BIRATIONAL GEOMETRY OF THE MODULI SPACE OF RANK 2 PARABOLIC VECTOR BUNDLES ON A RATIONAL CURVE

HAN-BOM MOON AND SANG-BUM YOO

Abstract. We investigate the birational geometry (in the sense of Mori’s program) of the moduli space of rank 2 semistable parabolic vector bundles on a rational curve. We compute the effective cone of the moduli space and show that all birational models obtained by Mori’s program are also moduli spaces of parabolic vector bundles with certain parabolic weights.

1. Introduction

In the last decade, it has been proved that studying the geometry of a moduli space in the viewpoint of the minimal model program (or Mori’s program) for a moduli space is very fruitful. Mori’s program for a moduli space $M$ consists of the following steps: 1) Compute the cone of effective divisors $\text{Eff}(M)$. 2) For each divisor $D \in \text{Eff}(M)$, find the projective model $M(D) := \text{Proj} \bigoplus_{m \geq 0} H^0(M, \mathcal{O}(\lfloor mD \rfloor))$. 3) Study the moduli theoretic interpretation (if there is) of $M(D)$ and its relation with $M$.

There are several intensively studied examples. For the moduli space $\mathcal{M}_g$ of stable curves, the famous Hassett-Keel program is a study of birational models of the form $\mathcal{M}_g(K_{\mathcal{M}_g} + \alpha D)$ with the boundary $D$ of singular curves and $\alpha \leq 1$. It has been shown that many of these models are indeed moduli spaces of curves with worse singularities (for a nice overview, see [FS13]). For Hilbert scheme $\text{Hilb}^n(\mathbb{P}^2)$ of $n$ points on $\mathbb{P}^2$, many of its birational models appearing in Mori’s program are moduli spaces of Bridgeland stable objects in $D^b(\mathbb{P}^2)$ with certain stability condition ([ABCH13]). For the moduli space of stable sheaves $M_H(v)$ on a K3 surface $X$, all flips of $M_H(v)$ are moduli spaces of Bridgeland stable objects in $D^b(X)$ ([BM13]).

1.1. The main result of the paper. The aim of this paper is to investigate the birational geometry of the moduli space $\mathcal{M}(\vec{a})$ of rank 2 semistable parabolic vector bundles of degree 0 on $\mathbb{P}^1$, in the sense of Mori’s program. The moduli functor depends on a parabolic weight $\vec{a}$, which imposes a certain stability condition. If we vary $\vec{a}$, then the moduli space changes. The study of this change has been well understood by many authors in [Ber94, BH95, Tha96, Tha02]. All birational morphisms between them are able to be described in terms of smooth blow-ups/downs, or variation of GIT. In this paper we revisit these birational modifications in terms of Mori’s program.

The following is the first main result of this paper, which is the first step of Mori’s program.

Theorem 1.1 (Theorem 6.2). Let $\vec{a}$ be a parabolic weight such that $\mathcal{M}(\vec{a})$ has the maximal Picard number $n + 1$. Then the effective cone $\text{Eff}(\mathcal{M}(\vec{a}))$ is polyhedral and there are precisely $2^{n-1}$ extremal rays.

Note that the computation of the effective cone of a variety is a hard problem in general. Except toric varieties, there are few examples of varieties with large Picard number and known effective cone. Among moduli spaces, most of examples with known effective cone have Picard number $\leq 2$ or have a simplicial
effective cone (for example, the moduli space of \( n \)-unordered pointed rational curves \( \overline{M}_{0,n}/S_n \) ([KM13]), the moduli space of stable maps \( \overline{M}_{0,0}(\mathbb{P}^d, d) \) ([CHS08])). Theorem 1.1 provides one highly nontrivial example of an algebraic variety with completely known non-simplicial effective cone.

After the computation of \( \text{Eff}(\mathcal{M}(\vec{a})) \), the following theorem is a simple consequence of the work of Pauly on generalized theta divisors ([Pau96]).

**Theorem 1.2** (Theorem 7.3). For any divisor \( D \in \text{int} \, \text{Eff}(\mathcal{M}(\vec{a})) \), the birational model \( \mathcal{M}(\vec{a})(D) \) is isomorphic to \( \mathcal{M}(\vec{b}) \) for some parabolic weight \( \vec{b} \).

Indeed, even in the case that \( D \in \partial \text{Eff}(\mathcal{M}(\vec{a})) \), we can describe the projective models as moduli spaces of parabolic bundles with less parabolic points (Remark 7.4). In short, all projective models of \( \mathcal{M}(\vec{a}) \) appear in Mori’s program of \( \mathcal{M}(\vec{a}) \) are moduli spaces of parabolic vector bundles with certain degree and stability condition.

Therefore as opposed to the case of Hilbert schemes and moduli spaces of ordinary stable sheaves, there is no newly appeared moduli space parametrizing objects in (some) derived categories. In this sense, the condition.

1.3. **Structure of the paper.** The organization of this paper is as follows. In Section 2 we review the definition and basic properties of moduli spaces of semistable rank 2 parabolic vector bundles. Also we state some known results on the wall-crossing behaviors of them. In Section 3, we give an elementary construction of \( \mathcal{M}(\vec{a}) \) as a simple GIT quotient. In Section 4, we compute the Picard group and the effective cone
of the GIT quotient appeared in the previous section. Section 5 reviews an elementary definition and combinatorics of $sl_2$-conformal blocks. In Section 6, we compute $\text{Eff}(\mathcal{M}({\bar{a}}))$ for an effective parabolic weight $\bar{a}$. Finally in Section 7, we prove Theorem 1.2.

Notations and conventions. We work over an algebraically closed field of characteristic 0. In this paper, we fix $n$ distinct parabolic points $\vec{p} = (p_1, \cdots, p_n)$ on $\mathbb{P}^1$. The notion and combinatorics of parabolic bundles are significantly simplified for the rank 2 case. So in this paper, our discussions are focused on the rank 2 case only. We denote the set $\{1, \cdots, n\}$ by $[n]$.

Acknowledgements. We would like to thank Young-Hoon Kiem and David Swinarski, for many enlightening discussions.

2. Preliminaries on the moduli space of parabolic vector bundles

In this section, we review some of basics on parabolic vector bundles on $\mathbb{P}^1$ and the moduli spaces of them. After that we review some known results on birational geometry of the moduli spaces. For details and proofs of the results in this section, see [MS80, BH95, Yok95, Tha96, Tha02].

