Geometric Invariant Theory and Construction of Moduli Spaces

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Part I

Moduli Spaces

Classification problem

Problem

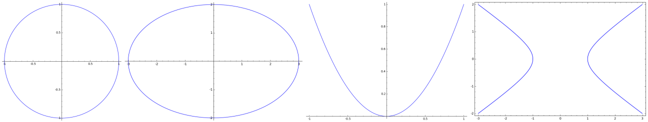
Find all possible mathematical objects with given conditions or axioms.

- Finite dimensional vector spaces
- cyclic groups
- finite simple groups
- Poincaré conjecture: a consequence of the classification of three dimensional compact manifolds

Classification problem

Question

Find all possible plane conics.



circle, ellipse, parabola, hyperbola, $+$ some degenerated cases including two lines.

All plane conics are obtained by varying parameters of the same equation.

Classification problem

Question

Find all possible plane conics.

Let's collect all possible equations of conics!

conic $= Z(h) = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) = 0\}$ for some degree 2 polynomial $h(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$

We need 6 numbers to give such a polynomial.

A nonzero scalar multiple gives the same conic.

The space of conics:

$$M_C = \{(a, b, c, d, e, f) \in \mathbb{R}^6 \mid a \neq 0 \text{ or } b \neq 0 \text{ or } c \neq 0\}/\sim .$$

Such M_C is called the moduli space of plane conics.

Moduli spaces

Definition

A moduli space is a space parametrizing a certain kind of geometric objects.

So a moduli space is an answer to a geometric classification problem. Informally, we may think a moduli space as a dictionary of geometric objects.

Examples:

- Projective space $\mathbb{P}^{n-1}$: moduli space of one-dimensional sub vector spaces of $\mathbb{R}^n$
- Grassmannian $Gr(r, n)$: moduli space of r-dimensional sub vector spaces of $\mathbb{R}^n$
- moduli space of circles on a plane: $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$

Moduli space of triangles

A triangle on $\mathbb{R}^2$ can be described by three vertices,
$v_1 = (x_1, y_1), v_2 = (x_2, y_2), v_3 = (x_3, y_3)$.

To get a triangle, three vertices v_1, v_2, v_3 must not be collinear.
Set

$$C = \{(v_1, v_2, v_3) \in \mathbb{R}^6 \mid v_1, v_2, v_3 \text{ are collinear}\}$$

Then the moduli space M_T of triangles seems to be

$$M_T = \mathbb{R}^6 - C.$$

But...

Moduli space of triangles

But three points v_2, v_3, v_1 define the same triangle.

More generally, a permutation of v_1, v_2, v_3 defines the same triangle.

In algebraic terms, there is a S_3 group action on $\mathbb{R}^6 - C$ and

$$M_T = (\mathbb{R}^6 - C)/S_3,$$

the quotient space (or orbit space).

Many moduli spaces are constructed in this way.

Lesson: Group actions and algebraic quotients are very important tools in moduli theory.

Moduli spaces of subvarieties

An algebraic variety (in $\mathbb{R}^n$) is a common zero set of polynomials $f_1, \cdots, f_m \in \mathbb{R}[x_1, \cdots, x_n]$.

$$Z(f_1, \cdots, f_m) = \{(x_1, \cdots, x_n) \mid f_1 = \cdots f_m = 0\}.$$

Note that $Z(f,g) = Z(2f, g) = Z(f+g, g) = Z(f+hg, g) = \cdots$

Two sets of polynomials $\{f_1, \cdots, f_m\}$ and $\{g_1, \cdots, g_k\}$ define the same algebraic variety if they generate the same ideal.

In summary:

variety (geometry) $\Leftrightarrow$ ideal (algebra)

Hilbert scheme

Recall that an algebraic variety is determined by an ideal.

Moduli space of algebraic varieties in $\mathbb{R}^n$ $\Leftrightarrow$ moduli space of ideals in $\mathbb{R}[x_1, \cdots, x_n]$

Definition

The Hilbert scheme $\mathrm{Hilb}(\mathbb{R}^n)$ is the moduli space of varieties in $\mathbb{R}^n$ (= moduli space of ideals in $\mathbb{R}[x_1, \cdots, x_n]$).

