Classical Invariant Theory and Birational Geometry of Moduli Spaces

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November 4, 2015

Part I

Invariant theory

Toy example

Let $\mathbb{Q}[x, y]$ be the polynomial ring with two variables.

Define $\mathbb{Z}/2\mathbb{Z} = \langle\sigma\rangle$-action on $\mathbb{Q}[x, y]$ by $\sigma \cdot x = -x, \ \sigma \cdot y = -y$.

For instance, $\sigma \cdot x^3 = (-x)^3 = -x^3$, $\sigma \cdot xy = (-x)(-y) = xy$.

Question

Find all polynomials such that $\sigma \cdot f = f$.

Toy example

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Find all polynomials such that $\sigma \cdot f = f$.

Example: x^2, xy, y^2, any polynomial $f(x^2, xy, y^2)$.

Answer: The set of such polynomials forms a subring

$$\mathbb{Q}[x^2, xy, y^2] \cong \mathbb{Q}[a, b, c]/\langle ac - b^2 \rangle.$$

Old question

G: group

V: G-respresentation, i.e., a vector space over k equipped with a linear G-action

$k[V]$: ring of polynomial functions on V

There is an induced G-action on $k[V]$. We say $f \in k[V]$ is a G-invariant (or simply invariant) if for every $\sigma \in G$, $\sigma \cdot f = f$. Or equivalently, $f(\sigma \cdot v) = f(v)$.

$k[V]^G$: subring of G-invariants

Question

Describe $k[V]^G$*.*

First example - Symmetric group

$G = S_n$, $V = k^n$ with a standard basis $\{v_1, v_2, \cdots, v_n\}$

S_n acts on V as permutations of $\{v_i\}$.

$k[V] = k[x_1, \cdots, x_n]$, and S_n-action on $k[V]$ is a permutation of $\{x_i\}$.

Examples of S_n-invariants:

$$
\begin{aligned}
e_1 &:= x_1 + x_2 + \cdots + x_n, \\
e_2 &:= \sum_{i<j} x_i x_j, \\
e_3 &:= \sum_{i<j<k} x_i x_j x_k, \\
&\vdots \\
e_n &:= x_1 x_2 \cdot \cdots \cdot x_n
\end{aligned}
$$

Theorem (Gauss, 1815)

As a k-algebra, $k[V]^{S_n}$ is generated by $e_1, e_2, \cdots, e_n$.

More examples - Linear algebra

$V = \mathrm{M}_{n\times n}$: set of $n \times n$ matrices, $G = \mathrm{GL}_n$

There is a conjugation action on V defined by $\sigma \cdot A := \sigma A \sigma^{-1}$ (basis change!).

$k[V] = k[x_{11}, x_{12}, \cdots, x_{nn}]$

Examples of G-invariants:

$$\text{trace} : x_{11} + x_{22} + \cdots + x_{nn},$$

$$\text{determinant} : \sum_{\tau \in S_n} sgn(\tau) \prod_{i=1}^{n} x_{i\tau(i)}.$$

More generally, coefficients of the characteristic polynomial are invariants.

Theorem

As a k-algebra, $k[V]^G$ is generated by coefficients of the characteristic polynomials. Therefore $k[V]^G \cong k[a_1, a_2, \cdots, a_n]$.

More examples - Homogeneous polynomials

k: char 0

$V_d = \{a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d\}$: set of degree d homogeneous polynomials of degree d with two variables x, y

$k[V_d] = k[a_0, a_1, \cdots, a_d]$

$G = \mathrm{SL}_2$ acts on V_d by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot g(x,y) = g(\alpha x + \beta y, \gamma x + \delta y)$

$k[V_2]^{\mathrm{SL}_2} = k[a_1^2 - 4a_0a_2]$

$k[V_3]^{\mathrm{SL}_2} = k[a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3]$

$k[V_4]^{\mathrm{SL}_2} = k[f_2, f_3]$, where

$$f_2 = a_0a_4 - \frac{a_1a_3}{4} + \frac{a_2^2}{12},\ f_3 = \begin{vmatrix} a_0 & a_1/4 & a_2/6 \\ a_1/4 & a_2/6 & a_3/4 \\ a_2/6 & a_3/4 & a_4 \end{vmatrix}.$$