2.1. Moduli space of parabolic vector bundles. A rank 2 parabolic vector bundle on $\mathbb{P}^1$ with parabolic structure at $\vec{p}$ is a collection $(E, \{V_i\}, \vec{a})$ where

1. $E$ is a vector bundle of rank 2 over $\mathbb{P}^1$;
2. for each $i \in [n], V_i \subset E|_{p_i}$ is a 1-dimensional subspace;
3. a weight sequence $\vec{a} = (a_1, \cdots, a_n)$, that is, a sequence of rational numbers such that $0 \leq a_i < 1$.

Sometimes we write $(E, \{V_i\}, \vec{a})$ simply as $(E, \{V_i\}), (E, \vec{a})$ or even $E$, if there is no confusion.

If we consider the moduli stack of parabolic vector bundles, it is highly non-separated even if we fix the rank and the degree of the underlying vector bundle. The notion of (semi)stability of parabolic vector bundles enables us to obtain a proper open substack.

A parabolic line bundle $(E, \vec{b})$ is simply a pair of line bundle $E$ and a parabolic weight $\vec{b} = (b_1, \cdots, b_n)$. Let $(E, \{V_i\}, \vec{a})$ be a rank 2 parabolic vector bundle on $\mathbb{P}^1$. A parabolic subbundle $(E', \vec{b})$ is a parabolic line bundle where $E' \subset E$ is a subbundle and

$$b_i = \begin{cases} a_i, & E'|_{p_i} = V_i \\ 0, & E'|_{p_i} \neq V_i. \end{cases}$$

A parabolic quotient bundle $(E'', \vec{c})$ is a parabolic line bundle where $E''$ is a quotient bundle and if $q : E \rightarrow E''$ is the quotient map,

$$c_i = \begin{cases} a_i, & q|_{p_i}(V_i) \neq 0 \\ 0, & q|_{p_i}(V_i) = 0. \end{cases}$$

For a rank 1 or 2 parabolic bundle $(E, \vec{a})$, the parabolic degree of $E$ is

$$\text{pardeg} E = \deg E + \sum_{i=1}^{n} a_i.$$

Finally, for a parabolic bundle $E$, the parabolic slope of $E$ is defined as

$$\mu(E) = \frac{\text{pardeg} E}{\text{rank} E}.$$
Definition 2.1. A rank 2 parabolic bundle \((E, \{V_i\}, \vec{a})\) is **(semi)stable** if for every parabolic subbundle \((E', \vec{b})\),

\[ \mu(E') (\leq) < \mu(E). \]

We say that two rank 2 semistable parabolic vector bundles are **S-equivalent** if they have the same factors on their Jordan-Hölder filtrations. In concrete terms, this equivalence relation is generated by the following: If \((E, \{V_i\}, \vec{a})\) is semistable and \((E', \vec{b}) \to (E, \{V_i\}, \vec{a})\) is a parabolic subbundle such that \(\mu(E') = \mu(E)\), then \(E \equiv E' \oplus E/E'\). By definition, if \(E\) is stable, then it is S-equivalent to itself only.

Let \(\mathcal{M}(\vec{a}, d)\) be the coarse moduli space of S-equivalent classes of rank 2, degree \(d\), semistable parabolic vector bundles on \(\mathbb{P}^1\) with parabolic structure \(\vec{a}\) at \(\vec{p}\). We denote \(\mathcal{M}(\vec{a}, 0)\) by \(\mathcal{M}(\vec{a})\).

Theorem 2.2 ([MS80, Theorem 4.1]). The moduli space \(\mathcal{M}(\vec{a}, d)\) is an irreducible normal projective variety of dimension \(n - 3\), if it is nonempty.

2.2. **Deformation theory of parabolic vector bundles.** The deformation theory of parabolic vector bundles has been worked out by Yokogawa in [Yok95] in great generality.

Let \((E, \{V_i\}, \vec{a})\) and \((F, \{W_i\}, \vec{b})\) be two rank 2 parabolic vector bundles. A bundle morphism \(f : E \to F\) is called **(strongly) parabolic** if \(f(V_i) = 0\) whenever \(a_i > b_i\). We shall denote by \(\text{ParHom}(E, E')\) and \(\text{SParHom}(E, E')\) the sheaves of parabolic and strongly parabolic morphisms, and by \(\text{ParHom}(E, E')\) and \(\text{SParHom}(E, E')\) their global sections respectively. We also use the notation \(\text{ParEnd}(E) := \text{ParHom}(E, E)\) and \(\text{ParEnd}(E) := \text{ParHom}(E, E)\).

The following fact is a standard consequence of the notion of the stability, as that of ordinary vector bundles. The proof is identical to that of [Fri98, Proposition 4.7, Corollary 4.8].

Proposition 2.3. Let \(E\) and \(F\) be stable parabolic bundles such that \(\mu(E) \geq \mu(F)\). Then dim \(\text{ParHom}(E, E') = 1\) if \(E\) and \(F\) are isomorphic, and 0 otherwise. In particular, \(\text{ParEnd}(E) = \mathbb{C} \cdot \text{id}\).

The category of parabolic bundles on \(\mathbb{P}^1\) is not abelian. However, Yokogawa showed that it is contained in an abelian category \(\mathcal{P}\) as a full subcategory using a generalized notion of parabolic sheaves, and \(\mathcal{P}\) has enough injective objects. For each parabolic vector bundle \(E\), \(\text{Ext}^i(E, -)\) is defined by the \(i\)-th right derived functor of \(\text{ParHom}(E, -)\) in \(\mathcal{P}\).

Lemma 2.4 ([Yok95, Theorem 1.4, 3.6]). Let \(E_1, E_2\) be two parabolic bundles. Then

\[ \text{Ext}^i(E_2, E_1) \cong H^i(\text{ParHom}(E_2, E_1)) \]

for \(i = 0, 1\).