In modern algebraic geometry, we prefer to use $\mathrm{Hilb}(\mathbb{P}^n)$ (moduli space of varieties in a complex projective space).

Theorem (Grothendieck)

$\mathrm{Hilb}(\mathbb{P}^n)$ *exists and it is a projective scheme.*

Moduli spaces

Alexander Grothendieck, 1928 $\sim$ 2014

More examples:

- $\mathcal{M}_g$: moduli space of smooth complex curves of genus g
- $\overline{\mathcal{M}}_g$: moduli space of stable curves of genus g
- $\overline{\mathcal{M}}_{g,n}$: moduli space of stable n-pointed curves of genus g
- $\overline{\mathcal{M}}_g(\mathbb{P}^r, d)$: moduli space of stable genus g degree d curves in $\mathbb{P}^r$
- $M_C(r, d)$: moduli space of rank r stable vector bundles of degree d on a curve C

Main question

Question

How can one construct such moduli spaces rigorously?

Part II

Invariant theory

Toy example

Let $\mathbb{Q}[x, y]$ be the polynomial ring with two variables.

Define $\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$-action on $\mathbb{Q}[x, y]$ by $\sigma \cdot x = -x$, $\sigma \cdot y = -y$.

For instance, $\sigma \cdot x^3 = (-x)^3 = -x^3$, $\sigma \cdot xy = (-x)(-y) = xy$.

Question

Find all polynomials such that $\sigma \cdot f = f$.

Toy example

Let $\mathbb{Q}[x, y]$ be the polynomial ring with two variables.

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Question

Find all polynomials such that $\sigma \cdot f = f$.

Example: x^2, xy, y^2, any polynomial $f(x^2, xy, y^2)$.

Answer: The set of such polynomials forms a subring

$$\mathbb{Q}[x^2, xy, y^2] \cong \mathbb{Q}[a, b, c]/\langle ac - b^2 \rangle.$$

Old question

G: group

V: G-representation, i.e., a vector space over k equipped with a linear G-action

$k[V]$: ring of polynomial functions on V

There is an induced G-action on $k[V]$. We say $f \in k[V]$ is a G-invariant (or simply invariant) if for every $\sigma \in G$, $\sigma \cdot f = f$. Or equivalently, $f(\sigma \cdot v) = f(v)$.

$k[V]^G$: subring of G-invariants

Question

Describe $k[V]^G$.

First example - Symmetric group

$G = S_n$, $V = k^n$ with a standard basis $\{v_1, v_2, \cdots, v_n\}$

S_n acts on V as permutations of $\{v_i\}$.

$k[V] = k[x_1, \cdots, x_n]$, and S_n-action on $k[V]$ is a permutation of $\{x_i\}$.

Examples of S_n-invariants:

$$e_1 := x_1 + x_2 + \cdots + x_n,$$
$$e_2 := \sum_{i<j} x_i x_j,$$
$$e_3 := \sum_{i<j<k} x_i x_j x_k,$$
$$\vdots$$
$$e_n := x_1 x_2 \cdot \cdots \cdot x_n$$

Theorem (Gauss, 1815)

As a k-algebra, $k[V]^{S_n}$ is generated by $e_1, e_2, \cdots, e_n$.

More examples - Linear algebra

$V = \mathrm{M}_{n\times n}$: set of $n \times n$ matrices, $G = \mathrm{GL}_n$

There is the conjugation action on V defined by $\sigma \cdot A := \sigma A \sigma^{-1}$ (basis change!).

$k[V] = k[x_{11}, x_{12}, \cdots, x_{nn}]$

Examples of G-invariants:

$$\text{trace} : x_{11} + x_{22} + \cdots + x_{nn},$$

$$\text{determinant} : \sum_{\tau \in S_n} sgn(\tau) \prod_{i=1}^{n} x_{i\tau(i)}.$$

More generally, coefficients of the characteristic polynomial are invariants.