More examples - Homogeneous polynomials

$k[V_5]^{\mathrm{SL}_2} = k[f_4, f_8, f_{12}, f_{18}]$, where

$$
\begin{aligned}
f_4 = &-2a_2^2a_3^2 + 6a_1a_3^3 + 6a_2^3a_4 - 19a_1a_2a_3a_4 - 15a_0a_3^2a_4 \\
&+ 9a_1^2a_4^2 + 40a_0a_2a_4^2 - 15a_1a_2^2a_5 + 40a_1^2a_3a_5 \\
&+ 25a_0a_2a_3a_5 - 250a_0a_1a_4a_5 + 625a_0^2a_5^2.
\end{aligned}
$$

$k[V_6]^{\mathrm{SL}_2} = k[f_2, f_4, f_6, f_{10}, f_{15}]$

$k[V_7]^{\mathrm{SL}_2}$ is generated by 30 generators.

$k[V_8]^{\mathrm{SL}_2} = k[f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}]$

$k[V_9]^{\mathrm{SL}_2}$ is generated by 92 generators.

$k[V_d]^{\mathrm{SL}_2}$ is unknown for $d \geq 11$.

Hilbert's 14th problem

Question

Can one always find finitely many generators $f_1, \cdots, f_r$ *such that* $k[V]^G = k[f_1, \cdots, f_r]$*?*

Observation: If $k[V] = k[x_1, \cdots, x_n]$, then $k[x_1, \cdots, x_n]^G = k(x_1, \cdots, x_n)^G \cap k[x_1, \cdots, x_n]$.

Question (Hilbert's 14th problem)

Suppose that L/k *is a subfield of* $k(x_1, \cdots, x_n)$*. Is* $L \cap k[x_1, \cdots, x_n]$ *finitely generated?*

Answer (Nagata, 1959): No. There are G and V such that $k[V]^G$ is not finitely generated.

Hilbert's 14th problem

David Hilbert

Hilbert himself had a positive result.

Theorem (Hilbert, 1890)

If G is a linearly reductive group, then $k[V]^G$ is finitely generated.

G is linearly reductive if for every surjective morphism of G-representations $f : V \to W$, $f^G : V^G \to W^G$ is surjective.

Examples (in char 0): finite groups, $(\mathbb{C}^*)^n$, GL_n, SL_n, SO_n, Sp_n, $\cdots$

Proof

$S := k[V] \cdots$ a polynomial ring

J: ideal generated by positive degree invariants

By Hilbert's basis theorem, $J = (f_1, \cdots, f_n)$.

Claim: As a k-algebra, S^G is generated by $f_1, \cdots, f_n$.

The S-module honomorphism $\phi : S^n \to J$, defined by $(h_1, \cdots, h_n) \mapsto \sum h_i f_i$ is surjective.

We use induction on the degree. Pick $h \in S^G$. Then $h \in J \cap S^G = J^G$.

By the linear reductivity, $(S^G)^n \to J^G$ is surjective, so we have $h = \sum h_i f_i$ where $h_i \in S^G$.

$\deg h_i < \deg h$. By induction hypothesis, h_i is generated by $f_1, \cdots, f_n$.

So is h. $\square$

This is exactly the reason why Hilbert proved the famous basis theorem.

Main example

$V = (k^2)^n$, SL_2 acts as $\sigma \cdot (v_1, \cdots, v_n) := (\sigma \cdot v_1, \cdots, \sigma \cdot v_n)$

$k[V] = k[x_1, y_1, x_2, y_2, \cdots, x_n, y_n]$

On $k[V]$, SL_2 acts as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x_i, y_i) = (\alpha x_i + \beta y_i, \gamma x_i + \delta y_i)$.

Examples of invariants:

$$x_i y_j - x_j y_i$$

Indeed, $\sigma \cdot (x_i y_j - x_j y_i) = \det(\sigma)(x_i y_j - x_j y_i) = x_i y_j - x_j y_i$.

Theorem (First fundamental theorem of invariant theory)

The invariant ring $k[V]^{\mathrm{SL}_2}$ is generated by $(x_i y_j - x_j y_i)$.

This result is true even for arbitrary commutative ring A instead of k (De Concini-Procesi, 1976).