For each parabolic weight \(\vec{a}\), let \(I_{\vec{a}} = \{i \in [n] | a_i \neq 0\}\). We say that two weights \(\vec{b}\) and \(\vec{c}\) are **complementary** if \(I_{\vec{b}} \cap I_{\vec{c}}\) defines a partition of \([1, \cdots, n]\). When two parabolic line bundles \((E, \vec{b})\) and \((F, \vec{c})\) have complementary weights, an extension of \((F, \vec{c})\) by \((E, \vec{b})\) is a short exact sequence of parabolic morphisms

\[
0 \longrightarrow (E, \vec{b}) \longrightarrow (G, \{V_i\}, \vec{a}) \longrightarrow (F, \vec{c}) \longrightarrow 0
\]

where

\[
a_i = \begin{cases} b_i, & i \in I_{\vec{b}} \\ c_i, & i \in I_{\vec{c}} \end{cases}
\]

and \(V_i = E|_{\vec{p}}\), if \(i \in I_{\vec{b}}\) and \(q(V_i) \neq 0\) if \(i \in I_{\vec{c}}\). It is obvious that \((E, \vec{b})\) (resp. \((F, \vec{c})\)) is a parabolic subbundle (resp. quotient bundle) of \((G, \{V_i\}, \vec{a})\). The proof of the following proposition is identical to the arguments in [Fri98, p. 31].
Proposition 2.5. For two parabolic line bundles \((E, \tilde{b})\) and \((F, \tilde{c})\) which have complementary weights, there is a one-to-one correspondence between the set of isomorphism classes of extensions of \(F\) by \(E\) and \(\text{Ext}^1(F, E)\).

We have a generalized Serre duality for parabolic bundles.

Proposition 2.6 ([Yok95, Proposition 3.7]). For any parabolic bundles \(E\) and \(F\), there are natural isomorphisms

\[
\text{Ext}^{1-i}(E, F \otimes \mathcal{O}(n-2)) \cong H^i(S\text{ParHom}(F, E))^\vee
\]

for \(i = 0, 1\).

Theorem 2.7 ([Yok95, Theorem 2.4]). Let \((E, \{V_i\}, \tilde{a})\) be a rank 2 parabolic bundle corresponding to a geometric point \(x\) of \(\mathcal{M}(\tilde{a})^s\). The Zariski tangent space of \(\mathcal{M}(\tilde{a})^s\) at \(x\) is naturally isomorphic to \(\text{Ext}^1(E, E)\). If \(\text{Ext}^2(E, E) = 0\), then \(\mathcal{M}(\tilde{a})^s\) is smooth at \(x\).

Because we are interested in parabolic bundles on a curve, \(\text{Ext}^2(E, E) = 0\) is automatic.

Corollary 2.8. \(\mathcal{M}(\tilde{a})^s\) is smooth.

2.3. Wall crossing. We devote this subsection to show that how \(\mathcal{M}(\tilde{a})\) changes when \(\tilde{a}\) varies. The birational map between \(\mathcal{M}(\tilde{a})\) and \(\mathcal{M}(\tilde{a}')\) with two adjacent parabolic weights \(\tilde{a}\) and \(\tilde{a}'\) is intensively studied in [BH95] and [Tha96, Section 7].

Remark 2.9. In [BH95, Tha96], the authors stated the result in the case that there is only one parabolic point. But the result is generalized to the case of an arbitrary number of parabolic points in a straightforward way.

The set of all possible parabolic weights is \(W := ([0, 1) \cap \mathbb{Q})^n\). The interior of \(W\) is denoted by \(W^0\).

Definition 2.10. A parabolic weight \(\tilde{a} \in W^0\) is called effective if \(\mathcal{M}(\tilde{a})^s \neq \emptyset\). An effective weight is called general if \(\mathcal{M}(\tilde{a}) = \mathcal{M}(\tilde{a})^s\).

By Corollary 2.8, for a general parabolic weight, \(\mathcal{M}(\tilde{a})\) is smooth.

Let us study stability walls on \(W\). Let \((E = \mathcal{O}(k) \oplus \mathcal{O}(-k), \{V_i\}, \tilde{a})\) be a parabolic vector bundle over \(\mathbb{P}^1\) for some nonnegative \(k\). If it is strictly semistable (hence it is on a wall), then there is a parabolic subbundle \((F = \mathcal{O}(-m), \tilde{b})\) such that \(\mu(F) = \mu(E)\). Let \(I = \{i \in \{1, \cdots, n\} | F|_{p_i} = V_i\}\). Then

\[-m + \sum_{i \in I} a_i = \mu(F) = \mu(E) = \frac{1}{2} \sum_{i=1}^n a_i,
\]

so

\[\mu(F) - \mu(E) = \sum_{i \in I} a_i - \sum_{i \in I^c} a_i = 2m.\]

Therefore all stability walls are defined by

\[(1) \quad \Delta_{I, m} = \{(a_1, \cdots, a_n) \in W \mid \sum_{i \in I} a_i - \sum_{i \in I^c} a_i = 2m\}.
\]

Lemma 2.11. The space of parabolic weights \(W\) is decomposed into finitely many chambers by walls \(\Delta_{I, m}\) for \(I \subset \{1, \cdots, n\}\) and \(m \in \mathbb{Z}\).

Note that \(\Delta_{I, m} = \Delta_{I^c, -m}\).

Next, we see what parabolic bundles become unstable as we cross a stability wall. It suffices to analyze the change under a simple wall-crossing along the relative interior of a wall. Choose a general point \(\tilde{a}\) in \(\Delta_{I, m}\). Let \(\Delta^+_{I, m}\) and \(\Delta^-_{I, m}\) be small neighborhoods of at \(\tilde{a}\) in

\[\{(b_1, \cdots, b_n) \in W \mid \sum_{i \in I} b_i - \sum_{i \in I^c} b_i > 2m\}\quad \text{and} \quad \{(b_1, \cdots, b_n) \in W \mid \sum_{i \in I} b_i - \sum_{i \in I^c} b_i < 2m\}\]
respectively. The stability coincide with the semistability on $\Delta_{I,m}^+$, A parabolic bundle is $\Delta_{I,m}^+$-stable (resp. $\Delta_{I,m}^{-}$-stable) if it is stable with respect to parabolic weights in $\Delta_{I,m}^+$ (resp. $\Delta_{I,m}^{-}$). We look for parabolic bundles which are $\Delta_{I,m}^{-}$-stable but $\Delta_{I,m}^{+}$-unstable.

**Lemma 2.12.** If $(\mathcal{O}(k) \oplus \mathcal{O}(-k), \{V_i\})$ is $\Delta_{I,m}^{-}$-stable but $\Delta_{I,m}^{+}$-unstable, then any destabilizing subbundle is of the form $(\mathcal{O}(-m), \tilde{b})$ and $I = I_{\tilde{b}} := \{i \in [n] \mid \mathcal{O}(-m)|_{I_{\tilde{b}}} = V_i\}$.