Theorem

As a k-algebra, $k[V]^G$ is generated by coefficients of the characteristic polynomials. Therefore $k[V]^G \cong k[a_1, a_2, \cdots, a_n]$.

More examples - Homogeneous polynomials

$k = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$

$V_d = \{a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d\}$: set of degree d homogeneous polynomials of degree d with two variables x, y

$k[V_d] = k[a_0, a_1, \cdots, a_d]$

$G = \mathrm{SL}_2$ acts on V_d by $\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \cdot g(x,y) = g(\alpha x + \beta y, \gamma x + \delta y)$

$k[V_2]^{\mathrm{SL}_2} = k[a_1^2 - 4a_0a_2]$

$k[V_3]^{\mathrm{SL}_2} = k[a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3]$

$k[V_4]^{\mathrm{SL}_2} = k[f_2, f_3]$, where

$$f_2 = a_0a_4 - \frac{a_1a_3}{4} + \frac{a_2^2}{12}, \ f_3 = \left|\begin{array}{ccc} a_0 & a_1/4 & a_2/6 \\ a_1/4 & a_2/6 & a_3/4 \\ a_2/6 & a_3/4 & a_4 \end{array}\right|.$$

More examples - Homogeneous polynomials

$k[V_5]^{\mathrm{SL}_2}$ is generated by f_4, f_8, f_{12}, f_{18}, where

$$\begin{aligned} f_4 = & -2a_2^2a_3^2 + 6a_1a_3^3 + 6a_2^3a_4 - 19a_1a_2a_3a_4 - 15a_0a_3^2a_4 \\ & + 9a_1^2a_4^2 + 40a_0a_2a_4^2 - 15a_1a_2^2a_5 + 40a_1^2a_3a_5 \\ & + 25a_0a_2a_3a_5 - 250a_0a_1a_4a_5 + 625a_0^2a_5^2. \end{aligned}$$

There is one relation (of degree 36) between them.

$k[V_6]^{\mathrm{SL}_2}$ is generated by $f_2, f_4, f_6, f_{10}, f_{15}$.

$k[V_7]^{\mathrm{SL}_2}$ is generated by 30 generators.

$k[V_8]^{\mathrm{SL}_2}$ is generated by $f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}$.

$k[V_9]^{\mathrm{SL}_2}$ is generated by 92 generators.

$k[V_d]^{\mathrm{SL}_2}$ is unknown for $d \geq 11$.

Masayoshi Nagata, 1927 $\sim$ 2008

Question (Hilbert's 14th problem)

Can one always find finitely many generators $f_1, \cdots, f_r$ such that $k[V]^G = k[f_1, \cdots, f_r]$?

Answer (Nagata, 1959): No. There are G and V such that $k[V]^G$ is not finitely generated.

Hilbert's 14th problem

David Hilbert, 1862 $\sim$ 1943

Hilbert himself had a positive result.

Theorem (Hilbert, 1890)

If G is a linearly reductive group, then $k[V]^G$ is finitely generated.

G is linearly reductive if for every surjective morphism of G-representations $f : V \to W$, $f^G : V^G \to W^G$ is surjective.

Examples (over $\mathbb{C}$): finite groups, $(\mathbb{C}^*)^n$, GL_n, SL_n, SO_n, Sp_n, $\cdots$

Proof

$S := k[V] \cdots$ a polynomial ring

J: ideal generated by positive degree invariants

By Hilbert's basis theorem, $J = (f_1, \cdots, f_n)$.

Claim: As a k-algebra, S^G is generated by $f_1, \cdots, f_n$.

The S-module honomorphism $\phi : S^n \to J$, defined by

$(h_1, \cdots, h_n) \mapsto \sum h_i f_i$ is surjective.

We use induction on the degree. Pick $h \in S^G$. Then $h \in J \cap S^G = J^G$.

By the linear reductivity, $(S^G)^n \to J^G$ is surjective, so we have

$h = \sum h_i f_i$ where $h_i \in S^G$.

$\deg h_i < \deg h$. By induction hypothesis, h_i is generated by $f_1, \cdots, f_n$.

So is h. $\square$

This is exactly the reason why Hilbert proved the famous basis theorem.

