Algebraic Geometry, Moduli Spaces, and Invariant Theory

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Part I

Algebraic Geometry

Algebraic geometry

From wikipedia:

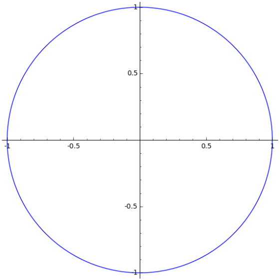
Algebraic geometry is a branch of mathematics, classically studying zeros of multivariate polynomials.

Algebraic geometry in highschool

The zero set of a two-variable polynomial provides a plane curve.

Example:

$$f(x,y) = x^2 + y^2 - 1 \Leftrightarrow Z(f) := \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$$



$\cdots$ a unit circle

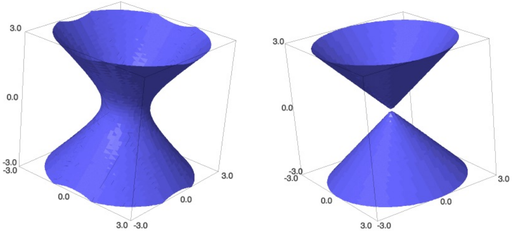
Algebraic geometry in college calculus

The zero set of a three-variable polynomial give a surface.

Examples:

$$f(x, y, z) = x^2 + y^2 - z^2 - 1 \Leftrightarrow Z(f) := \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$$

$$g(x, y, z) = x^2 + y^2 - z^2 \Leftrightarrow Z(g) := \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = 0\}$$

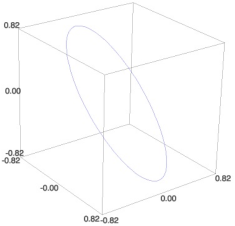


Algebraic geometry in college calculus

The common zero set of two three-variable polynomials gives a space curve.

Example:

$$f(x, y, z) = x^2 + y^2 + z^2 - 1,\ g(x, y, z) = x + y + z$$
$$\Leftrightarrow Z(f, g) := \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = g(x, y, z) = 0\}$$



Definition

An algebraic variety is a common zero set of some polynomials.

Algebraic geometry in college calculus

Two different sets of polynomials may give the same algebraic variety.

$$Z(f) = Z(2f)$$
$$Z(f, g) = Z(f + g, g) = Z(f + hg, g) = \cdots$$

Indeed, if an ideal I is generated by $(f_1, \cdots, f_n)$ and also it is generated by $(g_1, \cdots, g_m)$, then

$$Z(f_1, \cdots, f_n) = Z(g_1, \cdots, g_m).$$

So we may write $Z(I)$ and can say:

Definition

An algebraic variety is a common zero set of an ideal.

Geometry (of varieties) $\Leftrightarrow$ Algebra (ideals)

Main goal of algebraic geometry

1. Study the geometric property of given algebraic variety (connected? compact? dimension? topological invariants? $\cdots$)
2. Study the arithmetic property of given algebraic variety (is there any integer point? rational point? $\cdots$)
   Example: Fermat's last theorem (= Is there any non-trivial integer point on the variety $Z(x^n + y^n - z^n)$?)
3. Construct a tool to distinguish different varieties (homology, cohomology, motive, derived category, $\cdots$)
4. Classify all possible algebraic varieties under some equivalence relation (moduli problem)

Part II

Moduli spaces

Classification problem

Problem

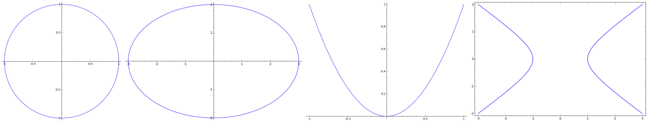
Find all possible mathematical objects with given conditions or axioms.

- Finite dimensional vector spaces
- cyclic groups
- finite simple groups
- Poincaré conjecture: a consequence of the classification of three dimensional compact manifolds

Classification problem

Question

Find all possible plane conics.



circle, ellipse, parabola, hyperbola, + some degenerated cases including two lines.

All plane conics are obtained by varying parameters of the same equation.

Classification problem

Question

Find all possible plane conics.

Let's collect all possible equations of conics!

conic $= Z(h)$ for some degree 2 polynomial

$h(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$

We need 6 numbers to give such a polynomial.

A nonzero scalar multiple gives the same conic.