**Proof.** Since $(\mathcal{O}(k) \oplus \mathcal{O}(-k), \{V_i\})$ is $\Delta_{I,m}^{-}$-unstable, we have a destabilizing subbundle $\mathcal{O}(-m')$ of $(\mathcal{O}(k) \oplus \mathcal{O}(-k), \{V_i\})$ such that

$$-m' + \sum_{i \in I_{\tilde{b}}} a_i > \frac{1}{2} \sum_{i=1}^{n} a_i$$

for any $\tilde{a} \in \Delta_{I,m}^+$. Since $(\mathcal{O}(k) \oplus \mathcal{O}(-k), \{V_i\})$ is $\Delta_{I,m}^{-}$-stable,

$$-m' + \sum_{i \in I_{\tilde{b}}} a_i < \frac{1}{2} \sum_{i=1}^{n} a_i$$

for any $\tilde{a} \in \Delta_{I,m}^-$. Then $\Delta_{I,m}^- \subset \Delta_{I,m}^+$ and $\Delta_{I,m}^- \subset \Delta_{I,m}^+$. Hence $\Delta_{I,m}^- = \Delta_{I,m}^+$. Thus $m' = m$ and $I = I_{\tilde{b}}$. \qed

The uniqueness of the maximal destabilizing subbundle can be shown as in the case of ordinary bundles.

Suppose that $\tilde{a}$ is a general point of $\Delta_{I,m}^-$. Let $\tilde{a}^-$ (resp. $\tilde{a}^+$) be in $\Delta_{I,m}^-$ (resp. $\Delta_{I,m}^+$). Assume that both $\mathcal{M}(\tilde{a}^-)$ and $\mathcal{M}(\tilde{a}^+)$ are nonempty. Since $\tilde{a}^\pm$ are general parabolic weights, $\mathcal{M}(\tilde{a}^\pm)$ are smooth by Corollary 2.8. There are two birational morphisms

$$\mathcal{M}(\tilde{a}^-) \xrightarrow{\phi^-} \mathcal{M}(\tilde{a}) \xleftarrow{\phi^+} \mathcal{M}(\tilde{a}^+).$$

The image $Y$ of the exceptional locus of $\phi^\pm$ is the locus parameterizes S-equivalent classes of $(\mathcal{O}(-m), \tilde{b}) \oplus (\mathcal{O}(m), \tilde{c})$, where $(\mathcal{O}(-m), \tilde{b})$ is the destabilizing bundle for $\Delta_{I,m}^+$ and $\tilde{c} = \tilde{a} - \tilde{b}$. The moduli of parabolic line bundles of a fixed degree on $\mathbb{P}^1$ is a point because there is a unique line bundle for each degree. So $Y$ is always a single point. For the same $I$, define $\tilde{b}^\pm$ and $\tilde{c}^\pm$ by using $\tilde{a}^\pm$. The exceptional fiber of $\phi^-$ (resp. $\phi^+$) is a projective space $Y^- := \mathbb{P}\text{Ext}^1((\mathcal{O}(m), \tilde{c}^-), (\mathcal{O}(-m), \tilde{b}^-))$ (resp. $Y^+ := \mathbb{P}\text{Ext}^1((\mathcal{O}(m), \tilde{b}^+), (\mathcal{O}(m), \tilde{c}^+))$).

**Proposition 2.13 ([Tha96, Section 7]).** The blow-up of $\mathcal{M}(\tilde{a}^-)$ along $Y^-$ is isomorphic to the blow-up of $\mathcal{M}(\tilde{a}^+)$ along $Y^+$. In particular, $\dim Y^- + \dim Y^+ = \dim \mathcal{M}(\tilde{a}) - 1$.

We will use the following dimension computation later.

**Proposition 2.14.** Let $\tilde{a}^-$ be a general point of $\Delta_{I,m}^-$. Then

$$\dim \text{Ext}^1((\mathcal{O}(m), \tilde{c}^-), (\mathcal{O}(-m), \tilde{b}^-)) = 2m + n - 1 - |I|.$$

**Proof.** By Proposition 2.6, we have natural isomorphisms

$$\text{Ext}^1((\mathcal{O}(m), \tilde{c}^-), (\mathcal{O}(-m), \tilde{b}^-)) \cong \text{SParHom}((\mathcal{O}(m), \tilde{c}^-), (\mathcal{O}(m), \tilde{c}^-))^\vee.$$

Consider the following short exact sequence of sheaves

$$0 \to \text{SParHom}((\mathcal{O}(-m - (n - 2)), \tilde{b}^-), (\mathcal{O}(m), \tilde{c}^-)) \to \text{Hom}((\mathcal{O}(-m - (n - 2)), \tilde{b}^-), (\mathcal{O}(m), \tilde{c}^-))$$

$$\to \bigoplus_{i=1}^{n} \text{Hom}((\mathcal{O}(-m - (n - 2)), \tilde{b}^-)|_{p_i}, (\mathcal{O}(m), \tilde{c}^-)|_{p_i}) \to 0.$$
where \( N_p((\mathcal{O}(k), \bar{x}), (\mathcal{O}(\ell), \bar{y})) \) is the subspace of strictly parabolic maps in \( \text{Hom}((\mathcal{O}(k), \bar{x}), (\mathcal{O}(\ell), \bar{y}))_{|p} \) at a point \( p \in \mathbb{P}^1 \). For \( \bar{a}^+ \in \Delta_{I,m}, \mu(\mathcal{O}(m), \bar{c}^-) > \mu(\mathcal{O}(-m), \bar{b}^-) \). Thus

\[
H^1(S\text{ParHom}((\mathcal{O}(-m - (n - 2)), \bar{b}^-), (\mathcal{O}(m), \bar{c}^-))) = \text{Ext}^0((\mathcal{O}(m), \bar{c}^-), (\mathcal{O}(-m), \bar{b}^-))^\vee = \text{ParHom}((\mathcal{O}(m), \bar{c}^-), (\mathcal{O}(-m), \bar{b}^-))^\vee = 0
\]

by Proposition 2.3. Hence we have a short exact sequence of vector spaces

\[
0 \to S\text{ParHom}((\mathcal{O}(-m - (n - 2)), \bar{b}^-), (\mathcal{O}(m), \bar{c}^-)) \to \text{Hom}((\mathcal{O}(-m - (n - 2)), \bar{b}^-), (\mathcal{O}(m), \bar{c}^-)) \to \bigoplus_{i=1}^{n} \text{Hom}((\mathcal{O}(m), \bar{c}^-)) \to 0.
\]

