MORI'S PROGRAM FOR $\overline{\mathrm{M}}_{0,6}$ WITH SYMMETRIC DIVISORS

HAN-BOM MOON

ABSTRACT. We complete Mori's program with symmetric divisors for the moduli space of stable six-pointed rational curves. As an application, we give an alternative proof of the complete Mori's program of the moduli space of genus two stable curves, which was first done by Hassett.

1. INTRODUCTION

Since Hassett and Hyeon initiated the study of birational geometry of moduli spaces in the viewpoint of Mori's program in [Has05, HH09, HH13], there have been a tremendous amount of results in this direction. **Mori's program** for a moduli space M consists of: 1) Compute the cone of effective divisors of M. 2) For an effective $\mathbb{Q}$-divisor D, find the birational model

$$M(D) := \mathrm{Proj} \bigoplus_{m \ge 0} H^0(M, \mathcal{O}(mD)).$$

3) Finally, study the moduli theoretic meaning of $M(D)$ and its relation with M.

In this paper, we complete Mori's program with symmetric divisors for the moduli space $\overline{\mathrm{M}}_{0,6}$ of stable six-pointed rational curves. On the effective cone of $\overline{\mathrm{M}}_{0,n}$, only the subcone generated by $K_{\overline{\mathrm{M}}_{0,n}}$ and ψ_i-classes has been studied intensively in [Sim08, AS12, FS11, KM11, Moo13]. One obstacle of the completion of Mori's full program for $\overline{\mathrm{M}}_{0,n}$ is that the cone of effective divisors is huge and unknown. As an initial step, we will focus on symmetric divisors, i.e., divisors which are invariant under the natural S_n-action on $\overline{\mathrm{M}}_{0,n}$.

To state our result neatly, we use the interval notation from [Che08]. For two divisors D_1 and D_2, (D_1, D_2) is the set of divisors $aD_1 + bD_2$ such that $a, b > 0$ and $[D_1, D_2]$ is the set of divisors $aD_1 + bD_2$ with $a, b \ge 0$. We can define $[D_1, D_2)$ and $(D_1, D_2]$ in a similar way. For the description of relevant divisor classes, see Definition 2.1.

Theorem 1.1 (Theorem 5.2)**.** *Let D be a symmetric effective divisor on $\overline{\mathrm{M}}_{0,6}$. Then:*

1. *(1) If $D \in (-K_{\overline{\mathrm{M}}_{0,6}}, K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi)$, $\overline{\mathrm{M}}_{0,6}(D) \cong \overline{\mathrm{M}}_{0,6}$.*
2. *(2) If $D \in [K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi, B_3)$, $\overline{\mathrm{M}}_{0,6}(D) \cong (\mathbb{P}^1)^6 /\!/_L SL_2$ with the symmetric linearization L.*
3. *(3) If $D \in (B_2, -K_{\overline{\mathrm{M}}_{0,6}}]$, $\overline{\mathrm{M}}_{0,6}(D)$ is the Veronese quotient V_A^2 with symmetric weight data $A = (\frac{1}{2}, \cdots, \frac{1}{2})$.*
4. *(4) Both $\overline{\mathrm{M}}_{0,6}(B_2)$ and $\overline{\mathrm{M}}_{0,6}(B_3)$ are a point.*

Date: July 30, 2014.

It is interesting that both $(\mathbb{P}^1)^6 /\!/_L SL_2$ and V_A^2 are classically known varieties. $(\mathbb{P}^1)^6 /\!/_L SL_2$ is isomorphic to the **Segre cubic** $\mathcal{S}_3$ (Remark 5.5) and V_A^2 is isomorphic to the **Igusa quartic** $\mathcal{I}_4$ (Remark 4.2) or **Castelnuovo-Richmond quartic**. Also $\mathcal{I}_4$ is isomorphic to the Satake compactification $\overline{\mathcal{A}}_2(2)$ of the moduli space of principally polarized abelian surfaces with level two structures ([Igu64]). Moreover, these two varieties are known to be projectively dual to each other ([DO88, Remark I.3]).

All birational models appear here are classically known varieties with or without their modular interpretations. For example, see [Dol12, Section 9.4] and references therein. So this article is an interpretation of the relation between classically known varieties using a modern viewpoint of Mori's program. Also Items (1) and (2), which are in the direction toward the canonical divisor, are proved in [Sim08, FS11, AS12, KM11, Moo13].

The author wants to point out a simple but important observation. As we can see in the definition of V_A^2 in Section 3, the birational model $\mathcal{I}_4$ is not a moduli space of abstract pointed curves, but that of (equivalent classes of) configurations of points. In general to find birational models of $\overline{\mathrm{M}}_{0,n}$ in the direction towards the anti-canonical divisor, it is insufficient to study moduli spaces of rational curves with worse singularities. The reason is that all moduli spaces of pointed rational curves (with worse singularities) are contractions of $\overline{\mathrm{M}}_{0,n}$ ([Smy13]), as opposed to the case of the moduli stack $\overline{\mathcal{M}}_g$. In the viewpoint of symmetric Mori's program, $\overline{\mathrm{M}}_{0,6}$ we discuss here is very simple in the sense that there is no flip. Thus we can explain everything by using well-known contractions. But for $n \ge 7$, there must be several flips even for symmetric divisors. So to understand Mori's program for $\overline{\mathrm{M}}_{0,n}$ for larger n, we need to 'find' completely new modular interpretations of birational models.

As a quick application of our results, we give a complete description of Mori's program for $\overline{\mathcal{M}}_2$. All smooth genus two curves are hyperelliptic, thus the coarse moduli space $\overline{\mathrm{M}}_2$ of the moduli space of genus two stable curves is isomorphic to $\overline{\mathrm{M}}_{0,6}/S_6$ ([AL02, Corollary 2.5]). Therefore we can directly translate symmetric Mori's program, as Mori's program for $\overline{\mathrm{M}}_2$ and that for $\overline{\mathcal{M}}_2$. The investigation of Mori's program for $\overline{\mathcal{M}}_2$ was done by Hassett in [Has05], as an initial step of the Hassett-Keel program. It has been one of the most influential projects on the birational geometry of moduli spaces in the last several years. As a consequence of Theorem 1.1, we give a different proof of Hassett's theorem ([Has05, Theorem 4.10]) for $\overline{\mathcal{M}}_2$.

Theorem 1.2 (Theorem 6.3)**.** *Let D be an effective divisor on $\overline{\mathcal{M}}_2$. Then:*

1. *If $D \in (\lambda, \delta_0 + 12\delta_1)$, $\overline{\mathcal{M}}_2(D) \cong \overline{\mathrm{M}}_2$.*
2. *If $D \in [\delta_0 + 12\delta_1, \delta_1)$, $\overline{\mathcal{M}}_2(D) \cong \mathbb{P}^6 /\!/ SL_2$.*
3. *If $D \in (\delta_0, \lambda]$, $\overline{\mathcal{M}}_2(D)$ is the Satake compactification $\overline{\mathcal{A}}_2^{\mathrm{Sat}}$ of the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces.*
4. *Both $\overline{\mathcal{M}}_2(\delta_0)$ and $\overline{\mathcal{M}}_2(\delta_1)$ are a point.*

We can summarize Mori's program for $\overline{\mathrm{M}}_{0,6}$ and $\overline{\mathcal{M}}_2$ with Figure 1. The diagonal maps are divisorial contractions (contracted divisors are indicated on arrows) and vertical maps are S_6-quotients.

$$\begin{tikzcd}
 & \overline{\mathrm{M}}_{0,6} \arrow[dl, "B_2"'] \arrow[dd] \arrow[dr, "B_3"] & \\
\mathcal{I}_4 = \overline{\mathcal{A}}_2(2) = V_A^2 \arrow[dd] & & (\mathbb{P}^1)^6//_L SL_2 = \mathcal{S}_3 \arrow[dd] \\
 & \overline{\mathrm{M}}_2 \arrow[dl, "\Delta_0"'] \arrow[dr, "\Delta_1"] & \\
\overline{\mathcal{A}}_2^{\mathrm{Sat}} & & \mathbb{P}^6//SL_2
\end{tikzcd}$$

FIGURE 1. Mori's program for $\overline{\mathrm{M}}_{0,6}$ and $\overline{\mathcal{M}}_2$

After the author finished the preparation of this manuscript, he noticed that Lange and Ortega recently ran the log minimal model program for a compactification $\overline{S}_2^+$ of the moduli space S_2^+ of even spin curves of genus 2 in [LO13]. On the level of coarse moduli spaces, there is an M_2-morphism $\pi : \mathrm{M}_{0,6} \to S_2^+$. This map π is a quotient map for the G-action where G is a subgroup of S_6 of order 72 (For more details, see [LO13, Section 7]). Thus one can regard the main result of this paper as a theorem parallel to [LO13, Theorem 2] on a finite cover.

