials. Therefore $k[V]^G \cong k[a_1, a_2, \cdots, a_n]$.

Example - Homogeneous forms

$V_d := \{a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d\}$: set of degree d homogeneous polynomials of degree d with two variables X, Y

$k[V_d] = k[a_0, a_1, \cdots, a_d] \curvearrowleft \mathrm{SL}_2$

$k[V_2]^{\mathrm{SL}_2} = k[a_1^2 - 4a_0a_2]$

$k[V_3]^{\mathrm{SL}_2} = k[a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3]$

$k[V_4]^{\mathrm{SL}_2}$ is generated by f_2 and f_3, where

$$f_2 = a_0a_4 - \frac{a_1a_3}{4} + \frac{a_2^2}{12}, \quad f_3 = \begin{vmatrix} a_0 & a_1/4 & a_2/6 \\ a_1/4 & a_2/6 & a_3/4 \\ a_2/6 & a_3/4 & a_4 \end{vmatrix}.$$

$k[V_5]^{\mathrm{SL}_2}$ is generated by f_4, f_8, f_{12}, and f_{18}, where

$$\begin{aligned} f_4 = & -2a_2^2a_3^2 + 6a_1a_3^3 + 6a_2^3a_4 - 19a_1a_2a_3a_4 - 15a_0a_3^2a_4 \\ & + 9a_1^2a_4^2 + 40a_0a_2a_4^2 - 15a_1a_2^2a_5 + 40a_1^2a_3a_5 \\ & + 25a_0a_2a_3a_5 - 250a_0a_1a_4a_5 + 625a_0^2a_5^2, \end{aligned}$$

and $f_8 = ☹$, etc.

Example - Homogeneous forms

$k[V_6]^{\mathrm{SL}_2}$ is generated by f_2, f_4, f_6, f_{10}, and f_{15}.

Cayley (1856): $k[V_7]^{\mathrm{SL}_2}$ is not finitely generated.

Gordan (1868): $k[V_d]^{\mathrm{SL}_2}$ is finitely generated for all d.

$k[V_7]^{\mathrm{SL}_2}$ is generated by 30 generators (Dixmier-Lazard, 1986).

$k[V_8]^{\mathrm{SL}_2}$ is generated by 9 generators (Shioda, 1967).

$k[V_9]^{\mathrm{SL}_2}$ is generated by 92 generators (Brouwer-Popoviciu, 2010).

$k[V_{10}]^{\mathrm{SL}_2}$ is generated by 106 generators (Brouwer-Popoviciu, 2010).

$k[V_d]^{\mathrm{SL}_2}$ is unknown for $d \geq 11$.

Theorem (Hilbert, 1890)

Let G be a linearly reductive group. Then for any finite dimensional G-representation V, $k[V]^G$ is finitely generated.

Fundamental theorems of invariant theory

$V = k^d$

$\mathrm{SL}_d \curvearrowright V \Rightarrow \mathrm{SL}_d \curvearrowright V^n = V \oplus V \oplus \cdots \oplus V$

$k[V^n] = k[X_{ij}]_{1 \le i \le d, 1 \le j \le n}$

Invariants: $[j_1, j_2, \cdots, j_d] := \det(X_{ij_1}, X_{ij_2}, \cdots, X_{ij_d})$

Theorem (First fundamental theorem of invariant theory)

The invariant subring $k[V^n]^{\mathrm{SL}_d}$ is generated by $[j_1, j_2, \cdots, j_d]$ for all multiindices $1 \le j_1 < j_2 < \cdots < j_d \le n$.

Theorem (Second fundamental theorem of invariant theory)

The ideal of relations in $k[V^n]^{\mathrm{SL}_d}$ is generated by Plücker relations:

$$\sum_{k=1}^{d+1} (-1)^k [i_1, i_2, \cdots, i_{d-1}, j_k][j_1, \cdots, j_{k-1}, j_{k+1}, \cdots, j_{d+1}].$$

True over any base ring (de Concini-Procesi).