Summary

- (Classical) Invariant theory is one of origins of modern development of abstract algebra.
- It is computationally very complicated. So it was once died in early 20th century.
- It is now one of active research areas because of computers.
- Its geometric counterpart, geometric invariant theory is an important tool in modern algebraic geometry, in particular moduli theory.

Part III

Geometric Invariant Theory

Quotient space

X: topological space

G acts on X

$$X/G := \text{orbit space} = \{[x] \mid x \in X,\ [x] = [y] \Leftrightarrow \exists g \in G, g \cdot x = y\}$$

Two natural properties that we desire:

1. There is a surjective continuous map (quotient map)

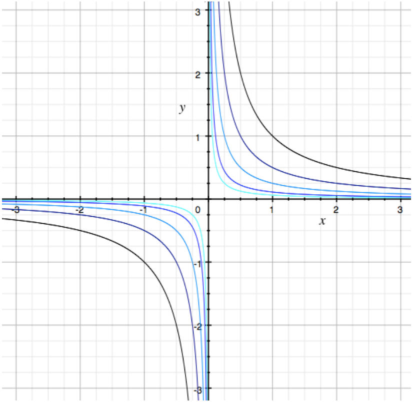
$$\begin{array}{rcl} \pi : X & \to & X/G \\ x & \mapsto & [x] \end{array}$$

2. If X is an algebraic variety, X/G is also an algebraic variety.

Quotient space

There is no such a quotient algebraic variety!

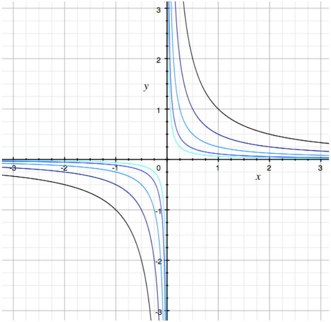
Example: $\mathbb{C}^*$ acts on $\mathbb{C}^2$ as $t \cdot (x, y) = (t^{-1}x, ty)$.



Quotient space

Orbits:

- $[(x, y)] = [(x^{-1}x, xy)] = [(1, x^{-1}y)]$ if $x \neq 0$.
- $[(0, y)] = [(y^{-1}0, y^{-1}y)] = [(0, 1)]$ if $y \neq 0$.
- $[(0, 0)]$



$\lim_{t\to 0}(t, t) = (0, 0)$ on $\mathbb{C}^2 \Rightarrow \lim_{t\to 0}[(t, t)] = [(0, 0)]$ on $\mathbb{C}^2/\mathbb{C}^*$
$\lim_{t\to 0}(t^2, 1) = (0, 1)$ on $\mathbb{C}^2 \Rightarrow \lim_{t\to 0}[(t^2, 1)] = [(0, 1)]$ on $\mathbb{C}^2/\mathbb{C}^*$
But $[(t, t)] = [(t^2, 1)] \Rightarrow \lim_{t\to 0}[(t, t)] = \lim_{t\to 0}[(t^2, 1)]$...?

$\mathbb{C}^2/\mathbb{C}^*$ cannot be Hausdorff!

Philosophy - Definition

Algebraic geometry is a study of geometric objects via (polynomial) functions.

If two points cannot be distinguished by a function, we cannot distinguish them!

Definition

Let $X = \mathrm{Spec}\, R$ and suppose that G acts on X. Let R^G be the ring of invariant functions.
The affine GIT quotient is $X//G := \mathrm{Spec}\, R^G$.

R is the space of functions on X.
R^G is the space of invariant functions on X $\Leftrightarrow$ the space of functions on X/G.

Examples - revisited

$\mathbb{C}^*$ acts on $\mathbb{C}^2$ as $t \cdot (x, y) = (t^{-1}x, ty)$.

$\mathbb{C}^2 = \mathrm{Spec}\ \mathbb{C}[x, y]$

$\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[xy]$

$$\mathbb{C}^2 // \mathbb{C}^* = \mathrm{Spec}\ \mathbb{C}[xy] = \mathbb{C}$$

GL_n acts on $\mathrm{M}_{n \times n} = \mathrm{Spec}\ \mathbb{C}[x_{11}, \cdots, x_{nn}]$ as $\sigma \cdot A = \sigma A \sigma^{-1}$.