This paper is organized as follows. In Section 2, we review basic facts about divisor and curve classes on $\overline{\mathrm{M}}_{0,n}$. Section 3 is for the background about Veronese quotients. We will study the geometry of a particular Veronese quotient V_A^2 with symmetric weight data A in Section 4. By using them, we prove Theorem 1.1 in Section 5. Section 6 contains a proof of Theorem 1.2.

We will work over an algebraically closed field of characteristic 0.

2. DIVISORS AND CURVES ON $\overline{\mathrm{M}}_{0,n}$

We begin by reviewing general facts about divisors and curves on $\overline{\mathrm{M}}_{0,n}$. The moduli space $\overline{\mathrm{M}}_{0,n}$ of stable n-pointed rational curves is a smooth projective variety of dimension $n-3$ with a natural S_n-action permuting marked points. A divisor D on $\overline{\mathrm{M}}_{0,n}$ is called symmetric if it is invariant under the S_n-action. The Neron-Severi vector space $N^1(\overline{\mathrm{M}}_{0,n})$ has dimension $2^{n-1} - \binom{n}{2} - 1$, but its S_n-invariant part $N^1(\overline{\mathrm{M}}_{0,n})^{S_n} \cong N^1(\overline{\mathrm{M}}_{0,n}/S_n)$ is $(\lfloor n/2 \rfloor - 1)$-dimensional ([KM13, Theorem 1.3]). The following is a list of natural symmetric divisors on $\overline{\mathrm{M}}_{0,n}$.

Definition 2.1. (1) For $i = 2, 3, \cdots, n-2$, let B_i be the closure of the locus of curves C with two irreducible components C_1 and C_2 such that C_1 (resp. C_2) contains i (resp. $n-i$) marked points. B_i is called a symmetric boundary divisor. By definition, $B_i = B_{n-i}$. Let $B = \sum_{i=2}^{\lfloor n/2 \rfloor} B_i$ be the total boundary divisor.

(2) Fix $1 \le i \le n$. Let $\mathbb{L}_i$ be a line bundle on $\overline{\mathrm{M}}_{0,n}$ such that over $(C, x_1, \cdots, x_n) \in \overline{\mathrm{M}}_{0,n}$, the fiber is Ω_{C,x_i}, the cotangent space of C at x_i. Let $\psi_i = c_1(\mathbb{L}_i)$. If we denote $\psi = \sum_{i=1}^n \psi_i$, then ψ is an S_n-invariant divisor.

(3) Let $K_{\overline{\mathrm{M}}_{0,n}}$ be the canonical divisor of $\overline{\mathrm{M}}_{0,n}$.

The symmetric effective cone $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})^{S_n} \cong \mathrm{Eff}(\overline{\mathrm{M}}_{0,n}/S_n)$ is generated by symmetric boundary divisors ([KM13, Theorem 1.3]). Thus we can write down $K_{\overline{\mathrm{M}}_{0,n}}$ and ψ as nonnegative linear combinations of boundary divisors.

Lemma 2.2. [Pan97, Proposition 2], [Moo13, Lemma 2.9] *In $N^1(\overline{\mathrm{M}}_{0,n})$, the following relations hold.*

(1) $K_{\overline{\mathrm{M}}_{0,n}} = \sum_{i=2}^{\lfloor n/2 \rfloor} \left(\frac{i(n-i)}{n-1} - 2 \right) B_i.$
(2) $\psi = K_{\overline{\mathrm{M}}_{0,n}} + 2B.$

Now we move to curve classes on $\overline{\mathrm{M}}_{0,n}$. Let $S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4 = \{1, 2, \cdots, n\}$ be a partition. Let $F(S_1, S_2, S_3, S_4)$ be an F-curve class corresponding to the partition ([KM13, Section 4]).

Lemma 2.3. [KM13, Corollary 4.4] *Let $F = F(S_1, S_2, S_3, S_4)$ be an F-curve and let $a_j = |S_j|$. Then*

$$F \cdot \sum r_i B_i = -r_{a_1} - r_{a_2} - r_{a_3} - r_{a_4} + r_{a_1+a_2} + r_{a_1+a_3} + r_{a_1+a_4}$$

if we define $r_1 = 0$ and $r_{a+b} = r_{n-a-b}$.

We need to know another curve class C_j (see [KM13, Lemma 4.8]). Fix a j-pointed $\mathbb{P}^1$. And let x be a moving point on $\mathbb{P}^1$. By gluing a fixed $(n-j+1)$-pointed $\mathbb{P}^1$ whose last marked point is y to the j-pointed $\mathbb{P}^1$ along x and y and stabilizing it, we obtain an one parameter family of n-pointed stable curves over $\mathbb{P}^1$, i.e., a curve $C_j \cong \mathbb{P}^1$ on $\overline{\mathrm{M}}_{0,n}$.

Lemma 2.4. [KM13, Lemma 4.8]

$$C_j \cdot B_i = \begin{cases} j, & i = j-1, \\ -(j-2), & i = j, \\ 0, & \textit{otherwise.} \end{cases}$$

For the convenience of readers, we leave a special case of $\overline{\mathrm{M}}_{0,6}$ below. The proof is an easy combination of above results.

Corollary 2.5. *The S_6-invariant Neron-Severi space $N^1(\overline{\mathrm{M}}_{0,6})^{S_6}$ has dimension two. The symmetric effective cone $\mathrm{Eff}(\overline{\mathrm{M}}_{0,6})^{S_6}$ is generated by B_2 and B_3. Moreover,*

(1) $K_{\overline{\mathrm{M}}_{0,6}} = -\frac{2}{5}B_2 - \frac{1}{5}B_3$,
(2) $\psi = \frac{8}{5}B_2 + \frac{9}{5}B_3$,
(3) $B_2 = -\frac{9}{2}K_{\overline{\mathrm{M}}_{0,6}} - \frac{1}{2}\psi$,
(4) $B_3 = 4K_{\overline{\mathrm{M}}_{0,6}} + \psi$.

Figure 2 shows several rays in $N^1(\overline{\mathrm{M}}_{0,6})^{S_6}$ generated by special divisors.

On $\overline{\mathrm{M}}_{0,6}$, there are only two types of F-curves, whose partition is of the form $1+1+1+3$ or $1+1+2+2$. We will denote the corresponding F-curves by $F_{1,1,1,3}$ and $F_{1,1,2,2}$ respectively.

Corollary 2.6. *On $\overline{\mathrm{M}}_{0,6}$, the intersection of symmetric divisors and curve classes are given by the following table.*

	ψ	$K_{\overline{\mathrm{M}}_{0,6}}$	B_2	B_3
$F_{1,1,1,3}$	3	-1	3	-1
$F_{1,1,2,2}$	2	0	-1	2
C_4	4	0	-2	4

Note that on $\overline{\mathrm{M}}_{0,6}$, $C_3 = F_{1,1,1,3}$.