SL_2-case - Graphical algebra

$V = k^2 \curvearrowleft \mathrm{SL}_2$

$k[V^n] = k\left[\begin{array}{cccc} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{array}\right]$

$[i, j] = X_i Y_j - X_j Y_i =: Z_{ij}$

First fundamental theorem $\Rightarrow$ $k[V^n]^{\mathrm{SL}_2}$ is generated by Z_{ij}

Second fundamental theorem $\Rightarrow$ two types of relations:

- $Z_{ji} = -Z_{ij}$
- Plücker relations: $Z_{ij}Z_{kl} - Z_{ik}Z_{jl} + Z_{il}Z_{jk} = 0$

Definition

The graphical algebra of order n is the invariant ring $k[V^n]^{\mathrm{SL}_2}$.

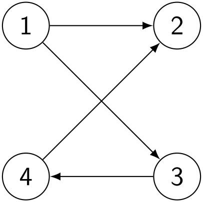
Graphical algebra - Combinatorial interpretation

Γ: a loopless directed graph on $[n] := \{1, 2, \cdots, n\}$

For an edge $e \in E_\Gamma$, $h(e)$: head of e, $t(e)$: tail of e.

For each Γ, let

$$Z_\Gamma := \prod_{e \in E_\Gamma} Z_{t(e)h(e)} = \prod_{e \in E_\Gamma} (X_{t(e)}Y_{h(e)} - X_{h(e)}Y_{t(e)}).$$



Γ

$$\begin{aligned} Z_\Gamma &= Z_{12}Z_{13}Z_{42}Z_{34} \\ &= (X_1Y_2 - X_2Y_1)(X_1Y_3 - X_3Y_1)(X_4Y_2 - X_2Y_4)(X_3Y_4 - X_4Y_3) \end{aligned}$$

Graphical algebra - Combinatorial interpretation

Γ, Γ': two graphs on $[n]$

$\Gamma \cdot \Gamma' :=$ the disjoint union of two graphs

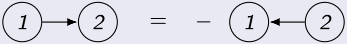
G_n: monoid of all directed graphs on $[n]$.

$Z_\Gamma \cdot Z_{\Gamma'} = Z_{\Gamma \cdot \Gamma'}$

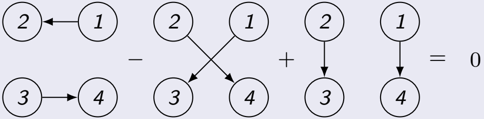
Theorem

The graphical algebra $k[V^n]^{\mathrm{SL}_2}$ is $k[G_n]$ modulo two types of relations:

- *Orientation reversing*



- *Plücker relations*



$$k[V^n] = k[X_i, Y_i] = \bigoplus_{\overrightarrow{d}} \mathrm{H}^0((\mathbb{P}^1)^n, \mathcal{O}(d_1, d_2, \cdots, d_n))$$

$$k[V^n]^{\mathrm{SL}_2} = k[X_i, Y_i]^{\mathrm{SL}_2} = \bigoplus_{\overrightarrow{d}} \mathrm{H}^0((\mathbb{P}^1)^n, \mathcal{O}(d_1, d_2, \cdots, d_n))^{\mathrm{SL}_2}$$

The latter one is the ring of all functions (divisors) on $(\mathbb{P}^1)^n//\mathrm{SL}_2$, which is, the Cox ring of $(\mathbb{P}^1)^n//\mathrm{SL}_2$.

Connection to moduli spaces?

There are (at least) two ways to interpret $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$:

1. A moduli space of n ordered marked points on $\mathbb{P}^1$ up to projective equivalence

   We will discuss the relation with the nef cone of $\overline{\mathrm{M}}_{0,n}$.