Graphical algebra - Combinatorial interpretation

Γ: a directed graph on n labeled vertices

$\mathbf{deg}\,\Gamma$: a sequence of degrees of vertices of Γ

For an edge $e \in E_\Gamma$, $h(e)$: head of e, $t(e)$: tail of e.

$\Gamma_1 \cdot \Gamma_2$: the union of two graphs on the same set of vertices

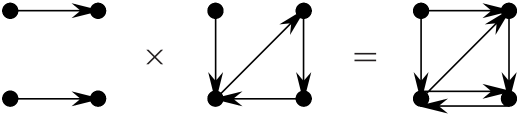
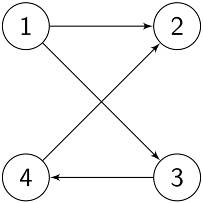


Figure from Howard, Millson, Snowden, Vakil, The equations for the moduli space of n points on the line

Graphical algebra - Combinatorial interpretation

For each Γ, let

$$X_\Gamma := \prod_{e \in E_\Gamma} (x_{h(e)} y_{t(e)} - x_{t(e)} y_{h(e)}).$$

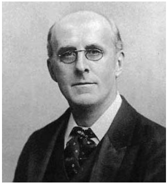


Γ

$$X_\Gamma = (x_2 y_1 - x_1 y_2)(x_3 y_1 - x_1 y_3)(x_2 y_4 - x_4 y_2)(x_4 y_3 - x_3 y_4)$$

By the fundamental theorem, as a vector space, $k[V]^{\mathrm{SL}_2}$ is generated by X_Γ.

Graphical algebra - Combinatorial interpretation



Alfred Kempe

$\mathbf{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{Z}^n_{\geq 0}$

$k[V]_{\mathbf{w}}$: subring of $k[V]$ consisting of multihomogeneous polynomials of multidegree $c\mathbf{w}$

$\mathbf{w} = \mathbf{1} = (1, \cdots, 1) \Rightarrow k[V]^{\mathrm{SL}_2}_{\mathbf{1}}$: generated by X_Γ for a regular graph Γ.

Theorem (Kempe, 1894)

The k-algebra $k[V]^{\mathrm{SL}_2}_{\mathbf{w}}$ is generated by the smallest degree elements.

The relations between generators have been computed recently (Howard-Millson-Snowden-Vakil, 2009).

Geometric interpretation

$\mathbf{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{Z}^n_{\geq 0}$

$k[V]_{\mathbf{w}}$: subspace of $k[V]$ consisting of multihomogeneous polynomials of multidegree $c\mathbf{w}$

$\mathbb{P}^1$: projective line $(k \cup \{\infty\})$

There is a natural SL_2-action on $\mathbb{P}^1$ (Möbius transform)

$k[V]_{\mathbf{w}}$: space of algebraic functions on $(\mathbb{P}^1)^n$ of multidegree $c\mathbf{w}$.

$$\begin{aligned} k[V]_{\mathbf{w}}^{\mathrm{SL}_2} &= \text{space of } \mathrm{SL}_2\text{-invariant algebraic functions on } (\mathbb{P}^1)^n \\ &= \text{space of algebraic functions on } (\mathbb{P}^1)^n/\mathrm{SL}_2 \end{aligned}$$

We use algebro-geometric quotient (or GIT quotient) $(\mathbb{P}^1)^n /\!/_{\mathbf{w}} \mathrm{SL}_2$.

Summary

Three aspects of the main example $k[V]^{\mathrm{SL}_2}$:

1. (Algebra) Invariant polynomials with respect to the SL_2-action
2. (Combinatorics) Directed labeled graphs
3. (Geometry) Functions on the quotient space $(\mathbb{P}^1)^n // \mathrm{SL}_2$

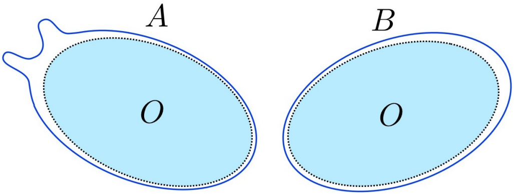
Part II

Birational geometry of moduli spaces

Birational geometry

One way to study a space (topological space, manifold, algebraic variety, etc.): compare it with other similar spaces.