The space of conics:

$$M_C = \{(a, b, c, d, e, f) \in \mathbb{R}^6 \mid a \neq 0 \text{ or } b \neq 0 \text{ or } c \neq 0\}/ \sim .$$

Such M_C is called the moduli space of plane conics.

Moduli spaces

Definition

A moduli space is a space parametrizing a certain kind of geometric objects.

So a moduli space is an answer to a geometric classification problem.

Examples:

- Projective space $\mathbb{P}^{n-1}$: moduli space of one-dimensional sub vector spaces of $\mathbb{R}^n$
- Grassmannian $Gr(r, n)$: moduli space of r-dimensional sub vector spaces of $\mathbb{R}^n$
- moduli space of circles on a plane: $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$

Moduli spaces

Recall that an algebraic variety is determined by an ideal.

Moduli space of algebraic varieties in $\mathbb{R}^n \Leftrightarrow$ moduli space of ideals in $\mathbb{R}[x_1, \cdots, x_n]$

Definition

The Hilbert scheme $\mathrm{Hilb}(\mathbb{R}^n)$ is the moduli space of varieties in $\mathbb{R}^n$ (= moduli space of ideals in $\mathbb{R}[x_1, \cdots, x_n]$).

In modern algebraic geometry, we prefer to use $\mathrm{Hilb}(\mathbb{P}^n)$ (moduli space of varieties in a complex projective space).

Moduli spaces

More examples:

- $\mathcal{M}_g$: moduli space of smooth complex curves of genus g
- $\overline{\mathcal{M}}_g$: moduli space of stable curves of genus g
- $\overline{\mathcal{M}}_{g,n}$: moduli space of stable n-pointed curves of genus g
- $\overline{\mathcal{M}}_g(\mathbb{P}^r, d)$: moduli space of stable genus g degree d curves in $\mathbb{P}^r$
- $M_C(r, d)$: moduli space of rank r stable vector bundles of degree d on a curve C

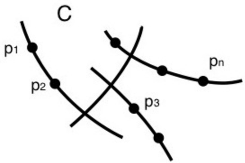
Moduli space of pointed genus 0 curves

$\mathrm{M}_{0,n} := \{(C, p_1, \cdots, p_n) \mid C \cong \mathbb{CP}^1,\ p_i \neq p_j\}/_\sim$

$\cdots$ moduli space of n-pointed smooth genus 0 curves.

It is an open subset of $\mathbb{C}^{n-3}$, so it is smooth but not compact.

$\overline{\mathrm{M}}_{0,n} :=$



$\Big\{ \ \ \mid$ C : connected projective curve with at worst nodal singularities, each component of $C \cong \mathbb{CP}^1$; p_i : distinct smooth points; $|\mathrm{Aut}| < \infty$ $\Big\}/_\sim$

$\cdots$ Deligne-Mumford compactification, or moduli space of n-pointed stable genus 0 curves.

Moduli space of pointed genus 0 curves

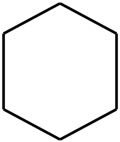
$\overline{\mathrm{M}}_{0,n}$ has many nice properties.

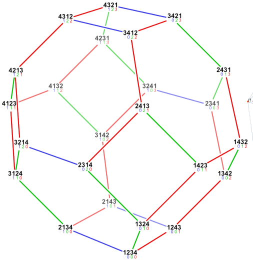
- $\overline{\mathrm{M}}_{0,n}$ is a smooth projective $(n-3)$-dimensional variety.
- $\overline{\mathrm{M}}_{0,n} - \mathrm{M}_{0,n} = D_2 \cup D_3 \cup \cdots \cup D_{\lfloor n/2 \rfloor}$ is a simple normal crossing divisor.
- An irreducible component of $\overline{\mathrm{M}}_{0,n} - \mathrm{M}_{0,n}$ is isomorphic to $\overline{\mathrm{M}}_{0,i} \times \overline{\mathrm{M}}_{0,j}$.
- Like toric varieties, there is a nice stratification of boundaries.
- Many topological invariants such as the cohomology ring, Hodge numbers are already known.

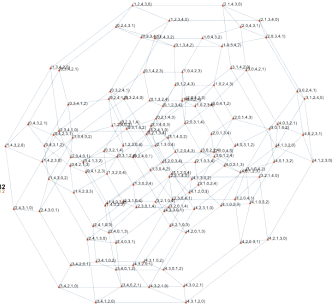
Moduli space of pointed genus 0 curves

But its combinatorial structure is extremely complicated!