2. A moduli space of rank 2, degree 0 parabolic bundles over $\mathbb{P}^1$

$$
\begin{aligned}
(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2 &= \{(p_1, \cdots, p_n) \mid p_i \in \mathbb{P}^1\}/_\sim \\
&= \{(V_1, \cdots, V_n) \mid V_i \subset k^2, \dim V_i = 1\}/_\sim \\
&= \{(E = \mathcal{O}_{\mathbb{P}^1}^2, (V_1, \cdots, V_n)) \mid V_i \subset E|_{x_i}\}/_\sim
\end{aligned}
$$

Theorem (M-Yoo)

Let $\mathrm{M}(\mathbf{a}, d)$ be the moduli space of $\mathbf{a}$-stable rank 2, degree d vector bundles on $\mathbb{P}^1$. For a general $\mathbf{a}$, the effective cone of $\mathrm{M}(\mathbf{a}, 0)$ is generated by 2^{n-1} extremal rays. ($\rho(\mathrm{M}(\mathbf{a}, 0)) = n + 1$).

Part II

$\overline{\mathrm{M}}_{0,n}$ and the F-conjecture

Positivity of divisors

Let D be an integral Cartier divisor on a projective variety X.

Let $\varphi_D : X \dashrightarrow \mathbb{P}^n$ be the map associated to $|D|$.

Definition

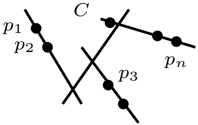
1. A divisor D is very ample if φ_D is an embedding.
2. A divisor D is ample if mD is very ample for some $m > 0$.
3. A divisor D is base-point-free if φ_D is regular.
4. A divisor D is semi-ample if mD is base-point-free for some $m > 0$.
5. A divisor D is nef if $D \cdot C \geq 0$ for every integral curve C on X.

$$\begin{tikzcd}
\text{ample} \arrow[r, Rightarrow] & \text{semi-ample} \arrow[r, Rightarrow] & \text{nef} \\
\text{very ample} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{base-point-free} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Question

Compute the cone of positive (ample, semi-ample, nef) divisors.

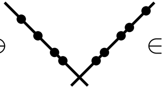
$\overline{\mathrm{M}}_{0,n} := \Bigg\{$



$\Bigg|$ C : connected projective curve
with at worst nodal singularities
p_i : distinct smooth points
$|\mathrm{Aut}| < \infty$ $\Bigg\}/\sim$

$\cdots$ moduli space of n-pointed stable rational curves.

$\exists$ a natural stratification indexed by the topological types of parametrized curves ($=$ dual graphs).

- $\overline{\Big\{ \; I \ni$  $\in I^c \; \Big\}} =: D_I \cdots$ boundary divisor
- The closure of any stratum is an intersection of boundary divisors, and is isomorphic to a product of $\overline{\mathrm{M}}_{0,k}$.
- F-curve: the closure of a one-dimensional stratum
- F-point: zero-dimensional stratum

F-conjecture

Guess: The geometry of $\overline{\mathrm{M}}_{0,n}$ is very similar to that of toric varieties.
Evidence (Kapranov): $\overline{\mathrm{M}}_{0,n}$ is an iterative blow-up of points, lines, planes, $\cdots$ of $\mathbb{P}^{n-3}$.

Conjecture (F-conjecture)

1. *The cone of curves of $\overline{\mathrm{M}}_{0,n}$ is generated by F-curves.*
2. *A divisor D on $\overline{\mathrm{M}}_{0,n}$ is nef if and only if $D \cdot F \geq 0$ for every F-curve F (F-nef).*

It turned out that $\overline{\mathrm{M}}_{0,n}$ is very different from toric varieties! (Vermeire, Castravet-Tevelev $\times 2$, Doran-Giansiracusa-Jensen, etc.)

There is a natural S_n-action on $\overline{\mathrm{M}}_{0,n}$.

Conjecture (S_n-invariant F-conjecture)

For an S_n-invariant divisor D, D is nef if and only if D is F-nef.

S_n-invariant F-conjecture for $\overline{\mathrm{M}}_{0,n}$ $\Rightarrow$ nef cone of $\overline{\mathrm{M}}_n$ (Gibney-Keel-Morrison)

Known results

Theorem (Keel-McKernan, 96)

The F-conjecture is true for $n \le 7$ in char 0.