$\mathbb{C}[x_{11}, \cdots, x_{nn}]^{\mathrm{GL}_n} = \mathbb{C}[a_1, \cdots, a_n]$ $\cdots$ polynomial ring generated by coefficients of the characteristic polynomial

$$\mathrm{M}_{n \times n} // \mathrm{GL}_n = \mathrm{Spec}\ \mathbb{C}[a_1, \cdots, a_n] = \mathbb{C}^n$$

$G := \mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$ acts on $\mathrm{Spec}\ \mathbb{C}[x, y]$ as $\sigma \cdot (x, y) = (-x, -y)$.

$\mathbb{C}[x, y]^G = \mathbb{C}[x^2, xy, y^2] \cong \mathbb{C}[a, b, c]/\langle ac - b^2 \rangle$

$$\mathbb{C}^2/G = \mathrm{Spec}\ \mathbb{C}[a, b, c]/\langle ac - b^2 \rangle = \text{quadric surface in } \mathbb{C}^3$$

Geometric Invariant Theory - Definition

David Mumford, 1937 $\sim$

Mumford wanted to obtain projective quotients of projective varieties.

Definition

$R = \bigoplus_{d \geq 0} R_d$ $\cdots$ graded ring

$X = \mathrm{Proj}\ R$: associated projective variety, G acts on X

The GIT quotient of X is $X/\!/G := \mathrm{Proj}\ R^G$.

There are three very important features of GIT quotient.

1. $X//G$ is NOT the quotient of X

$R^G \hookrightarrow R \Rightarrow \pi : X = \mathrm{Proj}\ R \dashrightarrow \mathrm{Proj}\ R^G = X//G$

π is defined only on an open subset of $\mathrm{Proj}\ R$.

$x \in X \Leftrightarrow m_x$: nontrivial maximal homogeneous ideal of R

$\pi(x)$ exists $\Leftrightarrow m_x \cap R^G \neq \bigoplus_{d>0} R^G_d$
$\Leftrightarrow \exists f \in \bigoplus_{d>0} R^G,\ f(x) \neq 0$

Definition

$X = \mathrm{Proj}\ R$ projective variety with G-action

$x \in X$ is semi-stable if there is a non-constant G-invariant homogeneous polynomial $f \in R$ such that $f(x) \neq 0$.

X^{ss}: the set of semi-stable points in X.

X^{us}: the set of unstable points $= X \setminus X^{ss}$

Then $\pi : X^{ss} \to X//G$ is a surjective continuous map.

2. $X//G$ is NOT the orbit space of X^{ss}

Two or more orbits may be identified in $X//G$.

For the quotient map $\pi : X^{ss} \to X//G$,

$\pi(x) = \pi(y) \Leftrightarrow \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$

Example:

$\mathbb{C}^*$ acts on $\mathbb{P}^2$ as $t \cdot (x : y : z) = (tx : t^{-1}y : z)$

$(0 : 0 : 1)$ and $(1 : 0 : 1)$ are in distinct orbits

$t \cdot (1 : 0 : 1) = (t : 0 : 1)$, $\lim_{t \to 0}(t : 0 : 1) = (0 : 0 : 1)$ $\Rightarrow$ orbit closures intersect

$\Rightarrow$ $\pi(0 : 0 : 1) = \pi(1 : 0 : 1)$

But for each $[x] \in X//G$, there is a unique closed orbit in X^{ss}

2. $X//G$ is NOT the orbit space of X^{ss}

Definition

We say $x \in X^{ss}$ is stable if $\dim G \cdot x = \dim G$ and $G \cdot x$ is closed in X^{ss}.

X^s: set of stable points $\subset X^{ss}$

$\pi : X^s \to \pi(X^s) \subset X//G$ is the orbit space of X^s

3. $X//G$ depends on the G-action on R

Recall that $X//G = \mathrm{Proj}\ R^G \cdots$ need G-action on R

Distinct G-actions on R may provide the same G-action on X!

