Classical Invariant Theory and Birational Geometry of Moduli Spaces of Parabolic Bundles

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Birational geometry and Mori's program

X: smooth projective variety

Question

Classify projective varieties (birational models) that share an open dense subset with X and are equivalent to or simpler than X.

Mori's program provides a theoretical framework.

1. Compute the cone $\mathrm{Eff}(X)$ of effective divisors of X.
2. For each $D \in \mathrm{int}(\mathrm{Eff}(X))$, construct and compute a birational model
$$X(D) := \mathrm{Proj} \bigoplus_{m \geq 0} \mathrm{H}^0(X, \mathcal{O}(mD)).$$
3. Study the relation between X and $X(D)$.

If X is a Mori dream space, at least theoretically this program can be completed.

Mori's program for moduli spaces

Problem

Apply Mori's program to moduli spaces.

Only non-trivial higher dimensional examples we can investigate in detail.

Many birational models of moduli spaces are again moduli spaces, too.

- Birational models of $\overline{\mathrm{M}}_g$: moduli spaces of curves with worse singularities.
- Birational models of $\overline{\mathrm{M}}_0(\mathbb{P}^d, d)$: Hilbert scheme, Chow variety, etc.
- Birational models of moduli spaces of sheaves: moduli spaces of Bridgeland stable objects.

Except toric varieties and the case that $\dim \mathrm{Eff}(X) = 2$, there are few completed examples.

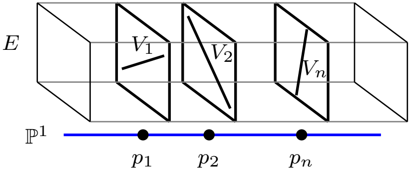
Moduli space of parabolic vector bundles

Fix n distinct points $\mathbf{p} = (p_1, \cdots, p_n)$ on $\mathbb{P}^1$.

Definition

A rank 2 parabolic bundle on $\mathbb{P}^1$ with parabolic points $\mathbf{p}$ is a collection of data $(E, \{V_i\})$ where

1. E is a rank 2 vector bundle on $\mathbb{P}^1$;
2. V_i is a 1-dimensional subspace of $E|_{p_i}$.



$\mathrm{M}_{\mathbf{p}}(d)$: moduli space of rank 2 degree d parabolic bundles $\cdots$ highly non-separated

To obtain a projective moduli space, we introduce a stability condition $\mathbf{a}$.

$\mathrm{M}_{\mathbf{p}}(d, \mathbf{a})$: moduli space of rank 2 degree d $\mathbf{a}$-semistable parabolic bundles

Main theorem

Theorem (M-Yoo, 16)

Let $\mathrm{M} := \mathrm{M}_{\mathbf{p}}(0, \mathbf{a})$ *be the moduli space of rank 2* $\mathbf{a}$*-semistable parabolic bundles on* $\mathbb{P}^1$ *with parabolic points* $\mathbf{p} = (p_1, \cdots, p_n)$*.*

1. $\dim \mathrm{Eff}(\mathrm{M}) \le n + 1$.
2. *When* $\dim \mathrm{Eff}(\mathrm{M}) = n + 1$, $\mathrm{Eff}(\mathrm{M})$ *is generated by* 2^{n-1} *extremal rays.*
3. *For every* $D \in \mathrm{Eff}(\mathrm{M})$, $\mathrm{M}(D) = \mathrm{M}_{\mathbf{q}}(d, \mathbf{b})$.

Outline of the proof:

1. Compute $\mathrm{Eff}(\mathrm{M})$ when $\mathbf{a}$ is small $\Leftarrow$ apply classical invariant theory
2. Scale up $\mathbf{a}$ $\Rightarrow$ measure the change of M (wall-crossing behavior)
3. Translate the question into the theory of $\mathfrak{sl}_2$-conformal blocks
4. Investigate their properties via combinatorics of (boxed) Catalan paths

Initial step of the proof and classical invariant theory

$$\begin{aligned}\mathrm{M}_{\mathbf{p}}(0,\mathbf{a}) &\text{“}=\text{”} \{(\mathcal{O}^2, (V_1, \cdots, V_n)) \mid V_i \subset \mathcal{O}^2|_{p_i}\}/\sim \\ &= \{(V_1, \cdots, V_n) \mid V_i \subset \mathbb{C}^2\}/\sim \\ &= \{(q_1, \cdots, q_n) \mid q_i \in \mathbb{P}^1\}/\sim \\ &\text{“}=\text{”} (\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2\end{aligned}$$

When $\mathbf{a}$ is small, $\mathrm{M}_{\mathbf{p}}(0,\mathbf{a}) = (\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2$.

$$\begin{aligned}\text{Divisors on } (\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2 &\Leftrightarrow \mathrm{SL}_2\text{-invariant divisors on } (\mathbb{P}^1)^n \\ &\Leftrightarrow \mathbb{C}[X_i, Y_i]_{1 \le i \le n}^{\mathrm{SL}_2}\end{aligned}$$

Theorem (Clebsch, 1872, First fundamental theorem of invariant theory)

The invariant ring $\mathbb{C}[X_i, Y_i]_{1 \le i \le n}^{\mathrm{SL}_2}$ is generated by $X_iY_j - X_jY_i$.

The effective cone of $(\mathbb{P}^1)^n /\!/_{\mathbf{a}} \mathrm{SL}_2$ is the cone over a hypersimplex $\Delta(2,n)$.

... and the story begins...

Thank you!