3. Veronese quotients and their geometric properties

In this section, we give a review about (a special case) of Veronese quotients introduced in [Gia13]. See [GJM13, GJMS13] for a generalization. The following description is different from the original one in [Gia13] (However, see [Gia13, Remark 2.4]). We will use a construction via moduli spaces of stable maps, which is useful for our purpose, in particular for the description of the morphism $\varphi_A : \overline{\mathrm{M}}_{0,n} \to V_A^d$.

3.1. **Veronese quotients.** Let $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$ be Kontsevich's moduli space of stable maps ([FP97]). It parametrizes maps $f : (C, x_1, \cdots, x_n) \to \mathbb{P}^d$ from an arithmetic genus 0 curve C with n-marked points to $\mathbb{P}^d$ such that $f_*[C] = d[L]$, where L is a line in $\mathbb{P}^d$, with the following stability conditions. Such a map f is called stable if

- C has at worst nodal singularities,
- x_i are distinct smooth points on C,
- $\omega_C + \sum x_i$ is f-ample.

There are n evaluation maps $e_i : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \mathbb{P}^d$. By taking the product of these maps, we have a map

$$e : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to (\mathbb{P}^d)^n.$$

Let $U_{d,n}$ be the image of e. Note that SL_{d+1} acts on both $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$ and $(\mathbb{P}^d)^n$ via $SL_{d+1} \to \mathrm{Aut}(\mathbb{P}^d)$ and e is SL_{d+1}-equivariant. Thus $U_{d,n}$ is an SL_{d+1}-invariant subvariety of $(\mathbb{P}^d)^n$.

For a choice of positive rational numbers $A = (a_1, \cdots, a_n)$, we can construct a $\mathbb{Q}$-linearization $L_A := \mathcal{O}(a_1) \otimes \mathcal{O}(a_2) \otimes \cdots \otimes \mathcal{O}(a_n)$. Since the stability does not change if we replace A by its scalar multiple, we will normalize it as $\sum_i a_i = d + 1$. For $(\mathbb{P}^d)^n$, the (semi)stability can be computed explicitly.

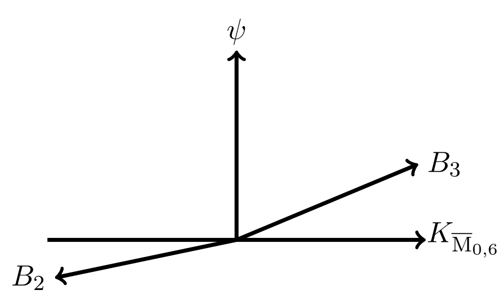


FIGURE 2. The Neron-Severi space of $\overline{\mathrm{M}}_{0,6}$

Theorem 3.1. [Dol03, Theorem 11.1] *Let $A = (a_1, \cdots, a_n)$ be a normalized linearization. A configuration $(x_1, \cdots, x_n) \in (\mathbb{P}^d)^n$ is (semi)stable if and only if for any proper linear subspace $W \subset \mathbb{P}^d$,*

$$\sum_{x_j \in W} a_j \ (\le) < \dim W + 1.$$

In particular, to guarantee the nonemptiness of (semi)stable locus, we need a necessary condition $a_i\ (\le) < 1$ for all i. Thus the hypersimplex

$$\Delta(d+1, n) = \{(a_1, \cdots, a_n) \in \mathbb{Q}^n \mid 0 \le a_i \le 1, \sum_{i=1}^n a_i = d+1\}$$

can be regarded as the space of effective linearizations.

Definition 3.2. Let $A = (a_1, \cdots, a_n) \in \Delta(d+1, n)$ such that $n \ge d+3$. The **Veronese quotient** is the GIT quotient

$$V_A^d := U_{d,n} /\!/_{L_A} SL_{d+1}.$$

Remark 3.3.
(1) It is called Veronese quotient because for a general configuration $(x_1, \cdots, x_n)$ with $n \ge d+3$, if there exists a rational normal curve C (a Veronese embedding of $\mathbb{P}^1$) in $\mathbb{P}^d$ such that $x_i \in C$ for all i, then such C is unique.
(2) This is a special case $\gamma = 0$ of general Veronese quotients described in [GJM13, GJMS13].
(3) Up to projective equivalence, there is a unique rational normal curve in $\mathbb{P}^d$. Thus after taking the quotient, we can regard it as a moduli space of configurations of points on an abstract rational curve and their degenerations. So V_A^d is birational to $\overline{\mathrm{M}}_{0,n}$.

Example 3.4. If $d = 1$, then $U_{1,n} = (\mathbb{P}^1)^n$ and the GIT quotient $(\mathbb{P}^1)^n /\!/_{L_A} SL_2$ itself is birational to $\overline{\mathrm{M}}_{0,n}$. In this case, the existence of a birational morphism $\rho : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n /\!/_{L_A} SL_2$ is proved in [Kap93].

When n is even and L_A is a symmetric linearization, $(\mathbb{P}^1)^n /\!/ SL_2$ has $\binom{n}{2}$ singular points. When $n = 6$, the map ρ contracts an irreducible component of a boundary divisor B_3 to a singular point. In particular, $F_{1,1,1,3}$ is contracted. Thus the semi-ample divisor $\rho^* L$ for an ample divisor L on $(\mathbb{P}^1)^6 /\!/_{L_A} SL_2$ is a scalar multiple of $K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi$ (see Corollary 2.6).

Example 3.5. For the purpose of this paper, the most important example is V_A^2, where $n = 6$ and $A = (\frac{1}{2}, \cdots, \frac{1}{2})$. In this case there are many strictly semistable points on V_A^2. We will study its (semi)stability in Section 4 in detail.

3.2. **Morphisms from $\overline{\mathrm{M}}_{0,n}$ and canonical polarizations.** One interesting common property of Veronese quotients is that they admit birational morphisms from $\overline{\mathrm{M}}_{0,n}$. This section is an outline of a proof in [GJM13]. You can find an original proof via Chow quotients in [Gia13, Section 3].

For $(f : (C, x_1, \cdots, x_n) \to \mathbb{P}^d) \in \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$, by forgetting the map f and stabilizing the domain, we can obtain a stable rational curve $(C^s, x_1, \cdots, x_n) \in \overline{\mathrm{M}}_{0,n}$. Thus there is a forgetful morphism $F : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \overline{\mathrm{M}}_{0,n}$.

$$\begin{tikzcd} \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \arrow[r, "e"] \arrow[d, "F"'] & (\mathbb{P}^d)^n \\ \overline{\mathrm{M}}_{0,n} & \end{tikzcd}$$

For an effective linearization L_A on $(\mathbb{P}^d)^n$, there exists an effective linearization L'_A on $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$ such that

$$e^{-1}(((\mathbb{P}^d)^n)^s(L_A)) \subset \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)^s(L'_A) \subset \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)^{ss}(L'_A) \subset e^{-1}(((\mathbb{P}^d)^n)^{ss}(L_A))$$

where $X^{ss}(L)$ (resp. $X^s(L)$) is the semistable (resp. stable) part of X with respect to the linearization L. In particular, we have a quotient morphism $\overline{e} : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)//_{L'_A} SL_{d+1} \to (\mathbb{P}^d)^n//_{L_A} SL_{d+1}$.