Since

\[
dim N_{p_i}((\mathcal{O}(-m - (n - 2)), \bar{b}^-), (\mathcal{O}(m), \bar{c}^-)) = \begin{cases} 0, & i \in I, \\ 1, & i \notin I. \end{cases}
\]

\[
dim S\text{ParHom}((\mathcal{O}(-m - (n - 2)), \bar{b}^-), (\mathcal{O}(m), \bar{c}^-)) = \dim \text{Hom}((\mathcal{O}(-m - (n - 2)), \bar{b}^-), (\mathcal{O}(m), \bar{c}^-)) - |I| = \dim H^0(\mathcal{O}(2m + n - 2)) - |I| = 2m + n - 1 - |I|.
\]

3. Elementary GIT Quotients and the Moduli Space of Parabolic Bundles

The ring of invariants of a product of projective lines have been studied since 19th century. In this section, we review some of the classical results and its relation with moduli spaces of rank 2 parabolic vector bundles on \( \mathbb{P}^1 \). For the basic of GIT, consult [MFK94].

3.1. The GIT quotient of a product of projective lines. Fix \( n \geq 3 \). For \( \bar{a} = (a_1, \cdots, a_n) \in \mathbb{Q}^n_{>0} \), consider an ample \( \mathbb{Q} \)-line bundle \( L := \mathcal{O}(a_1, \cdots, a_n) \) on \( (\mathbb{P}^1)^n \). On \( (\mathbb{P}^1)^n, \text{SL}_2 \) acts diagonally. We can take the GIT quotient with respect to \( L \),

\[
(\mathbb{P}^1)^n //_{L} \text{SL}_2 := \text{Proj} \bigoplus_{m \geq 0} H^0((\mathbb{P}^1)^n, [L^m])^{\text{SL}_2}.
\]

The (semi)stability of \((\mathbb{P}^1)^n\) with respect to \( L \) is obtained as the following theorem. We denote the stable (resp. semistable) locus by \((\mathbb{P}^1)^n)^s\) (resp. \((\mathbb{P}^1)^n)^{ss}\).

**Theorem 3.1 ([MFK94, Proposition 3.4]).** Let \( L = \mathcal{O}(a_1, \cdots, a_n) \) be a \( \mathbb{Q} \)-linearization. Let \( a := \sum_{i=1}^{n} a_i \). For a point \( x := (x_1, \cdots, x_n) \in (\mathbb{P}^1)^n \), \( x \in ((\mathbb{P}^1)^n)^s\) (resp. \( x \in ((\mathbb{P}^1)^n)^{ss} \)) if and only if for any \( y \in \mathbb{P}^1 \),

\[
\sum_{x_i = y} a_i \leq a/2 \quad \text{(resp.} < a/2). \)

**Corollary 3.2.**

1. For a linearization \( L = \mathcal{O}(a_1, \cdots, a_n), (\mathbb{P}^1)^n //_{L} \text{SL}_2 \) is nonempty if and only if \( a_i \leq a/2 \) for every \( 1 \leq i \leq n \).
2. The stable locus is nonempty (in particular, \((\mathbb{P}^1)^n //_{L} \text{SL}_2 \) is \((n - 3)\)-dimensional) if and only if \( a_i < a/2 \) for every \( 1 \leq i \leq n \).
3. The semi-stable locus coincides with the stable locus if and only if for any nonempty \( I \subset [n], \sum_{i \notin I} a_i \neq \sum_{i \in I} a_i \).

We say that a linearization \( L \) is **effective** if it satisfies (2). An effective linearization is **general** if it satisfies (3) as well. Compare with Definition 2.10.
3.2. The moduli space of parabolic bundles as an elementary GIT quotient. The readers are able to observe that the combinatorics of the stability is identical to that of the stability of rank 2 parabolic bundles on $\mathbb{P}^1$.

**Proposition 3.3.** Let $\vec{a} = (a_1, \cdots, a_n) \in W^0$ and let $L = \mathcal{O}(a_1, \cdots, a_n)$ be the corresponding $\mathbb{Q}$-linearization. Assume that $L$ has a nonempty stable locus and $a := \sum_{i=1}^n a_i < 2$. Then

$$\mathcal{M}(\vec{a}) \cong (\mathbb{P}^1)^n //_{L} SL_2.$$

**Proof.** First of all, let $(E, \{V_i\})$ be a semistable parabolic bundle of degree 0. By Grothendieck’s theorem, $E = \mathcal{O}(k) \oplus \mathcal{O}(-k)$ for some nonnegative integer $k$. If $k \geq 1$, then $\mu(\mathcal{O}(k)) \geq 1 > a/2 = \mu(E)$. Thus $E$ is not semistable unless it is a trivial bundle.

Let $X = (\mathbb{P}^1)^n$ and $\pi_i : X \to \mathbb{P}^1$ be the $i$-th projection. Let $E$ be a rank two trivial vector bundle on $X \times \mathbb{P}^1$. Then $\mathbb{P}(E)$ is isomorphic to $X \times \mathbb{P}^1 \times \mathbb{P}^1$. For each $i$, define a morphism

$$s_i : X \to X \times \mathbb{P}^1 \xrightarrow{\pi_i} X \times \{p_i\} \times \mathbb{P}^1$$

by the graph of $i$-th projection. Over each $X \times \{p_i\}$, define a line bundle $V_i$ as $[V_i]|_{(x, p_i)} = s_i(x)$. Then $V_i$ is a natural subbundle of $E|_{X \times \{p_i\}}$ of rank one. Now $(E, \{V_i\})$ is a family of rank 2 parabolic vector bundles on $\mathbb{P}^1$ over $X$. Consider the restricted family over $X^{ss}$ and use the same notation $(E, \{V_i\})$.

Let $(\mathcal{O}^2, \{V_i\})$ be the fiber of $(E, \{V_i\})$ over $x = (x_1, \cdots, x_n) \in X^{ss}$. Note that all subbundles of $\mathcal{O}^2$ is $\mathcal{O}(-k)$ for some nonnegative integer $k$. If $k \geq 1$, then

$$\mu(\mathcal{O}(-k)) = -k + \sum_{\mathcal{O}(-k)|_{p_i} = V_i} a_i < a/2 = \mu(\mathcal{O}^2)$$

because

$$\sum_{\mathcal{O}(-k)|_{p_i} = V_i} a_i - \sum_{\mathcal{O}(-k)|_{p_i} \neq V_i} a_i \leq a < 2 \leq 2k.$$

So it is not a destabilizing bundle.