Example:

Three $\mathbb{C}^*$ actions on $\mathbb{C}[x, y, z]$

1. $t \cdot (x, y, z) = (tx, t^{-1}y, t^{-1}z)$
2. $t \cdot (x, y, z) = (t^2x, y, z)$
3. $t \cdot (x, y, z) = (t^3x, ty, tz)$

They give the same action on $\mathbb{P}^2 = \mathrm{Proj}\ \mathbb{C}[x, y, z]$, because $(tx : t^{-1}y : t^{-1}z) = (t^2x : y : z) = (t^3x : ty : tz)$

3. $X//G$ depends on the G-action on R

1. $t \cdot (x, y, z) = (tx, t^{-1}y, t^{-1}z)$
   $\mathbb{C}[x, y, z]^{\mathbb{C}^*} = \mathbb{C}[xy, xz] \Rightarrow \mathbb{P}^2//\mathbb{C}^* = \mathrm{Proj}\ \mathbb{C}[xy, xz] = \mathbb{P}^1$
   $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ $\cdots$ unstable
2. $t \cdot (x, y, z) = (t^2x, y, z)$
   $\mathbb{C}[x, y, z]^{\mathbb{C}^*} = \mathbb{C}[y, z] \Rightarrow \mathbb{P}^2//\mathbb{C}^* = \mathrm{Proj}\ \mathbb{C}[y, z] = \mathbb{P}^1$
   $(1 : 0 : 0)$: unstable, $(0 : 1 : 0)$, $(0 : 0 : 1)$: semi-stable (but not stable)
3. $t \cdot (x, y, z) = (t^3x, ty, tz)$
   $\mathbb{C}[x, y, z]^{\mathbb{C}^*} = \mathbb{C} \Rightarrow \mathbb{P}^2//\mathbb{C}^* = \mathrm{Proj}\ \mathbb{C} = \emptyset$

3. $X//G$ depends on the G-action on R

Definition

G acts on $X = \mathrm{Proj}\ R$

A linearization of G-action on X is a G-action on R which induces the G-action on X.

It is not unique in general.

There is another issue on the choice of polarization - it is possible that $\mathrm{Proj}\ R = \mathrm{Proj}\ S$ even though $R \neq S$. We do not investigate it here.

Hilbert-Mumford criterion

The reason why GIT quotient has been so successful is that there is a combinatorial criterion for (semi-)stability.

$X = \mathrm{Proj}\, R \subset \mathbb{P}^n$. There is a linearized G-action on R

$\Rightarrow$ G-action on $\mathbb{C}^{n+1}$

Let $\lambda : \mathbb{C}^* \to G$ be a non-trivial homomorphism (one-parameter subgroup (1-PS)).

We have the induced $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$. For $x = (x_0 : x_1 : \cdots : x_n)$,

$t \cdot (x_0, x_1, \cdots, x_n) = (t^{m_0} x_0, t^{m_1} x_1, \cdots, t^{m_n} x_n)$

Definition

The Hilbert-Mumford index is $\mu(\lambda, x) := \min\{m_i \mid x_i \neq 0\}$.

Hilbert-Mumford criterion

Definition

The Hilbert-Mumford index is $\mu(\lambda, x) := \min\{m_i \mid x_i \neq 0\}$.

Theorem (Hilbert-Mumford criterion, Ver 1)

1. $x \in X^{ss} \Leftrightarrow \forall$ *1-PS* λ, $\mu(\lambda, x) \leq 0$
2. $x \in X^{s} \Leftrightarrow \forall$ *1-PS* λ, $\mu(\lambda, x) < 0$
3. $x \in X^{us} \Leftrightarrow \exists$ *1-PS* λ, $\mu(\lambda, x) > 0$

A direct computation is very trickly in general, but when G is a torus $(= (\mathbb{C}^*)^n)$, it is a purely combinatorial problem.

Hilbert-Mumford criterion

Let T be a maximal torus of G.

Example: $G = \mathrm{GL}_n, \mathrm{SL}_n \Rightarrow T =$ set of diagonal matrices in G $\cong (\mathbb{C}^*)^n$.