Since F is SL_{d+1}-invariant, there exists a quotient map $\overline{F}$.

$$\begin{array}{ccc} \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)//_{L'_A} SL_{d+1} & \xrightarrow{\overline{e}} & (\mathbb{P}^d)^n//_{L_A} SL_{d+1} \\ \Big\downarrow{\scriptstyle \overline{F}} & & \\ \overline{\mathrm{M}}_{0,n} & & \end{array}$$

In [GJM13, Proposition 4.6], it is proved that for a general effective linearization (stability and semistability on $(\mathbb{P}^d)^n$ coincide for L_A), $\overline{F}$ is an isomorphism. Thus we have a morphism

$$\varphi_A = \overline{ev} \circ \overline{F}^{-1} : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^d)^n//_{L_A} SL_{d+1}.$$

It is straightforward to check that the image of φ_A is V_A^d.

For any effective linearization L_A, if we perturb it slightly, we obtain an effective linearization L_{A_ϵ} such that the stability coincides with the semistability. From the general theory of the variation of GIT, we have a morphism

$$\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)//_{L'_{A_\epsilon}} SL_{d+1} \to (\mathbb{P}^d)^n//_{L_{A_\epsilon}} SL_{d+1} \to (\mathbb{P}^d)^n//_{L_A} SL_{d+1}.$$

We will denote it by $\overline{e}$, too. Also φ_A is defined as $\overline{e} \circ \overline{F}^{-1}$.

Remark 3.6. This morphism φ_A can be described in the following slightly different way. Note that for any effective linearization L_A on $(\mathbb{P}^d)^n$, there is a commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)^s(L'_{A_\epsilon}) & \hookrightarrow & \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)^{ss}(L'_A) \\ \Big\downarrow & & \Big\downarrow \\ ((\mathbb{P}^d)^n)^s(L_{A_\epsilon}) & \hookrightarrow & ((\mathbb{P}^d)^n)^{ss}(L_A) \end{array}$$

where L'_A, L'_{A_ϵ} are linearizations explained above. Finally, we have a quotient diagram

$$\begin{array}{ccc} \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)//_{L'_{A_\epsilon}} SL_{d+1} & \longrightarrow & \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)//_{L'_A} SL_{d+1} \\ \Big\downarrow & & \Big\downarrow \\ ((\mathbb{P}^d)^n)//_{L_{A_\epsilon}} SL_{d+1} & \longrightarrow & ((\mathbb{P}^d)^n)//_{L_A} SL_{d+1} \end{array}$$

which is commutative.

This result gives us a practical way to describe the contraction map $\varphi_{\mathcal{A}}$. Fix a curve $(C, x_1, \cdots, x_n) \in \overline{\mathrm{M}}_{0,n}$. First of all, find any stable map $f : (\tilde{C}, x_1, \cdots, x_n) \to \mathbb{P}^d$ up to projective transformation such that $f \in \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)^{ss}$ with respect to $L_{A'}$ and $F(f) = (C, x_1, \cdots, x_n)$. (If there is a strictly semistable point, then f and $\tilde{C}$ may not be unique.) Indeed, f can be determined by the degree of f on each irreducible component of $\tilde{C}$. Then we obtain a point configuration $x = (f(x_1), \cdots, f(x_n)) \in (\mathbb{P}^d)^n$. By the variation of GIT, there exists a unique closed orbit in $((\mathbb{P}^d)^n)^{ss}$ with respect to L_A which is contained in the closure of the orbit of x. This is $\varphi_A(C, x_1, \cdots, x_n)$. For more details, see [GJM13, Section 3].

A GIT quotient V_A^d has a canonical polarization $\overline{L}_A$ from its definition. Since $\varphi_A : \overline{\mathrm{M}}_{0,n} \to V_A^d$ is a regular morphism, we obtain a semi-ample line bundle $D_A := \varphi_A^*(\overline{L}_A)$ on $\overline{\mathrm{M}}_{0,n}$.

The numerical class of D_A is computed in [GJMS13, Theorem 2.1] in a broader context (In our situation, $\gamma = 0$ in the statement of the Theorem.). The following result is a special case we want to use in this article.

Lemma 3.7. *Suppose that $n = 6$ and $A = (\frac{1}{2}, \cdots, \frac{1}{2})$. Consider V_A^2 and the pull-back $D_A = \varphi_A^* \overline{L}_A$ of the canonical polarization. Then $F_{1,1,1,3} \cdot D_A = \frac{1}{2}$ and $F_{1,1,2,2} \cdot D_A = 0$.*

4. An explicit computation of the Veronese quotient V_A^2

When $n = 6$, with respect to the symmetric linearization L_A, $U_{2,6}$ has strictly semistable points. Thus to describe $V_A^2 = U_{2,6} /\!/_{L_A} SL_3$ concretely, we need to analyze the stability of $(\mathbb{P}^2)^6$ in detail. In this section, by computing the (semi)stable locus, we describe the morphism $\varphi_A : \overline{\mathrm{M}}_{0,6} \to V_A^2$ explicitly. From this section, we will use symmetric linearizations only. Note that there exists a unique symmetric linearization on $(\mathbb{P}^r)^d$ up to normalization. So we will not indicate the linearization for GIT quotients.

4.1. An explicit computation of stability on $(\mathbb{P}^2)^6 /\!/ SL_3$.

Due to Theorem 3.1, for a strictly semistable configuration on $(\mathbb{P}^2)^6$ there are two possibilities:

- on a point, there are exactly two marked points;
- on a line, there are four points.

Thus we can make a list of strictly semistable configurations. See Table 1. For each stratum, there is a figure for a typical element in the stratum. The three lines in the figure are standard coordinate lines of $\mathbb{P}^2$ and the symbol $\odot$ means a point with multiplicity two. In the next three rows, we describe the stabilizer in SL_3 of the configuration, the dimension and the orbit closure in the semistable locus for each stratum.

Since the set of geometric points in a GIT quotient bijectively corresponds to the set of closed orbits in the semistable locus, the set of geometric points in V_A^2 is in bijection with the orbits in $\mathrm{I} \sqcup \mathrm{VII} \sqcup U_{2,6}^s$.

4.2. A description of φ_A.

Now we can explicitly describe the morphism $\varphi_A : \overline{\mathrm{M}}_{0,6} \to V_A^2$ by following the recipe in Remark 3.6. For any $(C, x_1, \cdots, x_6) \in \overline{\mathrm{M}}_{0,6} - (B_2 \cup B_3)$, there is a degree 2 map $f : C \to \mathbb{P}^2$ whose image is a nonsingular conic. Note that all nonsingular conics are projectively equivalent, hence $\varphi_A(C, x_1, \cdots, x_6)$ is the image (up to projective equivalence) of six points in a conic.

For $(C, x_1, \cdots, x_6) \in B_3 - B_3 \cap B_2$, if we define a map $f : C \to \mathbb{P}^2$ such that $\deg f = 1$ on each irreducible component of C and the image is a union of two distinct lines, f is stable with

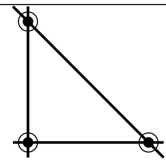
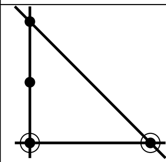
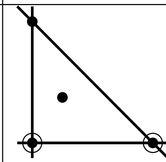
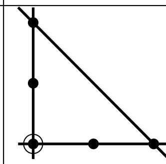
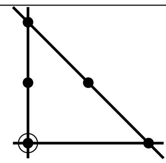
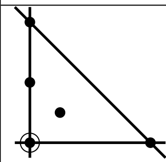
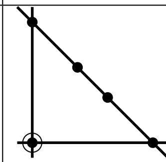
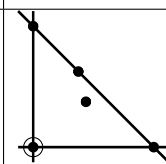
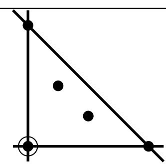
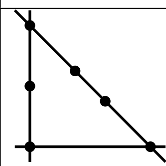
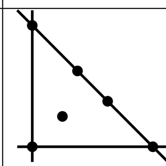
stratum	I	II	III	IV
				
stabilizer	$(\mathbb{C}^*)^2$	$\mathbb{C}^*$	1	1
dimension	6	7	8	8
orbit closure	closed	$\in$ I	$\in$ I, II	$\in$ I, II
stratum	V	VI	VII	VIII
				
stabilizer	1	1	$\mathbb{C}^*$	1
dimension	9	9	8	9
orbit closure	$\in$ I, II	$\in$ I, II	closed	$\in$ VII
stratum	IX	X	XI	
				
stabilizer	1	1	1	
dimension	10	9	10	
orbit closure	$\in$ VII	$\in$ VII	$\in$ VII	

TABLE 1. Strictly semistable configurations

respect to $L_{A'}$. So the image of each irreducible component is a line. Therefore $\varphi_A(C, x_1, \cdots, x_6)$ is a configuration of distinct points in the union of two lines such that 1) on each line there are three distinct points, and 2) at the intersection of the two lines there is no marked point.