Two spaces A, B are birational if they share a common open dense subset O. B is called a birational model of A.



$f : A \to B$ is called a birational morphism if it preserves O.

If B is simpler than A,

Geometric data of B $\Rightarrow$ Understand the geometry of A.

Mori's program

X: complex projective algebraic variety

Question

It is possible to classify all birational models of X?

No. By using an algebro-geometric surgery, the so-called blow-up, we can always make a birational model which is more complicate than X.

Question

Is it possible to classify all birational models of X which are equivalent or simpler than X?

Yes, if X is a Mori dream space.

There are many examples of Mori dream spaces.

Mori's program

An idea of the construction of a birational model is the following.

X: smooth complex projective variety

D: codimension 1 subvariety of X (it is called a divisor)

$L = \mathcal{O}(D)$: line bundle of X

$s_0, s_1, \cdots, s_k$: basis of the space of sections of L on X

Define a map $X \dashrightarrow \mathbb{P}^k$ as

$$x \mapsto (s_0(x) : s_1(x) : \cdots : s_k(x)).$$

By taking the closure of the image in $\mathbb{P}^k$, we obtain a projective variety.

Mori's program

Because of some technical reasons, we use an asymptote of the construction:

$$X(D) := \mathrm{Proj} \bigoplus_{m\geq 0} \mathrm{H}^0(X, \mathcal{O}(mD))$$

If D has some numerical positivity (the so-called bigness), then $X(D)$ is a birational model.

So it is important to understand the space of divisors, which is a convex cone in $\mathrm{H}^2(X, \mathbb{R})$.

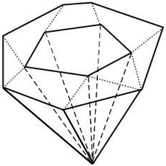
Mori's program

For a complex projective variety X, Mori's program consists of:

1. Study the cone of divisors $\mathrm{Eff}(X)$ in $\mathrm{H}^2(X, \mathbb{R})$;
2. For each divisor $D \in \mathrm{Eff}(X)$, compute $X(D)$;
3. Study the relation between X and $X(D)$.

If X is a Mori dream space, then

- For each $D \in \mathrm{Eff}(X)$, $X(D)$ is a well-defined projective variety;
- $\mathrm{Eff}(X)$ is a polyhedral cone;
- There is a finite chamber structure on $\mathrm{Eff}(X)$ which provides finitely many birational models.



Moduli spaces

A moduli space is a space parametrizing a certain kind of geometric objects.

Examples:

- Projective space $\mathbb{P}^{n-1}$: moduli space of one-dimensional sub vector spaces of k^n
- Grassmannian $Gr(r,n)$: moduli space of r-dimensional sub vector spaces of k^n
- moduli space of circles on a plane: $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$

Moduli spaces

More examples:

- $\mathrm{Hilb}(\mathbb{P}^n)$: moduli space of subschemes of $\mathbb{P}^n$
- $\overline{\mathcal{M}}_g$: moduli space of stable curves of genus g
- $\overline{\mathcal{M}}_{g,n}$: moduli space of stable n-pointed curves of genus g
- $\overline{\mathcal{M}}_g(\mathbb{P}^r, d)$: moduli space of stable genus g degree d curves in $\mathbb{P}^r$
- $M_C(r, d)$: moduli space of rank r stable vector bundles of degree d on a curve C

Mori's program of moduli spaces

Goal: Apply the framework of Mori's program to a given moduli space.

For a moduli space M,

1. Study the cone of divisors $\mathrm{Eff}(M)$ in $\mathrm{H}^2(M, \mathbb{R})$;
2. For each divisor $D \in \mathrm{Eff}(M)$, compute $M(D)$;
3. Study the relation between M and $M(D)$.

One reason why Mori's program of moduli spaces is interesting is that in many cases, birational models of M is also a moduli space of slightly different collection of geometric objects.

However, Step 1 is already very difficult in general.