X_T^{ss}: the semi-stable locus for the T-action.

Theorem (Hilbert-Mumford criterion, Ver 2)

1. $x \in X^{ss} \Leftrightarrow \forall\ g \in G,\ g \cdot x \in X_T^{ss}$
2. $x \in X^{s} \Leftrightarrow \forall\ g \in G,\ g \cdot x \in X_T^{s}$
3. $x \in X^{us} \Leftrightarrow \exists\ g \in G,\ g \cdot x \notin X_T^{ss}$

Summary

- The GIT quotient provides a method to construct an algebraic 'quotient' $X//G$ of a projective variety X.
- $X//G$ is a 'quotient' of $X^{ss} \subset X$, the semi-stable locus.
- $X//G$ contains a genuine quotient X^s/G of X^s, the stable locus.
- $X//G$ depends on the choice of linearization.

Part IV

GIT and Moduli Spaces

Moduli space of curves

$\mathcal{M}_g$: moduli space of smooth genus g complex curves ($=$ Riemann surfaces)

$\overline{\mathcal{M}}_g$: moduli space of stable genus g complex curves ($=$ compactification of $\mathcal{M}_g$)

$\overline{\mathcal{M}}_g = \{C \mid g(C) = g, C$ has at worst nodal singularities, $|\mathrm{Aut}(C)| < \infty\}$

Problem

Construct the moduli space $\overline{\mathcal{M}}_g$ as an algebraic variety.

Construction of moduli spaces of curves

Main idea: Moduli space of (abstract) varieties = moduli space of subvarieties in $\mathbb{P}^r/\mathrm{Aut}(\mathbb{P}^r) = \mathrm{Hilb}(\mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r)$

- Find a 'canonical' embedding of C in $\mathbb{P}^r$ $\Leftrightarrow$ Find a 'canonical' line bundle on C $\cdots$ ω_C (dualizing bundle)
- $C \in \overline{\mathcal{M}}_g \Rightarrow \omega_C^n$ is very ample if $n \geq 3$
- $C \overset{|\omega_C^n|}{\hookrightarrow} \mathbb{P}^r$

$r = \dim \mathrm{H}^0(C, \omega_C^n) - 1 = (2n-1)(g-1) - 1$

This embedding is unique up to a choice of ordered basis of $\mathrm{H}^0(C, \omega_C^n)$ $\Rightarrow$ need SL_{r+1}-quotient.

$\deg C = 2n(g-1) =: d,\ P(m) = dm + 1 - g$

$(C \subset \mathbb{P}^r) \in \mathrm{Hilb}^{P(m)}(\mathbb{P}^r) \subset \mathrm{Hilb}(\mathbb{P}^r)$ $\cdots$ subvariety parametrizing varieties with Hilbert polynomial $P(m)$

Construction of moduli spaces of curves

$$K := \{C \subset \mathbb{P}^{r-1} \mid \mathcal{O}(1)|_C \cong \omega_C^n\} \subset \mathrm{Hilb}^{P(m)}(\mathbb{P}^r)$$

It contains curves with very nasty singularities, too. However,

Theorem (Gieseker)

If $n \geq 5$, $K//\mathrm{SL}_{r+1} = K^{ss}/\mathrm{SL}_{r+1} = K^s/\mathrm{SL}_{r+1} \cong \overline{\mathcal{M}}_g$.

The point is that because $K//\mathrm{SL}_{r+1}$ is the quotient of K^{ss}, we can exclude many bad points!

Corollary

$\overline{\mathcal{M}}_g \subset \mathrm{Hilb}^{P(m)}(\mathbb{P}^r)//\mathrm{SL}_{r+1}$. *Therefore* $\overline{\mathcal{M}}_g$ *is a projective variety.*

Construction of moduli spaces of curves

The stability computation can be reduced to a combinatorial computation.

$$\mathrm{Hilb}^{P(m)}(\mathbb{P}^r) \hookrightarrow \mathrm{Gr}(\mathrm{Sym}^m \mathbb{C}^{r+1}, P(m)) \hookrightarrow \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m (\mathbb{C}^{r+1})^*)$$