The following pictures are 'approximations' (enable us to visualize them) of the decompositions of $\overline{\mathrm{M}}_{0,n}$ for $n = 4, 5, 6, 7$.







Moduli space of pointed genus 0 curves

$n = 8$

Main problem

Question

Study geometry of moduli spaces. For instance:

- *Understand the shape of given moduli space.*
- *Calculate its topological invariants.*
- *Find its subvarieties.*

Part III

Invariant theory

Toy example

Let $\mathbb{Q}[x, y]$ be the polynomial ring with two variables.

Define $\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$-action on $\mathbb{Q}[x, y]$ by $\sigma \cdot x = -x$, $\sigma \cdot y = -y$.

For instance, $\sigma \cdot x^3 = (-x)^3 = -x^3$, $\sigma \cdot xy = (-x)(-y) = xy$.

Question

Find all polynomials such that $\sigma \cdot f = f$.

Toy example

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Question

Find all polynomials such that $\sigma \cdot f = f$.

Example: x^2, xy, y^2, any polynomial $f(x^2, xy, y^2)$.

Answer: The set of such polynomials forms a subring

$$\mathbb{Q}[x^2, xy, y^2] \cong \mathbb{Q}[a, b, c]/\langle ac - b^2 \rangle.$$

Old question

G: group

V: G-representation, i.e., a vector space over k equipped with a linear G-action

$k[V]$: ring of polynomial functions on V

There is an induced G-action on $k[V]$. We say $f \in k[V]$ is a G-invariant (or simply invariant) if for every $\sigma \in G$, $\sigma \cdot f = f$. Or equivalently, $f(\sigma \cdot v) = f(v)$.

$k[V]^G$: subring of G-invariants

Question

Describe $k[V]^G$.

First example - Symmetric group

$G = S_n$, $V = k^n$ with a standard basis $\{v_1, v_2, \cdots, v_n\}$

S_n acts on V as permutations of $\{v_i\}$.

$k[V] = k[x_1, \cdots, x_n]$, and S_n-action on $k[V]$ is a permutation of $\{x_i\}$.

Examples of S_n-invariants:

$$\begin{aligned} e_1 &:= x_1 + x_2 + \cdots + x_n, \\ e_2 &:= \sum_{i<j} x_i x_j, \\ e_3 &:= \sum_{i<j<k} x_i x_j x_k, \\ &\vdots \\ e_n &:= x_1 x_2 \cdot \cdots \cdot x_n \end{aligned}$$

Theorem (Gauss, 1815)

As a k-algebra, $k[V]^{S_n}$ is generated by $e_1, e_2, \cdots, e_n$.

More examples - Linear algebra

$V = \mathrm{M}_{n\times n}$: set of $n \times n$ matrices, $G = \mathrm{GL}_n$

There is the conjugation action on V defined by $\sigma \cdot A := \sigma A \sigma^{-1}$ (basis change!).

$k[V] = k[x_{11}, x_{12}, \cdots, x_{nn}]$

Examples of G-invariants:

$$\text{trace} : x_{11} + x_{22} + \cdots + x_{nn},$$

$$\text{determinant} : \sum_{\tau \in S_n} sgn(\tau) \prod_{i=1}^{n} x_{i\tau(i)}.$$

More generally, coefficients of the characteristic polynomial are invariants.

Theorem

As a k-algebra, $k[V]^G$ is generated by coefficients of the characteristic polynomials. Therefore $k[V]^G \cong k[a_1, a_2, \cdots, a_n]$.

More examples - Homogeneous polynomials

$k = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$

$V_d = \{a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d\}$: set of degree d homogeneous polynomials of degree d with two variables x, y

$k[V_d] = k[a_0, a_1, \cdots, a_d]$

$G = \mathrm{SL}_2$ acts on V_d by $\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \cdot g(x,y) = g(\alpha x + \beta y, \gamma x + \delta y)$

$k[V_2]^{\mathrm{SL}_2} = k[a_1^2 - 4a_0a_2]$

$k[V_3]^{\mathrm{SL}_2} = k[a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3]$