- Studied consequences of the following situation: $\exists$ an ample divisor whose support is $D = \sum D_I$, and each D_I has a negative normal bundle.
- (Ray theorem) If R is an extremal ray of the curve cone, G is an effective divisor supported on D, and $(K_{\overline{\mathrm{M}}_{0,n}} + G) \cdot R < 0$, then R is generated by a curve $\subset D$.
- Used standard theorems in birational geometry in char 0: Cone theorem and Contraction theorem.

Known results

Theorem (Gibney, 03)

The S_n-invariant F-conjecture is true for $n \le 24$ in char 0.

- Ray theorem $\Rightarrow$ Suppose that $E = cK_{\overline{\mathrm{M}}_{0,n}} + G$ for some effective boundary G and $c \ge 0$. If $C \cdot E < 0$, then $C \subset G$.
- If E is an F-nef divisor such that for each inclusion of a boundary $b : \overline{\mathrm{M}}_{0,k} \hookrightarrow \overline{\mathrm{M}}_{0,n}$, $b^*(E) = c' K_{\overline{\mathrm{M}}_{0,k}} + G'$, then E is nef.
- Verified this condition for S_n-invariant F-nef divisors using a computer program.

Known results

Theorem (Fedorchuk, 14)

Over any field, the S_n-invariant F-conjecture is true for $n \leq 16$.

- Every divisor can be written as $D = \sum_{i=1}^{n} a_i \psi_i - \sum_I b_I D_I$.
- Studied carefully relations in the Picard group and the change of a_i, b_I when D is restricted to boundaries
- Combinatorial conditions on a_i, b_I for being stratally effective boundary ($= D$ is an effective linear combination of boundaries after taking the restriction to each boundary) and being boundary semi-ample ($=$ the sub linear system of $|mD|$ generated by effective linear combinations of boundaries is base-point-free).
- Defined several collections of functions describing the coefficients a_i and b_I of stratally effective boundary and boundary semi-ample divisors.
- Provided infinitely many examples.

Known results

The moduli space $\overline{\mathrm{M}}_{0,n}$ is embedded into a (non-proper) toric variety X_Δ where Δ is a fan, the so-called space of phylogenetic trees.

Let $i : \overline{\mathrm{M}}_{0,n} \hookrightarrow X_\Delta$ be the embedding.

- G_Δ: cone of semi-ample divisors on X_Δ
- L_Δ: cone of divisors on X_Δ so that the restriction to each orbit closure is effective
- F_Δ: cone of divisors on X_Δ so that the restriction to each $(d+1)$-dimensional orbit closure ($d =$ dimension of minimal orbits) is effective

Theorem (Gibney-Maclagan, 10)

$$i^*(G_\Delta) \subset i^*(L_\Delta) \subset \mathrm{Nef}(\overline{\mathrm{M}}_{0,n}) \subset i^*(F_\Delta)$$

and $i^*(F_\Delta)$ is the cone of F-nef divisors.

Proved the equality for $n \leq 6$.

Main theorem

$$\begin{tikzcd} \text{ample} \arrow[r, Rightarrow] & \text{semi-ample} \arrow[r, Rightarrow] & \text{nef} \\ \text{very ample} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{base-point-free} \arrow[u, Rightarrow] & \end{tikzcd}$$

Theorem (M-Swinarski)

Over $\operatorname{Spec}\mathbb{Z}$*, for every* S_n*-invariant F-nef integral divisor* D*, the following statements are true.*

1. *If* $n \le 19$, D *is semi-ample.*
2. *If* $n \le 16$, $2D$ *is base-point-free.*

Corollary

If $n \le 16$ *and* D *is ample, then* $2D$ *is very ample.*

It follows from the very ampleness of $K_{\overline{\mathrm{M}}_{0,n}} + D$ (Keel-Tevelev).