- $C \subset \mathbb{P}^r$ $\cdots$ curve in $\mathbb{P}^r$
- I_C $\cdots$ ideal of functions vanishing along C
- $\mathrm{H}^0(\mathbb{P}^r, I_C(m))$ $\cdots$ vector space of degree m polynomials vanishing along C
- If $m \gg 0$, we have a short exact sequence

$$0 \to \mathrm{H}^0(\mathbb{P}^r, I_C(m)) \to \mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) \to \mathrm{H}^0(C, \mathcal{O}_C(m)) \to 0$$

$\cdots$ $P(m)$-dimensional quotient space of a fixed $\binom{r+m}{m}$-dimensional vector space $\cdots$ a point on Grassmannian.

Construction of moduli spaces of curves

$$\mathrm{Hilb}^{P(m)}(\mathbb{P}^r) \hookrightarrow \mathrm{Gr}(\mathrm{Sym}^m \mathbb{C}^{r+1}, P(m)) \hookrightarrow \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m(\mathbb{C}^{r+1})^*)$$

- $\bigwedge^{P(m)} \mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(m)) \to \bigwedge^{P(m)} \mathrm{H}^0(C, \mathcal{O}_C(m)) \to 0 \ \cdots$ 1-dimensional quotient space of a fixed vector space
- $0 \to \bigwedge^{P(m)} \mathrm{H}^0(C, \mathcal{O}_C(m))^* \to \bigwedge^{P(m)} \mathrm{H}^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(m))^* \ \cdots$ 1-dimensional subspace of a fixed vector space $\cdots$ a point on a projective space.

Now $K \subset \mathrm{Hilb}^{P(m)}(\mathbb{P}^r) \subset \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m(\mathbb{C}^{r+1})^*)$ and

$$\overline{\mathcal{M}}_g = K//\mathrm{SL}_{r+1} \subset \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m(\mathbb{C}^{r+1})^*)//\mathrm{SL}_{r+1}.$$

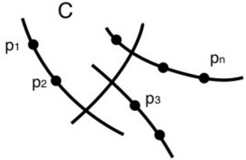
Do the GIT stability computation on the right hand side.

Genus 0, pointed curve cases

Consider moduli spaces of genus 0, n-pointed curves.

There is a 'canonical' compact moduli space

$$\overline{\mathrm{M}}_{0,n} := \left\{ \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right.$$



$$\left. \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right| \; \begin{array}{c} g(C) = 0 \\ p_i : \text{ distinct smooth points} \\ \mathrm{Aut} < \infty \end{array} \left. \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right\} /\sim$$

$\cdots$ Deligne-Mumford compactification, or moduli space of n-pointed stable rational curves.

There are also many alternative moduli spaces including Hassett's moduli spaces of weighted pointed stable curves $\overline{\mathrm{M}}_{0,A}$, Kontsevich-Boggi space $\overline{\mathrm{M}}_{0,n}^{Bog}$, and so on.

Question

Can one find a unified construction of all of such moduli spaces?

Genus 0, pointed curve cases

Natural construction:

$\mathrm{Chow}_{1,d}(\mathbb{P}^d) :=$ Chow variety of dimension 1, degree d cycles in $\mathbb{P}^d$

$U_{d,n} := \{(C, x_1, \cdots, x_n) \mid x_i \in C\} \subset \mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n$

$U_{d,n} // \mathrm{SL}_{d+1}$ is a candidate of a moduli space

Theorem (Giansiracusa, Jensen, M)

1. *We determined linearizations with* $U_{d,n}^{ss} \neq \emptyset$.
2. *We computed* $U_{d,n}^{ss}$ *for each linearization. There is a purely combinatorial description.*
3. *All currently known projective moduli spaces of genus 0 curves can be obtained as* $U_{d,n} // \mathrm{SL}_{d+1}$ *and there are more.*
4. $\overline{\mathrm{M}}_{0,n}$ *is dominant among them - there is a birational morphism* $\overline{\mathrm{M}}_{0,n} \to U_{d,n} // \mathrm{SL}_{d+1}$.

Thank you!