An interesting contraction happens to B_2. Let $(C = C_1 \cup C_2, x_1, \cdots, x_6)$ be a general point on B_2. Without loss of generality, suppose that $x_1, \cdots, x_4$ are on C_1. By the stability computation in Theorem 3.1, the images of them cannot be the same. Thus $\deg f|_{C_1}$ is one or two. If it is one, $\deg f|_{C_2} = 1$ so the image is a configuration of two lines such that on one line there are four distinct points and on the other line there are two distinct points. Also there is no marked point on the intersection of two lines. Thus it is an element of the stratum XI. If $\deg f|_{C_1} = 2$, then the image is a conic with five points (one of them has multiplicity two). So the image is a configuration of type IX. In both cases, the image has its orbit closure in the stratum VII. Note that $(C, x_1, \cdots, x_6)$ depends on the cross ratios of the five points, $x_1, \cdots, x_4$ and the singular points. But in the stratum VII, it depends only on the cross ratio of the four points $x_1, \cdots, x_4$. Therefore the image has dimension one and the contracted curve is exactly C_4. Since the cross ratio of four points is parameterized by $\mathbb{P}^1$, an irreducible component of B_2 maps to $\mathbb{P}^1$ by φ_A.

We can observe the contraction of $F_{1,1,2,2}$ in a similar way. Let $(C = C_1 \cup C_2 \cup C_3, x_1, \cdots, x_6)$ be a general element of $F_{1,1,2,2}$ where C_1 and C_2 are two tails. If a tail (say C_1) has degree 2, then the four marked points on the outside of C_1 are mapped to a point. Thus for a semistable map, the degree of a tail with 2 marked points is one or zero. So $(\deg f|_{C_1}, \deg f|_{C_2})$ can be $(1,1)$, $(1,0)$, $(0,1)$, or $(0,0)$. In each case, it is straightforward to check that the image configuration is of type IV, VI, VI, and III respectively. Therefore in any of these cases, the orbit closure contains the stratum I. Since a configuration in I does not have non-trivial moduli, $F_{1,1,2,2}$ is contracted.

Proposition 4.1. *The contraction map $\varphi_A : \overline{\mathrm{M}}_{0,6} \to V_A^2$ is an isomorphism outside of B_2. The image of B_2 is the union of 15 projective lines $L_1, \cdots, L_{15}$. Each L_i intersects the other L_j at three points, and at each intersection point there are three L_i's which pass through it. Finally, V_A^2 is singular along $\cup L_i$.*

Proof. The first statement is already discussed above. Note that there are 15 irreducible components of B_2. Each of them is contracted to a line, so the image is a union of 15 lines $L_1, \cdots, L_{15}$. An irreducible component of B_2 intersects with the other irreducible components of B_2 along three projective lines. They are F-curves $F_{1,1,2,2}$ and so are contracted. Note that $F_{1,1,2,2}$ has a point which is an intersection of three irreducible components of B_2. Therefore for each intersection point there are three L_i's passing through it.

Now it is enough to prove the last statement about the singularity. For a general curve C_4 in B_2, an irreducible component D of B_2 containing C_4 is isomorphic to $\overline{\mathrm{M}}_{0,5}$. Let p be the moving point on $\mathbb{P}^1$ which is used to define C_4 (see Section 2). Let $q_p : \overline{\mathrm{M}}_{0,5} \to \mathbb{P}^2$ be the Kapranov morphism for the marked point p ([Kap93, Section 4.2]). Then $q_p(C_4)$ is a conic in $\mathbb{P}^2$. Therefore by [KM13, Lemma 4.5],

$$N_{D/\overline{\mathrm{M}}_{0,6}}|_{C_4} \cong q_p^*\mathcal{O}_{\mathbb{P}^2}(-1)|_{C_4} \cong \mathcal{O}_{\mathbb{P}^2}(-1)|_{q_p(C_4)} \cong \mathcal{O}_{\mathbb{P}^1}(-2).$$

Thus locally φ_A is not isomorphic to a smooth blow-down and the image is singular along L_i. $\square$

Remark 4.2. In the literature, V_A^2 has had several alternative descriptions. First of all, note that $V_A^2 \subset (\mathbb{P}^2)^6 /\!/ SL_3$ is the closure of the locus of configurations of six points on smooth conics. In [DO88, p.17, Example 3] (also see [Dol03, Example 11.7]), it was proved that $(\mathbb{P}^2)^6 /\!/ SL_3$ is a double cover of $\mathbb{P}^4$ which is ramified over a quartic hypersurface so called the **Igusa quartic** (or **Castelnuovo-Richmond quartic**) $\mathcal{I}_4$. It is defined by

$$\sum_{i=1}^{6} X_i = 0, \quad \left(\sum_{i=1}^{6} X_i^2\right)^2 - 4\left(\sum_{i=1}^{6} X_i^4\right) = 0$$

in $\mathbb{P}^5$. $\mathcal{I}_4$ is exactly the locus of configurations on conics, thus V_A^2 is isomorphic to $\mathcal{I}_4$. On the other hand, $\mathcal{I}_4$ is the Satake compactification $\overline{\mathcal{A}}_2(2)$ of the moduli space $\mathcal{A}_2(2)$ of principally polarized abelian surfaces with level two structure ([Igu64]). To the author's knowledge, there have been no explicit constructions of a regular map from $\overline{\mathrm{M}}_{0,6}$ to $\overline{\mathcal{A}}_2(2)$.

Remark 4.3. From Proposition 4.1, V_A^2 is regular in codimension one. Since we can regard it as a complete intersection in $\mathbb{P}^5$, it is Cohen-Macaulay, in particular, it has S_2-property. Thus by Serre's criterion, V_A^2 is normal.

5. Proof of the main theorem

In this section, we run Mori's program for $\overline{\mathrm{M}}_{0,6}$ with symmetric divisors.

5.1. **Stable base locus decomposition.** For an effective divisor D, the stable base locus $\mathbf{B}(D)$ is defined as

$$\mathbf{B}(D) = \bigcap_{m \geq 0} \mathrm{Bs}(mD),$$

where $\mathrm{Bs}(D)$ is the set-theoretical base locus of D. As a first step toward Mori's program, we will compute the stable base locus decomposition of $\overline{\mathrm{M}}_{0,6}$, which dictates the difference of birational models of $\overline{\mathrm{M}}_{0,6}$.