New conjectures

Conjecture (Swinarski)

Over $\operatorname{Spec} \mathbb{Z}$*, for every* S_n*-invariant F-nef integral divisor* D*,* $2D$ *is base-point-free.*

true for $n \leq 16$

Conjecture (M)

Over $\operatorname{Spec} \mathbb{Z}$*, for every* S_n*-invariant F-nef integral divisor* D*,* D *is base-point-free.*

true for $n \leq 11$ and 13, only one unknown case for each $n = 12, 14, 15$

Part III

Graphical algebra and the S_n-invariant F-conjecture

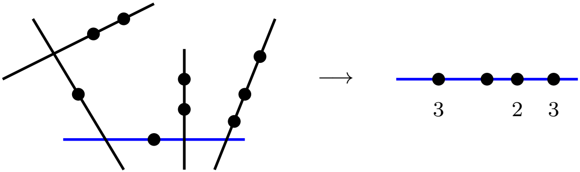
Kapranov's morphism

Theorem (Kapranov, 93)

There is a birational morphism $\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$.

$\overline{\mathrm{M}}_{0,n}$: moduli space of possibly singular curves with distinct marked points.

$(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$: moduli space of smooth curves with not necessarily distinct marked points.



Kapranov's morphism

Assume n is odd. (When n is even, need to do some extra work around strictly semistable locus.)

A subset $I \subset [n]$ always satisfies $2 \le |I| \le n/2$.

Observation 1. The morphism π preserves stratifications.
$\pi(D_I) = V(Z_{ij})_{i,j \in I} = \{(p_1, p_2, \cdots, p_n) \mid p_i = p_j, \ \forall i, j \in I\} =: V_I$
(here $Z_{ij} = X_i Y_j - X_j Y_i$)
$|I| = 2 \Rightarrow V_I$ is a divisor
$|I| > 2 \Rightarrow V_I$ is a higher codimensional subvariety

Observation 2. Any S_n-invariant F-nef divisor D can be written uniquely as

$$D = \pi^*(cL) - \sum_{i \ge 3} a_i D_i$$

$L = \sum_{|I|=2} V_I$: an ample divisor in $\mathrm{Pic}((\mathbb{P}^1)^n // \mathrm{SL}_2)^{S_n}$
$D_i = \sum_{|I|=i} D_I, \quad c, a_i \ge 0$

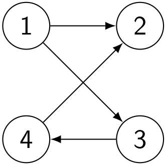
Identification of linear systems

$$D = \pi^*(cL) - \sum_{i \ge 3} a_i D_i$$

$$|D| = |\pi^*(cL) - \sum_{i \ge 3} a_i D_i| \cong |cL|_{\mathbf{a}} \subset |cL|$$

where $|cL|_{\mathbf{a}} := \{E \in |cL| \mid \text{multiplicity along } V_I \ge a_{|I|}\}$.
$\mathbf{a} := (a_3, a_4, \cdots, a_{\lfloor n/2 \rfloor})$

Recall that for each directed graph Γ on $[n]$, we can assign a monomial $Z_\Gamma := \prod_{\overrightarrow{ij} \in E_\Gamma} Z_{ij}$, so a divisor on $(\mathbb{P}^1)^n // \mathrm{SL}_2$.



Γ

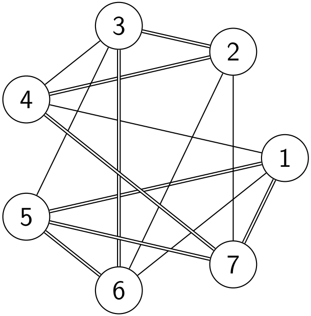
$$\Rightarrow \quad Z_\Gamma = Z_{12} Z_{13} Z_{42} Z_{34}$$

Combinatorial interpretation

$$|cL|_{\mathbf{a}} := \{E \in |cL| \mid \text{multiplicity along } V_I \geq a_{|I|}\}$$

$Z_\Gamma \in |cL|_{\mathbf{a}}$ if

1. For each $i \in [n]$, the valence of i (the number of edges incident to i) is $c(n-1)$ ($\Leftrightarrow Z_\Gamma \in |cL|$);
2. For each $I \subset [n]$, the number of edges connecting $i, j \in I$ is at least $a_{|I|}$ ($\Leftrightarrow Z_\Gamma \in |cL|_{\mathbf{a}}$).