Let $F = V \otimes \mathcal{O} \subset \mathcal{O}^2$ be a trivial subbundle for an one dimensional subspace $V \subset \mathcal{O}^2$. Note that if $F|_{x_i} = V$ for some $i \in I \subset [n]$ or $x_i = x_j$ for every $i, j \in I$. From the GIT stability in Theorem 3.1 (with $y = [V]$), $\sum_{E|_{x_i} = V} a_i \leq a/2$. Thus for $x = (x_1, \cdots, x_n) \in X^{ss}$,

$$\mu(E) = \sum_{E|_{x_i} = V} a_i \leq a/2 = \mu(\mathcal{O}^2).$$

Therefore $X^{ss}$ parametrizes semistable parabolic vector bundles with respect to the parabolic weight $\vec{a}$. Thus we have a classifying morphism $\mu : X^{ss} \to \mathcal{M}(\vec{a})$. There is a natural $SL_2$-action on $X^{ss}$ and each orbit parametrizes isomorphic parabolic bundles, since it acts as a canonical $SL_2$-action on each fiber of the trivial rank 2 bundle. Thus there is a quotient morphism $\bar{\mu} : X^{ss} / SL_2 \to \mathcal{M}(\vec{a})$.

One can check that $\bar{\mu}$ is injective. Indeed, the injectivity over the stable locus $\mathcal{M}(\vec{a})^s$ is obvious. For a strictly semistable point corresponding an $S$-equivalent class of $E := (\mathcal{O}, \vec{b}_1) \oplus (\mathcal{O}, \vec{b}_2) \in \mathcal{M}(\vec{a})$, $\mu^{-1}(E) = X_1 \cup X_2$ where $X_i = \{(x_1, \cdots, x_n) \mid x_j = x_k \text{ if } j, k \in I_{\vec{b}_i}\}$. Because the closure of the orbit of a point in $X_i$ contains an orbit $X_1 \cap X_2$ which is closed in $X^{ss}$, they are identified to a point in the GIT quotient. Since $\mathcal{M}(\vec{a})$ is irreducible and $\bar{\mu}$ is dominant, $\bar{\mu}$ is surjective. Finally, because $\mathcal{M}(\vec{a})$ is normal by Theorem 2.2, $\bar{\mu}$ is an isomorphism.

It is already known that $\mathcal{M}(\vec{a})$ is rational for any effective parabolic weight $\vec{a} \in W^0$ ([Bau91, BH95]). We provide another proof of the rationality of $\mathcal{M}(\vec{a})$ for any effective parabolic weight $\vec{a} \in W^0$, which is a simple consequence of Proposition 3.3.
Corollary 3.4. For any effective parabolic weight \( \vec{a} \in W^0 \), \( M(\vec{a}) \) is rational.

Proof. By Proposition 2.13 and Proposition 3.3, \( M(\vec{a}) \) is birational to \( (\mathbb{P}^1)^n//_{L}SL_2 \) where \( L \) is an effective linearization. It is known that \( (\mathbb{P}^1)^n//_{L}SL_2 \cong \mathbb{P}^{n-3} \), when \( L \) is proportional to \( O(n-2,1,\cdots,1) \). \( \square \)

3.3. General case. For a general parabolic weight \( \vec{a} \), we may find \( c > 1 \) such that \( \vec{a} = c\vec{b} \) and \( \sum b_i < 2 \). Thus to study the geometry of \( M(\vec{a}) \), it suffices to study the change of the moduli space when the parabolic weight changes from \( M(\vec{b}) \) to \( M(c\vec{b}) = M(\vec{a}) \) for \( c > 1 \). Note that if \( 1 \leq c < \min\{1/b_i\} \), then \( c\vec{b} \in W \), too.

Proposition 3.5. Let \( \vec{a} \) be a general parabolic weight in \( W^0 \) such that \( \sum a_i < 2 \). Consider the wall-crossings from \( M(\vec{a}) \) to \( M(c\vec{a}) \) as \( c \) increases in the range of \( 1 \leq c < \min\{1/a_i\} \). Suppose that all wall-crossings are simple ones. Then the first wall-crossing is a blow-up at the point \( [\vec{p}] \in (\mathbb{P}^1)^n//_{L}SL_2 \cong M(\vec{a}) \). All other wall-crossings are flips or blow-downs.

Proof. By Lemma 2.11, each stability wall is given by \( \Delta_{I,m} \). Then for \( c_0\vec{a} \in \Delta_{I,m}, \)

\[
c_0 \left( \sum_{i \in I} a_i - \sum_{i \in I^c} a_i \right) = 2m
\]

by (1). Then for \( c > c_0, c\vec{a} \in \Delta_{I,m}^+ \) and

\[
c \left( \sum_{i \in I} a_i - \sum_{i \in I^c} a_i \right) > 2m.
\]

This is true only if \( m \geq 0 \). Also, during the variation of stability conditions in the proposition, we do not meet a stability wall of type \( \Delta_{I^c,0} \), because the ratios between parabolic weights do not change. Thus \( m > 0 \).

Then by Proposition 2.13, the blow-up of \( M^- \) along \( Y^- \) is isomorphic to the blow-up of \( M^+ \) along \( Y^+ \). Furthermore, by Proposition 2.14, \( Y^- \) is a point and \( Y^+ = \mathbb{P}^{n-4} \), which is a divisor of \( M^+ \). Therefore \( M^+ \) is isomorphic to the blow-up of \( M^- = (\mathbb{P}^1)^n//_{L}SL_2 \) at the point \( Y^- \). Note that \( x = (x_1,\cdots,x_n) \in Y^- \) if and only if the corresponding parabolic bundle \( (O^2,\{V_i\}) \) has a subbundle \( O(-1) \) which contains all \( V_i \)'s. Since \( O(-1) \subset O \) is isomorphic to the tautological subbundle, \( (x_{1},\cdots,x_{n}) = ([V_1],\cdots,[V_n]) \) is equivalent to \( \vec{p} = (p_1,\cdots,p_n) \).

