Chow ring of the moduli space of stable sheaves supported on quartic curves

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Problem

Let $\mathrm{M}(d,\chi) = \mathrm{M}_{\mathbb{P}^2}(dm+\chi)$ be the moduli space of semistable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $dm+\chi$. Compute its topological invariants.

Motivation

1. Curve counting (GV invariants) on CY threefolds (Hosono-Saito-Takahashi, Katz, Kiem-Li, $\cdots$) requires the study of $\mathrm{M}_X(dm+1)$ for a CY threefold X. For a local CY $X = \mathcal{O}_{\mathbb{P}^2}(-3)$, $\mathrm{M}_X(dm+\chi) = \mathrm{M}(d,\chi)$.
2. Le Potier's strange duality conjecture:
   For $c, v \in \mathrm{K}(\mathbb{P}^2)$ such that $\chi(c\cdot v) = 0$,
   $$\mathrm{H}^0(\mathrm{M}_{\mathbb{P}^2}(c), \lambda(-v))^* \overset{?}{\cong} \mathrm{H}^0(\mathrm{M}_{\mathbb{P}^2}(v), \lambda(-c)).$$
   What is $\dim \mathrm{H}^0(\mathrm{M}_{\mathbb{P}^2}(c), \lambda(-v))$?

Known results

1. $d = 1,2 \Rightarrow \mathrm{M}(d,1) \cong |\mathcal{O}_{\mathbb{P}^2}(d)|$
2. $d = 3 \Rightarrow \mathrm{M}(3,1) =$ universal cubic on $\mathbb{P}^2$
3. $\mathrm{M}(d,\chi) \cong \mathrm{M}(d,\chi+d)$
4. (Maican [5]) $\mathrm{M}(d,\chi) \cong \mathrm{M}(d,-\chi)$,
5. Poincaré polynomial of $\mathrm{M}(d,1)$ is known for $d \le 6$.
6. (Markman [6]) $\mathrm{A}^*(\mathrm{M}(d,\chi),\mathbb{Q}) = \mathrm{H}^*(\mathrm{M}(d,\chi),\mathbb{Q})$

Main Theorem ([1])

Let $\mathrm{M}(4,1)$ be the moduli space of stable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $4m+1$. Then

$$\begin{gathered}\mathrm{A}^*(\mathrm{M}(4,1),\mathbb{Q}) \cong \mathbb{Q}[\alpha,\beta,x,y,z]/\langle xz-yz, \beta^2z-3yz-9z^2, 3\alpha^2z-\alpha\beta z+yz,\\ \beta^2y-3y^2-9yz, \beta^2x-xy-3y^2-3\alpha\beta z-9yz+9z^2,\\ \beta^4+3x^2-9xy-3y^2-54yz-81z^2, \beta yz+9\alpha z^2-3\beta z^2,\\ 2\beta xy-3\beta y^2-9\alpha yz-27\alpha z^2+9\beta z^2, 3\beta x^2-7\beta y^2-36\alpha yz-108\alpha z^2+36\beta z^2,\\ \alpha^{12}+3\alpha^{11}\beta+3\alpha^{10}(\beta^2+2x-y)+\alpha^9(-\beta^3+12\beta x+2\beta y)\\ +3\alpha^8(9x^2-16xy+27y^2)+28\alpha^7\beta y^2+56\alpha^6y^3+201\alpha\beta z^5-19yz^5-613z^6,\\ 6\alpha^{10}xy-12\alpha^{10}y^2-10\alpha^9\beta y^2-45\alpha^8y^3-104\alpha\beta z^6+2yz^6+310z^7\rangle\end{gathered}$$

where α, β are of degree 1 and x, y, z are of degree 2.