Proposition 5.1. *Let D be a symmetric effective divisor on $\overline{\mathrm{M}}_{0,6}$. Then:*

(1) *If $D \in [-K_{\overline{\mathrm{M}}_{0,6}}, K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi]$, D is semi-ample.*
(2) *If $D \in (K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi, B_3]$, $\mathbf{B}(D) = B_3$.*
(3) *If $D \in [B_2, -K_{\overline{\mathrm{M}}_{0,6}})$, $\mathbf{B}(D) = B_2$.*

Proof. For $n = 6$, it is well-known that a divisor D on $\overline{\mathrm{M}}_{0,6}$ is nef if and only if $D \cdot F \geq 0$ for all F-curves ([KM13, Theorem 1.2]). From Corollary 2.6, it is straightforward to check that $\mathrm{Nef}(\overline{\mathrm{M}}_{0,6})$ is generated by $-K_{\overline{\mathrm{M}}_{0,6}}$ and $K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi$. Thus for item (1), it is sufficient to show that $-K_{\overline{\mathrm{M}}_{0,6}}$ and $K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi$ are semi-ample. It is a direct consequence of the fact that $\overline{\mathrm{M}}_{0,6}$ is a Mori dream space ([Cas09]), but in this case furthermore we can write these divisors as pull-backs of ample divisors. $K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi$ is a scalar multiple of the pull-back of an ample line bundle on $(\mathbb{P}^1)^6 // SL_2$ by Example 3.4. From Lemma 3.7, $-K_{\overline{\mathrm{M}}_{0,6}}$ is a scalar multiple of a semi-ample divisor D_A. So both of them are semi-ample.

If $D \in (K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi, B_3]$, since $K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi$ is semi-ample, $\mathbf{B}(D) \subset B_3$. On the other hand, by Corollary 2.6, $F_{1,1,1,3} \cdot D < 0$ thus $F_{1,1,1,3} \subset \mathbf{B}(D)$. But $F_{1,1,1,3}$ covers an open dense subset of B_3. So $\mathbf{B}(D) = B_3$.

Finally, if $D \in [B_2, -K_{\overline{\mathrm{M}}_{0,6}})$, $\mathbf{B}(D) \subset B_2$ because $-K_{\overline{\mathrm{M}}_{0,6}}$ is semi-ample. The negative intersection $C_4 \cdot D < 0$ implies $C_4 \subset \mathbf{B}(D)$. Since C_4 covers an open dense subset of B_2, $\mathbf{B}(D) = B_2$. ☐

We can summarize the above result by Figure 3.

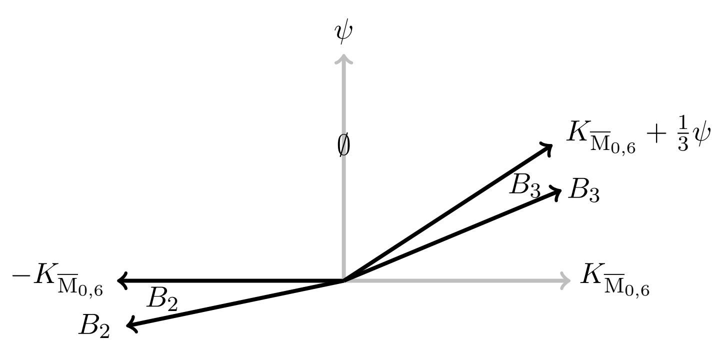


FIGURE 3. Stable base locus decomposition of $\overline{\mathrm{M}}_{0,6}$

5.2. **Mori's program for $\overline{\mathrm{M}}_{0,6}$.** Now we can perform Mori's program of $\overline{\mathrm{M}}_{0,6}$ for all symmetric divisors.

Theorem 5.2. *Let D be a symmetric effective divisor on $\overline{\mathrm{M}}_{0,6}$. Then:*

1. *If $D \in (-K_{\overline{\mathrm{M}}_{0,6}}, K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi)$, then $\overline{\mathrm{M}}_{0,6}(D) \cong \overline{\mathrm{M}}_{0,6}$.*
2. *If $D \in [K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi, B_3)$, then $\overline{\mathrm{M}}_{0,6}(D) \cong (\mathbb{P}^1)^6 /\!/ SL_2$.*
3. *If $D \in (B_2, -K_{\overline{\mathrm{M}}_{0,6}}]$, then $\overline{\mathrm{M}}_{0,6}(D)$ is V_A^2 where $A = (\frac{1}{2}, \cdots, \frac{1}{2})$.*
4. *Both $\overline{\mathrm{M}}_{0,6}(B_2)$ and $\overline{\mathrm{M}}_{0,6}(B_3)$ are a point.*

Proof. If $D \in (-K_{\overline{\mathrm{M}}_{0,6}}, K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi)$ it is ample by Proposition 5.1. Thus $\overline{\mathrm{M}}_{0,6}(D) \cong \overline{\mathrm{M}}_{0,6}$.

If $D \in [K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi, B_3)$, then $D = K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi + aB_3$ for some $a > 0$. By taking a sufficiently large multiple, we may assume that D is a linear combination of integral divisors. Note that $K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi$ is $\rho^* L$ for $\rho : \overline{\mathrm{M}}_{0,6} \to (\mathbb{P}^1)^6 /\!/ SL_2$ of an ample line bundle L (Example 3.4), and B_3 is the exceptional locus of ρ. Thus by [Deb01, Lemma 7.11],

$$\begin{aligned} H^0(\overline{\mathrm{M}}_{0,6}, \mathcal{O}(mD)) &= H^0(\overline{\mathrm{M}}_{0,6}, \mathcal{O}(m(K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi + aB_3))) \\ &\cong H^0((\mathbb{P}^1)^6 /\!/ SL_2, L^m). \end{aligned}$$

Therefore

$$\overline{\mathrm{M}}_{0,6}(D) \cong \mathrm{Proj} \bigoplus_{m \ge 0} H^0(\overline{\mathrm{M}}_{0,6}, \mathcal{O}(mD)) \cong \mathrm{Proj} \bigoplus_{m \ge 0} H^0((\mathbb{P}^1)^6 /\!/ SL_2, L^m) \cong (\mathbb{P}^1)^6 /\!/ SL_2.$$

Similarly, if $D \in (B_2, -K_{\overline{\mathrm{M}}_{0,6}}]$, then $D = -K_{\overline{\mathrm{M}}_{0,6}} + bB_2$ for some $b > 0$ and B_2 is the exceptional locus of $\varphi_A : \overline{\mathrm{M}}_{0,6} \to V_A^2$. Also $-K_{\overline{\mathrm{M}}_{0,6}}$ is the pull-back of an ample line bundle M on V_A^2. By the same argument,

$$\overline{\mathrm{M}}_{0,6}(D) \cong \mathrm{Proj} \bigoplus_{m \ge 0} H^0(V_A^2, M^m),$$

which is the normalization of V_A^2. But V_A^2 is normal by Remark 4.3 so it is isomorphic to its normalization.

The last assertion is obvious since both B_2, B_3 are fixed divisors. □

Remark 5.3. Therefore in symmetric Mori's program on $\overline{\mathrm{M}}_{0,6}$, there is no flip. In Mori's full program on $\overline{\mathrm{M}}_{0,6}$ there must be many flips because we can construct many modular small contractions of $\overline{\mathrm{M}}_{0,6}$ by using a general construction described in [GJM13].

On the other hand, for $n \ge 7$, even for symmetric Mori's program many flips should appear.

Remark 5.4. For divisors of the form $K_{\overline{\mathrm{M}}_{0,n}} + \alpha B$ with $\alpha \le 1$, Mori's program can be regarded as an analogy with the Hassett-Keel program for $\overline{\mathcal{M}}_g$. In this direction, Mori's program is done by many authors [Sim08, AS12, FS11, KM11] for all n. When $n = 6$ it covers a half of the symmetric effective cone.

The subcone of the effective cone generated by $K_{\overline{\mathrm{M}}_{0,n}}$ and ψ_i-classes has been studied intensively. There is a general picture for non-symmetric weight data and even for higher genera cases. See [Moo11].

Remark 5.5. It is well-known that $(\mathbb{P}^1)^6//SL_2$ is isomorphic to the **Segre cubic** $\mathcal{S}_3$, which is defined by

$$\sum_{i=1}^{6} X_i = 0, \quad \sum_{i=1}^{6} X_i^3 = 0$$

in $\mathbb{P}^5$ ([Dol03, Example 11.6]). One fascinating fact is that the Segre cubic is projectively dual to the Igusa quartic $\mathcal{I}_4$ ([DO88, Remark I.3]), as a hypersurface in $\mathbb{P}^4$. It would be interesting if this projective dual map can be described concretely in terms of $\overline{\mathrm{M}}_{0,6}$.