Mori's program of moduli spaces

Example: Moduli space of curves in $\mathbb{P}^r$

Four ways to think a curve $C \subset \mathbb{P}^r$:

1. an inclusion map $\iota : C \to \mathbb{P}^r$
2. a defining ideal I_C
3. an $\mathcal{O}_{\mathbb{P}^r}$-module (sheaf) $\mathcal{O}_C$
4. a homological cycle $[C]$

They give four different compact moduli spaces of curves in $\mathbb{P}^r$:

1. $\overline{\mathcal{M}}_g(\mathbb{P}^r, d)$: (Kontsevich) moduli space of maps with finite automorphisms
2. $\mathrm{Hilb}^{dm+1-g}(\mathbb{P}^r)$: (Grothendieck) moduli space of ideals
3. $\mathrm{Simp}^{dm+1-g}(\mathbb{P}^r)$: (C. Simpson) moduli space of sheaves
4. $\mathrm{Chow}_{1,d}(\mathbb{P}^r)$: (Chow) moduli space of algebraic cycles

Mori's program of moduli spaces

For the moduli space $\overline{\mathcal{M}}_0(\mathbb{P}^3, 3)$ of moduli space of degree 3, genus 0 space curves, if we apply Mori's program, we can obtain

$$\overline{\mathcal{M}}_0(\mathbb{P}^3, 3), \quad \mathrm{Hilb}^{3m+1}(\mathbb{P}^3) = \mathrm{Simp}^{3m+1}(\mathbb{P}^3), \quad \mathrm{Chow}_{1,3}(\mathbb{P}^3),$$

and one more moduli space: the space of net of quadrics (Chen, 08).

Application of birational geometry

One application of birational geometry of moduli spaces in this viewpoint is the computation of topological invariants.

Example (Kiem-M, 10, Chung-Kiem, 11)

$$\begin{tikzcd}
\mathrm{M}_1 \arrow[d, "3"'] \arrow[dr, "2"] & & \mathrm{M}_2 \arrow[dl, "3"'] \arrow[dr, "3"] & \mathrm{Hilb}^{3m+1}(\mathbb{P}^r) \arrow[d, "1"] \\
\mathcal{Q}_0(\mathbb{P}^r, 3) & \mathcal{M}_0(\mathbb{P}^r, 3) & & \mathrm{Simp}^{3m+1}(\mathbb{P}^r)
\end{tikzcd}$$

The number on an arrow refers the number of blow-ups.

By evaluating Poincaré polynomial of $\mathcal{Q}_0(\mathbb{P}^r, 3)$, and measuring the difference of these moduli spaces, we obtain Poincaré polynomial of $\mathrm{Hilb}^{3m+1}(\mathbb{P}^r)$.

Part III

… and they came together

Parabolic bundles

A rank 2 parabolic bundle on $\mathbb{P}^1$ with parabolic points $\mathbf{p} = (p_1, \cdots, p_n)$ is a collection of data $(E, \{U_i\}, \mathbf{a})$ where

- E is a rank 2 vector bundle on $\mathbb{P}^1$;
- U_i is a 1-dimensional subspace of $E|_{p_i}$.
- $\mathbf{a} = (a_1, \cdots, a_n)$ where $a_i \in (0,1) \cap \mathbb{Q}$.

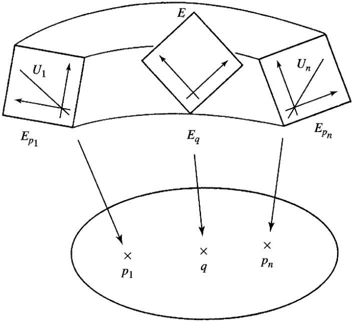


Figure from Mukai, Introduction to invariants and moduli

Moduli space of parabolic bundles

To make a compact Hausdorff moduli space, we need to impose a stability condition.

A parabolic bundle $(E, \{U_i\}, \mathbf{a})$ is (semi-)stable if for every sub line bundle $E' \subset E$,

$$\deg E' + \sum_{E'|_{p_i} = V_i} a_i \ (\leq) < \frac{1}{2}(\deg E + \sum a_i).$$

A weight $\mathbf{a}$ is general if the equality does not hold for every sub line bundle.

If we take a general weight $\mathbf{a}$, then the moduli space $M_{\mathbf{p}}(\mathbf{a}, d)$ of stable parabolic bundles of degree d is a compact Hausdorff moduli space.