$k[V_4]^{\mathrm{SL}_2} = k[f_2, f_3]$, where

$$f_2 = a_0a_4 - \frac{a_1a_3}{4} + \frac{a_2^2}{12},\ f_3 = \left|\begin{array}{ccc} a_0 & a_1/4 & a_2/6 \\ a_1/4 & a_2/6 & a_3/4 \\ a_2/6 & a_3/4 & a_4 \end{array}\right|.$$

More examples - Homogeneous polynomials

$k[V_5]^{\mathrm{SL}_2}$ is generated by f_4, f_8, f_{12}, f_{18}, where

$$\begin{aligned} f_4 = & -2a_2^2a_3^2 + 6a_1a_3^3 + 6a_2^3a_4 - 19a_1a_2a_3a_4 - 15a_0a_3^2a_4 \\ & + 9a_1^2a_4^2 + 40a_0a_2a_4^2 - 15a_1a_2^2a_5 + 40a_1^2a_3a_5 \\ & + 25a_0a_2a_3a_5 - 250a_0a_1a_4a_5 + 625a_0^2a_5^2. \end{aligned}$$

There is one relation (of degree 36) between them.

$k[V_6]^{\mathrm{SL}_2}$ is generated by $f_2, f_4, f_6, f_{10}, f_{15}$

$k[V_7]^{\mathrm{SL}_2}$ is generated by 30 generators.

$k[V_8]^{\mathrm{SL}_2}$ is generated by $f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}$

$k[V_9]^{\mathrm{SL}_2}$ is generated by 92 generators.

$k[V_d]^{\mathrm{SL}_2}$ is unknown for $d \geq 11$.

Hilbert's 14th problem

Masayoshi Nagata, 1927 $\sim$ 2008

Question (Hilbert's 14th problem)

Can one always find finitely many generators $f_1, \cdots, f_r$ such that $k[V]^G = k[f_1, \cdots, f_r]$?

Answer (Nagata, 1959): No. There are G and V such that $k[V]^G$ is not finitely generated.

Hilbert's 14th problem

David Hilbert, 1862 $\sim$ 1943

Hilbert himself had a positive result.

Theorem (Hilbert, 1890)

If G is a linearly reductive group, then $k[V]^G$ is finitely generated.

Examples (over $\mathbb{C}$): finite groups, $(\mathbb{C}^*)^n$, GL_n, SL_n, SO_n, Sp_n, $\cdots$

Main example

$V = (k^2)^n$, SL_2 acts as $\sigma \cdot (v_1, \cdots, v_n) := (\sigma \cdot v_1, \cdots, \sigma \cdot v_n)$

$k[V] = k[x_1, y_1, x_2, y_2, \cdots, x_n, y_n]$

On $k[V]$, SL_2 acts as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(x_i, y_i) = (\alpha x_i + \beta y_i, \gamma x_i + \delta y_i)$.

Examples of invariants:

$$x_i y_j - x_j y_i$$

Indeed, $\sigma \cdot (x_i y_j - x_j y_i) = \det(\sigma)(x_i y_j - x_j y_i) = x_i y_j - x_j y_i$.

Theorem (First fundamental theorem of invariant theory)

The invariant ring $k[V]^{\mathrm{SL}_2}$ is generated by $(x_i y_j - x_j y_i)$.

This result is true even for arbitrary commutative ring A instead of k (De Concini-Procesi, 1976).

Graphical algebra - Combinatorial interpretation

Γ: a directed graph on n labeled vertices

$\mathbf{deg}\,\Gamma$: a sequence of degrees of vertices of Γ

For an edge $e \in E_\Gamma$, $h(e)$: head of e, $t(e)$: tail of e.

$\Gamma_1 \cdot \Gamma_2$: the union of two graphs on the same set of vertices

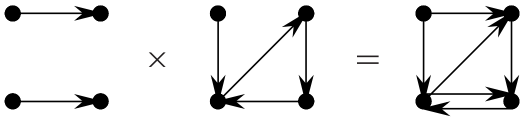
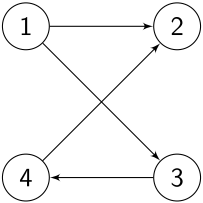


Figure from Howard, Millson, Snowden, Vakil, The equations for the moduli space of n points on the line

Graphical algebra - Combinatorial interpretation

For each Γ, let

$$X_\Gamma := \prod_{e \in E_\Gamma} (x_{h(e)} y_{t(e)} - x_{t(e)} y_{h(e)}).$$

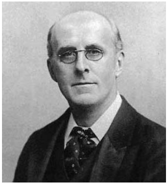


Γ

$$X_\Gamma = (x_2 y_1 - x_1 y_2)(x_3 y_1 - x_1 y_3)(x_2 y_4 - x_4 y_2)(x_4 y_3 - x_3 y_4)$$

By the fundamental theorem, $k[V]^{\mathrm{SL}_2}$ is generated by X_Γ.