Idea of Proof

We show that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{N}(3;2,3) & \mathrm{Hilb}^3(\mathbb{P}^2) \arrow[l] & \mathrm{M}^\infty(4,1) \arrow[l, Rightarrow] \arrow[d, rightsquigarrow] \\
\mathrm{Q} \arrow[u, Rightarrow] & \mathrm{M}^+(4,1) \arrow[l] \arrow[ur, dashed, leftrightarrow, "\beta"] \arrow[r, rightsquigarrow] \arrow[d] & \mathrm{M}^3(4,1) \arrow[d, Rightarrow] \\
 & \mathrm{M}(4,1) \arrow[ul, dashed, leftrightarrow, "\gamma"] \arrow[r, Rightarrow, "\pi"] & |\mathcal{O}_{\mathbb{P}^2}(4)|.
\end{tikzcd}$$

where

- $\mathrm{N}(3;2,3)$: moduli of Kronecker quiver representations,
- Q: projective bundle over $\mathrm{N}(3;2,3)$,
- $\mathrm{M}^\alpha(4,1)$: moduli space of α-stable pairs.

($\rightarrow$): smooth blow-up, ($\Rightarrow$): fibration, ($\rightsquigarrow$): small contraction, ($\dashrightarrow$): flip

1. $\mathrm{M}^\infty(4,1) \cong$ relative Hilbert scheme of 3 points on the universal quartic plane curve
2. β: wall-crossing of α-stable pairs
3. γ: Bridgeland wall-crossing
4. π: Fitting map

Description at a general point:

$$\begin{tikzcd}
{[\mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 3}]} & I_{Z,\mathbb{P}^2} \arrow[l] & I_{Z,C} \arrow[l] \arrow[dd] \\
{[\mathcal{O} \to I_{Z,\mathbb{P}^2}(4)]} \arrow[u] & {[\mathcal{O} \to \mathcal{O}_C(Z)]} \arrow[l] \arrow[ur, leftrightarrow] \arrow[d] & \\
 & \mathcal{O}_C(Z) \arrow[ul, leftrightarrow] \arrow[r] & \mathrm{supp}(C)
\end{tikzcd}$$

Description along exceptional loci:

$$\begin{tikzcd}
(\ell \subset \mathbb{P}^2) & & \\
(p \in \ell) \arrow[u] & (p \in \ell \cap C) \arrow[l] \arrow[d] & \\
 & I_{p,C}(1) \arrow[r] & \mathrm{supp}(C)
\end{tikzcd}$$

A key technique to construct a map $\mathrm{M}^+(4,1) \to \mathrm{Q}$ is the elementary modification of pairs ([3]).
By using standard formulas for blow-ups and projective bundles, we obtain $\mathrm{A}^*(\mathrm{M}(4,1),\mathbb{Q})$ from $\mathrm{A}^*(\mathrm{N}(3;2,3),\mathbb{Q})$, which was computed by Ellingsrud and Strømme ([2]).

Modular description of Chow classes

From the diagram, we also obtain a moduli theoretic description of some Chow classes. For a general $F \in \mathrm{M}(4,1)$, there is a unique (up to scalar) section $\mathcal{O}_{\mathbb{P}^2} \overset{s}{\to} F$ and $Q_F = \mathrm{coker}(s)$ is a finite scheme.

- $\alpha = \{F \mid p \in \mathrm{supp}(F)\}$ for a point $p \in \mathbb{P}^2$
- $-\beta = \overline{\{F \mid Q_F \cap \ell \neq \emptyset\}}$ for a line $\ell \subset \mathbb{P}^2$
- $x - y = \overline{\{F \mid p \in Q_F\}}$ for a point $p \in \mathbb{P}^2$
- $z = \{F \mid \mathrm{H}^0(F) = 2\}$, the Brill-Noether locus

Application to generalized Donaldson numbers

Corollary ([1])

The total Chern class of $\mathrm{M}(4,1)$ was computed.
$c_1 = 12\alpha$, $c_2 = 66\alpha^2 - 3\alpha\beta - 3\beta^2 + 6x + 2y + 34z, \cdots$

Thus we can compute the Euler characteristic (or generalized Donaldson number [4]) of every line bundle on $\mathrm{M}(4,1)$.

Conjecture

For the Fitting map $\pi : \mathrm{M}(d,1) \to |\mathcal{O}_{\mathbb{P}^2}(d)|$, let $\alpha_d := \pi^*\mathcal{O}_{|\mathcal{O}_{\mathbb{P}^2}(d)|}(1)$. Then
$$\chi(\mathrm{M}(d,1), m\alpha_d) = \binom{m+3d-1}{m}.$$

This is true for $d \le 4$.

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