Moduli space of parabolic bundles

Basic properties of $M_{\mathbf{p}}(\mathbf{a}, d)$:

- $M_{\mathbf{p}}(\mathbf{a}, d)$ depends on $\mathbf{a}$.
- $M_{\mathbf{p}}(\mathbf{a}, d)$ and $M_{\mathbf{p}}(\mathbf{b}, d)$ are birational.
- If $\mathbf{a}$ is general, $M_{\mathbf{p}}(\mathbf{a}, d)$ is a smooth projective variety of dimension $n-3$.
- If $\mathbf{a}$ moves but does not hit the stability walls

$$\deg E' + \sum_{E'|_{p_i} = V_i} a_i = \frac{1}{2}(\deg E + \sum a_i),$$

  then $M_{\mathbf{p}}(\mathbf{a}, d)$ does not change. So there are finitely many non-isomorphic $M_{\mathbf{p}}(\mathbf{a}, d)$'s.

Goal

Run Mori's program to $M_{\mathbf{p}}(\mathbf{a},0)$.

1. Study the cone of divisors $\mathrm{Eff}(M_{\mathbf{p}}(\mathbf{a},0))$ in $\mathrm{H}^2(M_{\mathbf{p}}(\mathbf{a},0),\mathbb{R})$;
2. For each divisor $D \in \mathrm{Eff}(M_{\mathbf{p}}(\mathbf{a},0))$, compute $M_{\mathbf{p}}(\mathbf{a},0)(D)$;
3. Study the relation between $M_{\mathbf{p}}(\mathbf{a},0)$ and $M_{\mathbf{p}}(\mathbf{a},0)(D)$.

Note that $M_{\mathbf{p}}(\mathbf{b},0)$ is a birational model of $M_{\mathbf{p}}(\mathbf{a},0)$. So we already have a plenty of models.

A special case

Suppose that $\sum a_i$ is very small.

By Grothendieck's theorem, if E is a rank 2, degree 0 bundle on $\mathbb{P}^1$, $E \cong \mathcal{O}(k) \oplus \mathcal{O}(-k)$ for some $k \in \mathbb{N}$.

Recall the stability inequality:

$$\deg E' + \sum_{E'|_{p_i} = V_i} a_i \leq \frac{1}{2}(\deg E + \sum a_i),$$

If $k > 0$, then for $E' = \mathcal{O}(k)$, the left hand side is always greater and $(E, \{U_i\}, \mathbf{a})$ is unstable.

So if $(E, \{U_i\}, \mathbf{a})$ is stable, then $k = 0$ and $E = \mathcal{O}^2$.

A special case

Thus if $\sum a_i$ is small,

$$M_{\mathbf{p}}(\mathbf{a},0) = \{(\mathcal{O}^2, \{U_i\}, \mathbf{a})\}/_\sim = \{(\mathcal{O}^2, \{U_i\})\}/_\sim$$

$$= \{(\mathbb{P}^1, \{[U_i]\})\}/\mathrm{SL}_2 = (\mathbb{P}^1)^n//\mathrm{SL}_2.$$

By the GIT stability analysis, one can check that

$$M_{\mathbf{p}}(\mathbf{a},0) = (\mathbb{P}^1)^n//_{\mathbf{a}}\mathrm{SL}_2.$$

General cases

We can increase $\mathbf{a}$ by multiplying $c > 1$.

1. If $\sum a_i$ is small, $M_{\mathbf{p}}(\mathbf{a}, 0) = (\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2$.
2. The first change appears when $c \sum a_i = 2$. There is exactly one bundle (up to isomorphism) which becomes unstable. It is $(\mathcal{O}^2, \{U_i\}, \mathbf{a})$ where $(\mathbb{P}^1, \{[U_i]\}) = (\mathbb{P}^1, \{p_i\})$.
3. If c increases more, then $M_{\mathbf{p}}(c\mathbf{a}, 0)$ is a single point blow-up $\mathrm{Bl}_x (\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2$.
4. All of the next changes are flips or contractions. So they are equivalent or simpler than $\mathrm{Bl}_x (\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2$.

Therefore it is sufficient to study Mori's program of $\mathrm{Bl}_x (\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2$.

Mori's program for moduli of parabolic bundles

Step 1: Study the cone $\mathrm{Eff}(M_{\mathbf{p}}(\mathbf{a},0)) = \mathrm{Eff}(\mathrm{Bl}_x(\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2)$.