Graphical algebra - Combinatorial interpretation



Alfred Kempe, 1849 $\sim$ 1922

$\mathbf{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{Z}^n_{\geq 0}$

$k[V]_{\mathbf{w}}$: subring of $k[V]$ consisting of polynomials of multidegree $c\mathbf{w}$

$\mathbf{w} = \mathbf{1} = (1, \cdots, 1) \Rightarrow k[V]^{\mathrm{SL}_2}_{\mathbf{1}}$: generated by X_Γ for a regular graph Γ.

Theorem (Kempe, 1894)

The k-algebra $k[V]^{\mathrm{SL}_2}_{\mathbf{w}}$ is generated by the smallest degree elements.

The relations between generators have been computed recently (Howard-Millson-Snowden-Vakil, 2009).

Geometric interpretation

$\mathbf{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{Z}^n_{\geq 0}$

$k[V]_{\mathbf{w}}$: subspace of $k[V]$ consisting of multihomogeneous polynomials of multidegree $c\mathbf{w}$

$\mathbb{P}^1$: projective line ($k \cup \{\infty\}$)

There is a natural SL_2-action on $\mathbb{P}^1$ (Möbius transform)

$k[V]_{\mathbf{w}}$: space of algebraic functions on $(\mathbb{P}^1)^n$ of multidegree $c\mathbf{w}$.

$$\begin{aligned} k[V]_{\mathbf{w}}^{\mathrm{SL}_2} &= \text{space of } \mathrm{SL}_2\text{-invariant algebraic functions on } (\mathbb{P}^1)^n \\ &= \text{space of algebraic functions on } (\mathbb{P}^1)^n/\mathrm{SL}_2 \end{aligned}$$

We use algebro-geometric quotient (or GIT quotient) $(\mathbb{P}^1)^n /\!/_{\mathbf{w}} \mathrm{SL}_2$.

Summary

Three aspects of the main example $k[V]^{\mathrm{SL}_2}$:

1. (Algebra) Invariant polynomials with respect to the SL_2-action
2. (Combinatorics) Directed labeled graphs
3. (Geometry) Functions on the quotient space $(\mathbb{P}^1)^n//\mathrm{SL}_2$

Part IV

... and they came together

Moduli space of stable genus 0 curves

Question

What kind of subvarieties are in the moduli space $\overline{\mathrm{M}}_{0,n}$*?*

The initial case is:

Question

Classify all possible curves on the moduli space $\overline{\mathrm{M}}_{0,n}$.

The F-conjecture

William Fulton, 1939 $\sim$

Natural curves on $\overline{\mathrm{M}}_{0,n}$: one dimensional boundary strata (the so-called F-curves)

Conjecture (F-conjecture, mid-90s, Fulton-Faber)

Every curve class is a non-negative linear combination of F-curves.

The F-conjecture

We can dualize this question, by intersection of subvarieties.

D: divisor ($=$ a codimension 1 subvariety) of $\overline{\mathrm{M}}_{0,n}$

Intersection theory: a method generalizing the number of intersection points

D: divisor, C: curve

$$(D, C) \mapsto (D \cdot C) \in \mathbb{Z}$$

Definition

- A divisor D is nef if $D \cdot C \geq 0$ for every curve C.
- A divisor D is F-nef if $D \cdot F \geq 0$ for every F-curve F.

Conjecture (F-conjecture)

A divisor D is nef if and only if it is F-nef.

The F-conjecture

There is a natural S_n-action on $\overline{\mathrm{M}}_{0,n}$.

We may formulate an S_n-invariant version of the F-conjecture.

Conjecture (S_n-invariant F-conjecture)

An S_n-invariant divisor D is nef if and only if it is F-nef.

This version is important because it implies a similar statement for $\overline{\mathcal{M}}_g$ (Gibney-Keel-Morrison, 01).