6. Mori's program for $\overline{\mathcal{M}}_2$

Let $\overline{\mathcal{M}}_2$ be the moduli stack of genus two stable curves. Essentially all birational contractions of $\overline{\mathcal{M}}_2$ were described in [Has05], even though Mori's program was described for only one half of the effective cone of $\overline{\mathcal{M}}_2$. Indeed, since all genus two smooth curves are hyperelliptic, the coarse moduli space $\overline{\mathrm{M}}_2$ is isomorphic to $\overline{\mathrm{M}}_{0,6}/S_6$ ([AL02, Corollary 2.5]). Thus Theorem 5.2 gives Mori's full program for $\overline{\mathrm{M}}_2$. Also since $\mathrm{Pic}(\overline{\mathrm{M}}_2)_{\mathbb{Q}} \cong \mathrm{Pic}(\overline{\mathcal{M}}_2)_{\mathbb{Q}}$, this result can be regarded as Mori's program of $\overline{\mathcal{M}}_2$.

First of all, we have natural isomorphisms

$$\mathrm{Pic}(\overline{\mathcal{M}}_2)_{\mathbb{Q}} \cong \mathrm{Pic}(\overline{\mathrm{M}}_2)_{\mathbb{Q}} \cong \mathrm{Pic}(\overline{\mathrm{M}}_{0,6})_{\mathbb{Q}}^{S_6}.$$

For the coarse moduli map $q : \overline{\mathcal{M}}_2 \to \overline{\mathrm{M}}_2$, we have the first isomorphism $q^* : \mathrm{Pic}(\overline{\mathrm{M}}_2)_{\mathbb{Q}} \to \mathrm{Pic}(\overline{\mathcal{M}}_2)_{\mathbb{Q}}$. This is an isomorphism only if we take $\mathbb{Q}$-Picard groups. The second isomorphism comes from $\pi^* : \mathrm{Pic}(\overline{\mathrm{M}}_2) \to \mathrm{Pic}(\overline{\mathrm{M}}_{0,6})^{S_6}$ where $\pi : \overline{\mathrm{M}}_{0,6} \to \overline{\mathrm{M}}_2$ is the S_6-quotient map. By following the notations in [Has05], we denote by δ_0 (resp. δ_1) the boundary divisor of irreducible nodal curves (resp. that of a union of two elliptic curves respectively) on $\overline{\mathcal{M}}_2$. Let Δ_0, Δ_1 be the corresponding boundary divisors on the coarse moduli space $\overline{\mathrm{M}}_2$. Let λ be the Hodge class on $\overline{\mathrm{M}}_2$ and also its pull-back on $\overline{\mathcal{M}}_2$. The effective cone of $\overline{\mathcal{M}}_2$ (resp. $\overline{\mathrm{M}}_2$) is generated by δ_0 and δ_1 (resp. Δ_0 and Δ_1). Since $q : \overline{\mathcal{M}}_2 \to \overline{\mathrm{M}}_2$ is ramified along Δ_1, $q^*(\Delta_0) = \delta_0$, $q^*(\Delta_1) = 2\delta_1$.

The following simple lemma tells the relation between S_6-symmetric Mori's program of $\overline{\mathrm{M}}_{0,6}$ and Mori's program of $\overline{\mathrm{M}}_2$. For a projective variety X and a divisor D, let

$$R(X, D) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}(mD))$$

be the section ring of D.

Lemma 6.1. *Let G be a finite group acting on a projective variety X. Let $\pi : X \to X/G$ be the quotient map and D be a $\mathbb{Q}$-Cartier divisor on X/G. Assume that $R(X, \pi^*D)$ is finitely generated and $Y :=$* $\mathrm{Proj}\ R(X, \pi^*D)$*. Then $R(X/G, D)$ is finitely generated and* $\mathrm{Proj}\ R(X/G, D) = Y/G$.

Proof. We may assume that D is a Cartier divisor, because $\mathrm{Proj}\ R(X, D)$ does not change after replacing D by kD. Note that $H^0(X/G, \mathcal{O}(mD)) \cong H^0(X, \pi^*\mathcal{O}(mD))^G$. Thus

$$R(X/G, D) = \bigoplus_{m \geq 0} H^0(X/G, \mathcal{O}(mD)) \cong \bigoplus_{m \geq 0} H^0(X, \pi^*\mathcal{O}(mD))^G = R(X, \pi^*D)^G.$$

Since G is a finite group and $R(X, \pi^* D)$ is finitely generated, the G-invariant subring is finitely generated, too. By the definition of projective quotient, $\mathrm{Proj}\ R(X/G, D) \cong \mathrm{Proj}\ R(X, \pi^* D)^G = Y/G$. $\square$

Theorem 6.2. *Let D be an effective divisor on $\overline{\mathrm{M}}_2$. Then:*

(1) If $D \in (\lambda, \Delta_0 + 6\Delta_1)$, $\overline{\mathrm{M}}_2(D) \cong \overline{\mathrm{M}}_2$.
(2) If $D \in [\Delta_0 + 6\Delta_1, \Delta_1)$, $\overline{\mathrm{M}}_2(D) \cong \mathbb{P}^6 // SL_2$.
(3) If $D \in (\Delta_0, \lambda]$, $\overline{\mathrm{M}}_2(D) \cong \overline{\mathcal{A}}_2^{\mathrm{Sat}}$, the Satake compactification of $\mathcal{A}_2$.
(4) Both $\overline{\mathrm{M}}_2(\Delta_0)$ and $\overline{\mathrm{M}}_2(\Delta_1)$ are a point.

Proof. First of all, for the quotient map $\pi : \overline{\mathrm{M}}_{0,6} \to \overline{\mathrm{M}}_2$, set-theoretically $\pi^{-1}(\Delta_0) = B_2$ and $\pi^{-1}(\Delta_1) = B_3$. Note that S_6 acts freely on a general point of $\overline{\mathrm{M}}_{0,6} - (B_2 \cup B_3)$ and a general point of B_3, but there is an order two stabilizer for a general point of B_2, which exchanges the two marked points in the two-pointed irreducible component. Thus $\pi^*(\Delta_0) = 2B_2$ and $\pi^*(\Delta_1) = B_3$.

Since $\lambda \equiv \frac{1}{10}(\Delta_0 + \Delta_1)$ ([HM98, Exercise 3.143]), $\pi^*\lambda = \frac{1}{5}B_2 + \frac{1}{10}B_3 = -\frac{1}{2}K_{\overline{\mathrm{M}}_{0,6}}$. Also $\pi^*(\Delta_0 + 6\Delta_1) = 2B_2 + 6B_3 = 15(K_{\overline{\mathrm{M}}_{0,6}} + \frac{1}{3}\psi)$. Thus the decomposition of the effective cone of $\overline{\mathrm{M}}_2$ in the statement is exactly the image of the stable base locus decomposition of $\overline{\mathrm{M}}_{0,6}$. So we obtain the result from Theorem 5.2. For example, for $D \in (\lambda, \Delta_0 + 6\Delta_1)$, $\overline{\mathrm{M}}_2(D) = \overline{\mathrm{M}}_{0,6}(\pi^* D)/S_6 = \overline{\mathrm{M}}_{0,6}/S_6 = \overline{\mathrm{M}}_2$. For $D \in [\Delta_0 + 6\Delta_1, \Delta_1)$, $\overline{\mathrm{M}}_2(D) = \overline{\mathrm{M}}_{0,6}(\pi^* D)/S_6 = (\mathbb{P}^1)^6 // SL_2 / S_6 = (\mathbb{P}^1)^6 / S_6 // SL_2 = \mathbb{P}^6 // SL_2$ since the SL_2-action and S_6-action commute. Finally, we already know that $\overline{\mathrm{M}}_2(\lambda) = \overline{\mathcal{A}}_2^{\mathrm{Sat}}$ from the definition of λ. Now for $D \in (\Delta_0, \lambda]$, $\overline{\mathrm{M}}_2(D) \cong \overline{\mathrm{M}}_{0,6}(\pi^* D)/S_6$ is independent of the choice of D. Therefore $\overline{\mathrm{M}}_2(D) \cong \overline{\mathcal{A}}_2^{\mathrm{Sat}}$. $\square$

Finally, we can obtain an alternative proof of the main theorem of [Has05], as a restatement of Theorem 6.2. Note that $q^*(\Delta_0 + 6\Delta_1) = \delta_0 + 12\delta_1$.