Step 2: For each divisor $D \in \mathrm{Eff}(M_{\mathbf{p}}(\mathbf{a},0))$, compute $M_{\mathbf{p}}(\mathbf{a},0)(D)$.

Step 3: Study the relation between $M_{\mathbf{p}}(\mathbf{a},0)$ and $M_{\mathbf{p}}(\mathbf{a},0)(D)$.

Theorem (M-Yoo, 14)

1. *The effective cone $\mathrm{Eff}(M_{\mathbf{p}}(\mathbf{a},0))$ is generated by 2^{n-1} level one $\mathfrak{sl}_2$ conformal blocks.*
2. *For each divisor $D \in \mathrm{Eff}(M_{\mathbf{p}}(\mathbf{a},0))$,*

$$M_{\mathbf{p}}(\mathbf{a},0)(D) = M_{\mathbf{q}}(\mathbf{b},d)$$

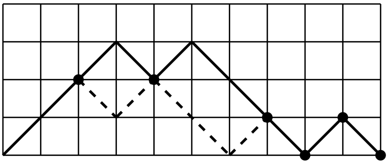
for some $\mathbf{b}$, d, and possibly smaller number of parabolic points.

Step 3 has been well-understood by Boden, Hu, and Thaddeus.

Mori's program for moduli of parabolic bundles

Ingredients: Identifications of some objects

- (Geometry) A divisor on $\mathrm{Bl}_x(\mathbb{P}^1)^n/\!/_{\mathbf{a}}\mathrm{SL}_2$;
- (Invariant theory) SL_2-invariant polynomials in $k[V]_{\mathbf{a}}^{\mathrm{SL}_2}$ with some vanishing conditions;
- (Representation theory) $\mathfrak{sl}_2$-conformal blocks in representation theory of affine Lie algebra;
- (Combinatorics) Boxed Catalan paths.



Moduli space of pointed rational curves

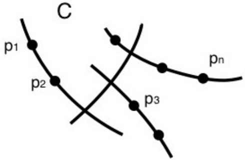
$\mathrm{M}_{0,n} := \{(C, p_1, \cdots, p_n) \mid C \cong \mathbb{P}^1,\ p_i \neq p_j\}/_\sim$

$\cdots$ moduli space of n-pointed smooth rational curves.

It is an open subset of $\mathbb{A}^{n-3}$, so it is smooth but not compact.

$\overline{\mathrm{M}}_{0,n} :=$

$\Bigg\{$



$\Bigg|$

C : connected projective curve

with at worst nodal singularities

p_i : distinct smooth points

$|\mathrm{Aut}| < \infty$

$\Bigg\}/_\sim$

$\cdots$ Deligne-Mumford compactification, or moduli space of n-pointed stable rational curves.

Moduli space of pointed rational curves

$\overline{\mathrm{M}}_{0,n}$ has many nice properties.

- $\overline{\mathrm{M}}_{0,n}$ is a smooth projective $(n-3)$-dimensional smooth variety.
- $\overline{\mathrm{M}}_{0,n} - \mathrm{M}_{0,n} = D_2 \cup D_3 \cup \cdots \cup D_{\lfloor n/2 \rfloor}$ is a simple normal crossing divisor.
- An irreducible component of $\overline{\mathrm{M}}_{0,n} - \mathrm{M}_{0,n}$ is isomorphic to $\overline{\mathrm{M}}_{0,i} \times \overline{\mathrm{M}}_{0,j}$.
- Like toric varieties, there is a stratification of boundaries.
- Many topological invariants such as the cohomology ring, Hodge numbers are already known.

Moduli space of stable rational curves

To run Mori's program for $\overline{\mathrm{M}}_{0,n}$, as a first step we need to know the cone of divisors $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})$.

It is known only for $n \le 6$ (Hassett-Tschinkel, 02).

A subcone of $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})$, the so-called nef cone $(\mathrm{Nef}(\overline{\mathrm{M}}_{0,n}))$ corresponds to the birational models admit a map from $\overline{\mathrm{M}}_{0,n}$.

$$D \in \mathrm{Nef}(\overline{\mathrm{M}}_{0,n}) \Leftrightarrow \forall \text{ curve } C \subset \overline{\mathrm{M}}_{0,n},\ D \cdot C \ge 0.$$

Question

Is there a finite list of curves which is sufficient to check the nefness?