The F-conjecture - known results

- The F-conjecture is known for $n \leq 7$ in char 0 (Keel-McKernan, 96).
- The S_n-invariant F-conjecture is known for $n \leq 24$ in char 0 (Gibney, 09).
- The S_n-invariant F-conjecture is known for $n \leq 16$ in char p (Fedorchuk, 15).
- All computer calculation have been frozen for $n = 7$.

Several notations on divisors

A divisor D on X is used to define a rational map $\phi_{|D|} : X \dashrightarrow \mathbb{P}^n$.

Definition

Let D be a divisor on X.

1. D is very ample if $\phi_{|D|} : X \to \mathbb{P}^n$ is an embedding.
2. D is ample if mD is very ample for some $m \gg 0$.
3. D is base-point-free if $\phi_{|D|} : X \to \mathbb{P}^n$ is regular.
4. D is semi-ample if mD is base-point-free for some $m \gg 0$.
5. D is nef if $D \cdot C \geq 0$ for every curve C.

$$\begin{tikzcd}
\text{base-point-free} \arrow[r, Rightarrow] & \text{semi-ample} \arrow[r, Rightarrow] & \text{nef} \\
\text{very ample} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{ample} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Result

Theorem (M-Swinarski, 16)

Let $n \leq 18$. Let D be an S_n-invariant divisor.

1. *Over $\operatorname{Spec} \mathbb{Z}$, D is semi-ample if and only if it is F-nef.*
2. *Over a field k, D is base-point-free if and only if it is F-nef.*
3. *Over a field k, D is very ample if and only if it is ample.*

- These are stronger statements than the S_n-invariant F-conjecture.
- This project is based on computer calculation. We expect that we can finish the computation up to $n = 20$ in this summer.

Idea of proof (of semi-ampleness)

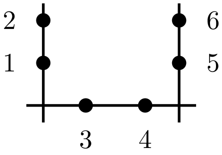
Theorem (Kapranov, 93)

There is a reduction morphism

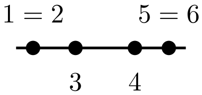
$$\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2.$$

$\overline{\mathrm{M}}_{0,n}$: moduli space of singular curves, but the marked points must be distinct.

$(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$: moduli space of smooth curves with marked points so that some collisions are allowed.



versus



Idea of proof (of semi-ampleness)

1. $\pi_*(D_2) = D_2$ and D_2 is ample on $(\mathbb{P}^1)^n//\mathrm{SL}_2$.
2. Every S_n-invariant divisor on $\overline{\mathrm{M}}_{0,n}$ can be written as $\pi^*(cD_2) - \sum_{i\geq 3} a_i D_i$ for some $c, a_i \geq 0$.
3. The linear system $|\pi^*(cD_2) - \sum_{i\geq 3} a_i D_i|$ is identified with a sub linear system $|cD_2|_{\mathbf{a}} \subset |cD_2|$ on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ such that $D \in |cD_2|_{\mathbf{a}}$ vanishes on $\pi(D_i)$ with at least multiplicity a_i.
4. A graphical divisor X_Γ is in $|cD_2|_{\mathbf{a}}$ if and only if
    - Γ is a regular graph with a vertex set $[n]$ and of degree $c(n-1)$;
    - For each $I \subset [n]$, the number of edges connecting vertices in I is at least $a_{|I|}$.

Idea of proof (of semi-ampleness)

We can obtain a purely combinatorial condition guarantees the base-point-freeness of a linear system.

Proposition

A linear system $|\pi^*(cD_2) - \sum_{i \ge 3} a_i D_i|$ *is base-point-free if for every trivalent labeled tree* T*, there is a graph* Γ *such that*

1. Γ *is a regular graph with a vertex set* $[n]$ *and of degree* $c(n-1)$*;*
2. *For each* $I \subset [n]$*, the number of edges connecting vertices in* I *is at least* $a_{|I|}$*;*
3. *For each* $J \subset [n]$ *such that* J *spans a tail of* T*, the number of edges connecting vertices in* J *is exactly* $a_{|J|}$*.*

This is a linear programming problem! Computers can solve it. Moreover, it is characteristic independent.

Idea of proof

We can strengthen the statement from semi-ampleness to base-point-freeness and very ampleness by using:

- Cox ring
- embedding into a toric variety
- finite graph theory
- tropical compactification

Thank you!