Theorem 6.3. *Let D be an effective divisor on $\overline{\mathcal{M}}_2$. Then:*

(1) If $D \in (\lambda, \delta_0 + 12\delta_1)$, $\overline{\mathcal{M}}_2(D) \cong \overline{\mathrm{M}}_2$.
(2) If $D \in [\delta_0 + 12\delta_1, \delta_1)$, $\overline{\mathcal{M}}_2(D) \cong \mathbb{P}^6 // SL_2$.
(3) If $D \in (\delta_0, \lambda]$, $\overline{\mathcal{M}}_2(D) \cong \overline{\mathcal{A}}_2^{\mathrm{Sat}}$, the Satake compactification of $\mathcal{A}_2$.
(4) Both $\overline{\mathcal{M}}_2(\delta_0)$ and $\overline{\mathcal{M}}_2(\delta_1)$ are a point.

Remark 6.4. (1) The divisor classes of the form $K_{\overline{\mathcal{M}}_2} + \alpha\delta$ are more familiar to people who are interested in the Hassett-Keel program. In this setup, Theorem 6.3 can be translated as follows: For $2 > \alpha > 9/11$, $\overline{\mathcal{M}}_2(\alpha) := \overline{\mathcal{M}}_2(K_{\overline{\mathcal{M}}_2} + \alpha\delta) \cong \overline{\mathrm{M}}_2$. For $9/11 \ge \alpha > 7/10$, $\overline{\mathcal{M}}_2(\alpha) \cong \mathbb{P}^6 // SL_2$. For $\alpha \ge 2$, $\overline{\mathcal{M}}_2(\alpha) \cong \overline{\mathcal{A}}_2^{\mathrm{Sat}}$. Finally, $\overline{\mathcal{M}}_2(7/10)$ is a point. But it does not cover the part $[\delta_0, \delta_0 + \delta_1]$ of the effective cone .

(2) It is well-known that $\mathbb{P}^6 // SL_2 \cong \mathbb{P}(2, 4, 6, 10)$ ([Dol03, Section 10.2]).

Acknowledgement. The author would like to thank Angela Gibney and the anonymous referee for thorough and helpful comments on earlier drafts of this paper. Also he thanks Igor Dolgachev for indicating some references.

References

[AL02] D. Avritzer and H. Lange. The moduli spaces of hyperelliptic curves and binary forms. *Math. Z.*, 242(4):615–632, 2002. 2, 13

[AS12] Valery Alexeev and David Swinarski. Nef divisors on $\overline{M}_{0,n}$ from GIT. In *Geometry and arithmetic*, EMS Ser. Congr. Rep., pages 1–21. Eur. Math. Soc., Zürich, 2012. 1, 2, 12

[Cas09] Ana-Maria Castravet. The Cox ring of $\overline{M}_{0,6}$. *Trans. Amer. Math. Soc.*, 361(7):3851–3878, 2009. 11

[Che08] Dawei Chen. Mori's program for the Kontsevich moduli space $\overline{M}_{0,0}(\mathbb{P}^3, 3)$. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 067, 17, 2008. 1

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001. 12

[DO88] Igor Dolgachev and David Ortland. Point sets in projective spaces and theta functions. *Astérisque*, (165):210 pp. (1989), 1988. 2, 10, 13

[Dol03] Igor Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003. 6, 10, 13, 14

[Dol12] Igor V. Dolgachev. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012. A modern view. 2

[FP97] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997. 5

[FS11] Maksym Fedorchuk and David Ishii Smyth. Ample divisors on moduli spaces of pointed rational curves. *J. Algebraic Geom.*, 20(4):599–629, 2011. 1, 2, 12

[Gia13] Noah Giansiracusa. Conformal blocks and rational normal curves. *J. Algebraic Geom.*, 22(4):773–793, 2013. 5, 6

[GJM13] Noah Giansiracusa, David Jensen, and Han-Bom Moon. GIT compactifications of $\bar{M}_{0,n}$ and flips. *Adv. Math.*, 248:242–278, 2013. 5, 6, 7, 8, 12

[GJMS13] Angela Gibney, David Jensen, Han-Bom Moon, and David Swinarski. Veronese quotient models of $\overline{\mathrm{M}}_{0,n}$ and conformal blocks. *Michigan Math. J.*, 62(4):721–751, 2013. 5, 6, 8

[Has05] Brendan Hassett. Classical and minimal models of the moduli space of curves of genus two. In *Geometric methods in algebra and number theory*, volume 235 of *Progr. Math.*, pages 169–192. Birkhäuser Boston, Boston, MA, 2005. 1, 2, 13, 14

[HH09] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009. 1

[HH13] Brendan Hassett and Donghoon Hyeon. Log minimal model program for the moduli space of stable curves: the first flip. *Ann. of Math. (2)*, 177(3):911–968, 2013. 1

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998. 14

[Igu64] Jun-ichi Igusa. On Siegel modular forms genus two. II. *Amer. J. Math.*, 86:392–412, 1964. 2, 10

[Kap93] M. M. Kapranov. Chow quotients of Grassmannians. I. In *I. M. Gelfand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 29–110. Amer. Math. Soc., Providence, RI, 1993. 6, 10

[KM11] Young-Hoon Kiem and Han-Bom Moon. Moduli spaces of weighted pointed stable rational curves via GIT. *Osaka J. Math.*, 48(4):1115–1140, 2011. 1, 2, 12

[KM13] Seán Keel and James McKernan. Contractible extremal rays on $\overline{M}_{0,n}$. In *Handbook of moduli. Vol. II*, volume 25 of *Adv. Lect. Math. (ALM)*, pages 115–130. Int. Press, Somerville, MA, 2013. 3, 4, 10, 11

[LO13] Herbert Lange and Angela Ortega. Pryms of non-cyclic triple coverings and log canonical models of the spin moduli space of genus 2. arXiv:1309.6984, 2013. 3

[Moo11] Han-Bom Moon. Log canonical models for $\bar{M}_{g,n}$. arXiv:1111.5354, 2011. 12

[Moo13] Han-Bom Moon. Log canonical models for the moduli space of stable pointed rational curves. *Proc. Amer. Math. Soc.*, 141(11):3771–3785, 2013. 1, 2, 4

[Pan97] Rahul Pandharipande. The canonical class of $\overline{M}_{0,n}(\mathbf{P}^r, d)$ and enumerative geometry. *Internat. Math. Res. Notices*, (4):173–186, 1997. 4

[Sim08] Matthew Simpson. *On log canonical models of the moduli space of stable pointed genus zero curves*. PhD thesis, 2008. Thesis (Ph.D.)–Rice University. 1, 2, 12

[Smy13] David Ishii Smyth. Towards a classification of modular compactifications of $M_{g,n}$. *Invent. Math.*, 192(2):459–503, 2013. 2

DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NY 10458

E-mail address: hmoon8@fordham.edu