F-conjectures

Natural curves on $\overline{\mathrm{M}}_{0,n}$: one dimensional boundary strata (so-called F-curves)

Conjecture (F-conjecture, mid-90s)

$D \in \mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$ if and only if $D \cdot F \geq 0$ for every F-curve F.

Conjecture (strong F-conjecture)

D is semi-ample if and only if $D \cdot F \geq 0$ for every F-curve F.

Conjecture (stronger F-conjecture)

An integral divisor D is base-point-free if and only if $D \cdot F \geq 0$ for every F-curve F.

F-conjecture is known for $n \leq 7$ in char 0 (Keel-McKernan, 96).

F-conjectures

There is a natural S_n-action on $\overline{\mathrm{M}}_{0,n}$.

Conjecture (S_n-invariant F-conjecture)

An S_n-invariant divisor D is in $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$ if and only if $D \cdot F \geq 0$ for every F-curve F.

We also have strong and stronger S_n-invariant F-conjectures.

(Gibney-Keel-Morrison, 01) If S_n-invariant F-conjecture is true for $n = g$, then we obtain $\mathrm{Nef}(\overline{\mathcal{M}}_g)$.

(Gibney, 09) In char 0, S_n-invariant F-conjecture is true for $n \leq 24$.

(Fedorchuk, 15) In char p, S_n-invariant F-conjecture is true for $n \leq 16$.

Reduction to GIT

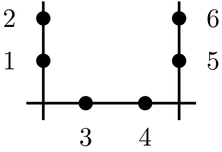
Theorem (Kapranov, 93)

There is a reduction morphism

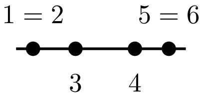
$$\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2.$$

$\overline{\mathrm{M}}_{0,n}$: moduli space of singular curves, but the marked points must be distinct.

$(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$: moduli space of smooth curves with marked points so that some collisions are allowed.



versus



Identification of divisors with graphs

1. Every S_n-invariant divisor on $\overline{\mathrm{M}}_{0,n}$ can be written as $\pi^*(cD_2) - \sum_{i\geq 3} a_i D_i$ for some $c, a_i \in \mathbb{Z}_{\geq 0}$.
2. The linear system $|\pi^*(cD_2) - \sum_{i\geq 3} a_i D_i|$ is identified with a sub linear system $|cD_2|_{(a_i)} \subset |cD_2|$ on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ such that $D \in |cD_2|_{(a_i)}$ vanishes on $\pi(D_i)$ with at least multiplicity a_i.
3. $|cD_2|_{(a_i)}$ is generated by X_Γ where
   - Γ is a regular graph with a vertex set $[n]$ and of degree $c(n-1)$;
   - For each $I \subset [n]$ with $|I| = i$, the number of edges connecting vertices in I is at least a_i.

Graph theoretic interpretation of F-conjecture

We can obtain a purely combinatorial condition guarantees the base-point-freeness of a linear system.

Proposition

A linear system $|\pi^*(cD_2) - \sum_{i\geq 3} a_i D_i|$ *is base-point-free if for every trivalent labeled tree* T, *there is a graph* Γ *such that*

1. Γ *is a regular graph with a vertex set* $[n]$ *and of degree* $c(n-1)$;
2. *For each* $I \subset [n]$ *with* $|I| = i$, *the number of edges connecting vertices in* I *is at least* a_i;
3. *For each* $J \subset [n]$ *with* $|J| = i$ *and* J *spans a tail of* T, *the number of edges connecting vertices in* J *is exactly* a_i.

This is a linear programming problem! Computers can solve it.

Moreover, it is characteristic independent.

Current result

Theorem (M-Swinarski, 15)

In arbitrary characteristic (even over $\operatorname{Spec}\mathbb{Z}$), the strong S_n-invariant F-conjecture is true for $n \leq 15$.

The proof is using computers. $n \leq 10$ cases took seconds to check, but $n = 14$ case took 19 hours and $n = 15$ case took 4 days.

This project is ongoing. We are trying to reduce the computational complexity by refining the idea. At the same time we are finding a proof for arbitrary n.

Thank you!