Γ

$$\Rightarrow \qquad Z_\Gamma \in |L|_{a_3=2}$$

Combinatorial interpretation

$$\begin{array}{ccc} |D| & \Longleftrightarrow & |cL|_{\mathbf{a}} \\ \uparrow & & \uparrow \\ |D|_G & \Longleftrightarrow & |cL|_{\mathbf{a},G} \end{array}$$

$|cL|_{\mathbf{a},G}$: sub linear system of $|cL|_{\mathbf{a}}$ generated by graphical monomials.

$|D|_G$: sub linear system of $|D|$ which is identified with $|cL|_{\mathbf{a},G}$.

Definition

An S_n-invariant F-nef divisor D is G-base-point-free if $|D|_G$ is base-point-free.

- D is G-base-point-free $\Rightarrow$ D is base-point-free.
- $\mathrm{Bs}(|D|_G)$ is a union of boundary strata.
- If $\mathrm{Bs}(|D|_G)$ is nonempty, there must be an F-point in $\mathrm{Bs}(|D|_G)$.

Combinatorial interpretation

D is G-base-point-free
$\Leftrightarrow \forall$ F-point $F = \cap_{I \in T} D_I$, $\exists\ E \in |D|_G$, $F \notin E$
$\Leftrightarrow \forall$ F-point $F = \cap_{I \in T} D_I$, $\exists\ E' \in |cL|_{\mathbf{a},G}$, E' has the minimum multiplicity along V_I for $I \in T$.
$\Leftrightarrow \forall$ F-point $F = \cap_{I \in T} D_I$, $\exists\ Z_\Gamma \in |cL|_{\mathbf{a},G}$, Z_Γ has the minimum multiplicity ($= a_{|I|}$) along V_I for $I \in T$.

Proposition

For an S_n-invariant F-nef divisor $D = \pi^*(cL) - \sum_{i \ge 3} a_i D_i$, $|D|$ is G-base-point-free if for every F-point $F = \cap_{I \in T} D_I$, there is a graph Γ such that:

1. *For each $i \in [n]$, the valence of i is $c(n-1)$;*
2. *For each $I \subset [n]$, the number of edges connecting $i, j \in I$ is at least $a_{|I|}$;*
3. *For each $I \in T$, the number of edges connecting $i, j \in I$ is precisely $a_{|I|}$.*

Feasibility problem

For each $c, a_3, a_4, \cdots, a_{\lfloor n/2 \rfloor}$,

What we need to find: for each F-point $F = \cap_{I \in T} D_I$, a graph weighting $w : E_{K_n} \to \mathbb{Z}$, such that:

1. $w(ij) \geq 0$;
2. For each $i \in [n]$, $\sum_{j \neq i} w(ij) = c(n-1)$;
3. For each $I \subset [n]$, $\sum_{i,j \in I} w(ij) \geq a_{|I|}$;
4. For each $I \in T$, $\sum_{i,j \in I} w(ij) = a_{|I|}$.

This is a feasibility problem ($=$ the nonemptiness of a polytope).

Computational problem:

1. Choose an F-point F for each S_n-orbit. Let P be the set of such representatives.
2. For each $F \in P$, make a polytope $Q(F)$ described by the above equations and inequalities.
3. Check $Q(F) \neq \emptyset$.

Final remarks

1. This computation is faster than expectation: For instance, for small n, for most of extremal rays of the S_n-invariant F-nef cone, $\cap_{F \in P} Q(F) \neq \emptyset$.
2. The cone of G-semi-ample divisors is a polyhedral lower bound of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})^{S_n}$.
3. D is G-semi-ample $\Leftrightarrow$ D is S_n-invariant boundary semi-ample (Fedorchuk) $\Leftrightarrow$ $D \in i^*(G_\Delta)^{S_n}$ (Gibney-Maclagan)

   So for $n \le 19$,

$$i^*(G_\Delta)^{S_n} = i^*(L_\Delta)^{S_n} = \mathrm{Nef}(\overline{\mathrm{M}}_{0,n})^{S_n} = i^*(F_\Delta)^{S_n}.$$

Thank you!