After the first wall-crossing, since \( m > 1 \) or \( |I| < n, 2m + n - 1 - |I| > 1 \). Thus by Proposition 2.14 and Proposition 2.13 again, the modification is not a blow-up anymore. \( \square \)

4. The effective cone of the GIT quotient

As a first step toward Mori’s program of \( M(\vec{a}) \) and \( (\mathbb{P}^1)^n//_{L}SL_2 \), we compute the effective cone of \( (\mathbb{P}^1)^n//_{L}SL_2 \).

4.1. Rational Picard group. The Picard group of \( (\mathbb{P}^1)^n \) is generated by the pull-backs \( \pi_i^*O(1) \) for \( 1 \leq i \leq n \) where \( \pi_i : (\mathbb{P}^1)^n \to \mathbb{P}^1 \) is the \( i \)-th projection. We denote the tensor product \( \pi_i^*O(b_1) \otimes \cdots \otimes \pi_i^*O(b_n) \) by \( O(b_1,\cdots,b_n) \) or \( O(\sum_{i=1}^n b_ie_i) \) where \( e_i \) is the \( i \)-th standard basis in \( \mathbb{Q}^n \). So Pic((\mathbb{P}^1)^n) \cong \mathbb{Z}^n \) and the nef cone Neff((\mathbb{P}^1)^n) \subset Pic((\mathbb{P}^1)^n)_{\mathbb{Q}} \cong \mathbb{Q}^n \) is generated by \( O(e_i) \). The effective cone is equal to the nef cone so it is simplicial.
Let $L = \mathcal{O}(a_1, \cdots, a_n)$ be a $\mathbb{Q}$-linearization of $(\mathbb{P}^1)^n$. Consider the GIT quotient $(\mathbb{P}^1)^n//_{L} SL_2$. Since it is a quotient of semistable locus, there is a natural diagram

\[
\begin{array}{ccc}
((\mathbb{P}^1)^n)^{ss} & \longrightarrow & (\mathbb{P}^1)^n \\
\iota & \searrow & \\
\downarrow & & \\
(\mathbb{P}^1)^n//_{L} SL_2 & & 
\end{array}
\]

where $\iota$ is the inclusion and $\pi$ is the quotient map.

For any two indices $1 \leq i < j \leq n$, let $\Delta_{(i,j)} = \{(x_1, \cdots, x_n) \in (\mathbb{P}^1)^n | x_i = x_j\}$. It is $SL_2$-invariant, so it descends to an effective cycle

$D_{(i,j)} = \pi(\iota^*(\Delta_{(i,j)})) = \{(x_1, \cdots, x_n) \in (\mathbb{P}^1)^n//_{L} SL_2 | x_i = x_j\}$

on the quotient, if $\Delta_{(i,j)}$ intersects the semistable locus, i.e., $a_i + a_j \leq a/2$. Furthermore, if it intersects the stable locus (so $a_i + a_j < a/2$), then $D_{(i,j)}$ is a divisor on $(\mathbb{P}^1)^n//_{L} SL_2$. If $a_i + a_j = a/2$, $\Delta_{(i,j)}$ has a unique semistable orbit $\{(x_1, \cdots, x_n) | x_i = x_j, x_k = x_\ell \text{ for all } k, \ell \neq i, j\}$ which is closed in $(\mathbb{P}^1)^n^{ss}$. Thus in this case $D_{(i,j)}$ is a single point.

Note that $\mathcal{O}(\Delta_{(i,j)}) = \mathcal{O}(e_i + e_j)$.

**Proposition 4.1.** Suppose that $n \geq 5$. Let $L = \mathcal{O}(a_1, \cdots, a_n)$ be an effective $\mathbb{Q}$-linearization on $(\mathbb{P}^1)^n$. Let $a = \sum a_i$.

1. The rational Picard group $\text{Pic}((\mathbb{P}^1)^n//_{L} SL_2)_Q$ is naturally identified with the quotient space $\text{Pic}((\mathbb{P}^1)^n)_Q/(\Delta_{(i,j)} | a_i + a_j \geq a/2)$, via the identification $D_{(i,j)} \mapsto \Delta_{(i,j)}$.

2. The rank of $\text{Pic}((\mathbb{P}^1)^n//_{L} SL_2)_Q$ is $n - k$, where $k$ is the number of $\Delta_{(i,j)}$ with $a_i + a_j \geq a/2$.

**Remark 4.2.** When $n = 4$, for any effective linearization $L$, the GIT quotient $(\mathbb{P}^1)^4//_{L} SL_2$ is isomorphic to $\mathbb{P}^1$.

**Proof of Proposition 4.1.** Let $((\mathbb{P}^1)^n//_{L} SL_2)^s := \pi((\mathbb{P}^1)^n)^s)$. Since the image of strictly semistable locus is the union of finitely many points, $\text{Pic}((\mathbb{P}^1)^n//_{L} SL_2) \cong \text{Pic}((\mathbb{P}^1)^n//_{L} SL_2)^s)$.

If we denote the nonstable locus $(\mathbb{P}^1)^n - ((\mathbb{P}^1)^n)^s$ by $j : ((\mathbb{P}^1)^n)^{ns} \hookrightarrow (\mathbb{P}^1)^n$, we have a natural exact sequence

\[A_{n-1}(((\mathbb{P}^1)^n)^{ns}) \xrightarrow{j} \text{Pic}((\mathbb{P}^1)^n)^s \xrightarrow{\iota^*} \text{Pic}((\mathbb{P}^1)^n)^s \rightarrow 0.\]

After tensoring $\mathbb{Q}$, the sequence is exact too. Each $(n-1)$-dimensional irreducible component of $((\mathbb{P}^1)^n)^{ns}$ is of the form $\Delta_{(i,j)}$ with $a_i + a_j \geq a/2$. Therefore we have

$\text{Pic}(((\mathbb{P}^1)^n)^{s})_Q = \text{Pic}((\mathbb{P}^1)^n)_Q/(\Delta_{(i,j)} | a_i + a_j \geq a/2)$.

Let $\text{Pic}(X)^{SL_2}$ be the group of isomorphism classes of $SL_2$-invariant line bundles on $X$. Since

$\mathcal{O}(e_i) = \frac{1}{2} (\mathcal{O}(e_i + e_j) \otimes \mathcal{O}(e_i + e_k) \otimes \mathcal{O}(e_j + e_k)^{-1}) = \frac{1}{2} (\mathcal{O}(\Delta_{(i,j)}) + \Delta_{(i,k)} - \Delta_{(j,k)})$,

and the right hand side is $SL_2$-invariant, $\text{Pic}((\mathbb{P}^1)^n)_Q \cong \text{Pic}((\mathbb{P}^1)^n)^{SL_2}_{\mathbb{Q}}$. The same identity is true for $((\mathbb{P}^1)^n)^s$, too.

By Kempf’s descent lemma ([DN89, Theorem 2.3]), an $SL_2$-linearized (in particular, $SL_2$-invariant) line bundle $E$ on $((\mathbb{P}^1)^n)^s$ descends to $(\mathbb{P}^1)^n//_{L} SL_2$ if and only if for every closed orbit $SL_2 \cdot x$, the stabilizer
Stab\(_x\) acts on \(E_x\) trivially. Because Hom(SL\(_2\), \(\mathbb{C}\))\(^*\) is trivial, for each SL\(_2\)-invariant line bundle there is at most one linearization. Furthermore since \((\mathbb{P}^1)^n\) is normal, for any SL\(_2\)-invariant line bundle \(E, E^n\) admits a linearization for some \(n \in \mathbb{Z}\) ([MFK94, Corollary I.1.6]). Therefore
\[
\text{Pic}((\mathbb{P}^1)^n/L SL_2) \cong \text{Pic}((\mathbb{P}^1)^n/L SL_2)^{\ast}_Q \cong \text{Pic}((\mathbb{P}^1)^n)^{\ast}_{SL_2}.
\]
This isomorphism is given by \(\pi^\ast\). Thus \(D_{\{i,j\}}\) maps to \(\Delta_{\{i,j\}}\). This proves Item (1).

To show Item (2), it suffices to show that the set of divisorial nonstable components are linearly independent in Pic((\(\mathbb{P}^1\))^n)_Q. Let \(G\) be a finite simple graph of which vertex set is \([n]\) and edge set is \(\Delta_{\{i,j\}}| a_i + a_j \geq a/2\), the set of nonstable divisors. Two vertices \(i\) and \(j\) are connected by \(\Delta_{\{i,j\}}\). If there are two disjoint edges \(\Delta_{\{i,j\}}, \Delta_{\{k,l\}}\) in \(G\), \(a > a_i + a_j + a_k + a_l \geq a\). Thus there are no disjoint edges. Then \(G\) must be a star shaped graph (all vertices are connected to a central vertex) or a complete graph \(K_3\) of degree 3. In these cases, it is straightforward to check that the edge set is linearly independent.

**Definition 4.3.** A \(\mathbb{Q}\)-linearization \(L = \mathcal{O}(a_1, \cdots, a_n)\) is called a linearization with a maximal stable locus if \(a_i + a_j < a/2\) for any \(\{i, j\} \subset [n]\).

Note that if \(L\) is a linearization with a maximal stable locus, there is no nonstable divisor. In particular, we have the maximal possible Picard rank. It includes the case of symmetric linearization \(L = \mathcal{O}(b, b, \cdots, b)\) for some \(b \in \mathbb{Q}_{>0}\).

**Corollary 4.4.** Suppose that \(n \geq 5\). Let \(L\) be a \(\mathbb{Q}\)-linearization with a maximal stable locus. Then Pic((\(\mathbb{P}^1\))^n/L SL_2)_Q is isomorphic to Pic((\(\mathbb{P}^1\))^n)_Q. In particular, it has rank \(n\) and \(D_{\{i,j\}}\) generates the rational Picard group.

### 4.2. The effective cone.

The following proposition is a translation of a result in the classical invariant theory.

**Proposition 4.5.** Suppose that \(n \geq 5\). Let \(L\) be an effective \(\mathbb{Q}\)-linearization \(\mathcal{O}(a_1, \cdots, a_n)\) on \((\mathbb{P}^1)^n\). Then the effective cone \(\text{Eff}((\mathbb{P}^1)^n/L SL_2)\) of \((\mathbb{P}^1)^n/L SL_2\) is generated by \(D_{\{i,j\}}| 1 \leq i < j \leq n, a_i + a_j < a/2\).

**Proof.** Let \(D\) be an effective divisor on \((\mathbb{P}^1)^n/L SL_2\). Then \(\pi^\ast D\) is an SL\(_2\)-invariant divisor on \((\mathbb{P}^1)^n\)^\(\ast\). By taking the closure, it is extended to an SL\(_2\)-invariant divisor on \((\mathbb{P}^1)^n\). Therefore we have a natural linear map
\[
f : \text{Pic}((\mathbb{P}^1)^n/L SL_2)_Q \to \text{Pic}((\mathbb{P}^1)^n)^{\ast}_{SL_2} \to \text{Pic}((\mathbb{P}^1)^n)_Q
\]
given by \(f(D) = \pi^\ast D\). Note that \(f\) is injective (see for instance [Laz04, Example 2.1.14]). As a \(\mathbb{Q}\)-vector space, Pic((\(\mathbb{P}^1\))^n/L SL_2)_Q is generated by \(D_{\{i,j\}}\) with \(a_i + a_j < a/2\). Since \(f(D_{\{i,j\}}) = \Delta_{\{i,j\}}\), \(\text{im } f\) is generated by \(\Delta_{\{i,j\}}\) with \(a_i + a_j < a/2\). Let \(V = \text{im } f\). Obviously, \(f\) sends an effective divisor to an effective divisor. So it induces a map
\[
f : \text{Eff}((\mathbb{P}^1)^n/L SL_2) \to \text{Eff}((\mathbb{P}^1)^n) \cap V.
\]

Now \(f(D)\) is an SL\(_2\)-invariant effective divisor in \(\text{Eff}((\mathbb{P}^1)^n) \cap V\). If we denote the homogeneous coordinates of the \(i\)-th factor of \((\mathbb{P}^1)^n\) by \(s_i : t_i\), then by the first fundamental theorem of invariant theory ([HMSV09, Section 2]), for any line bundle \(L\) on \((\mathbb{P}^1)^n\), every SL\(_2\)-invariant element of \(\text{H}^0(L)\) is generated by products of \((s_i t_j - s_j t_i)\), which is precisely \(\Delta_{\{i,j\}}\). In particular, the divisor class of \(f(D)\) is an effective linear combination of \(\{\Delta_{\{i,j\}}\}\). Moreover, since \(f(D)\) is in \(V\), on this linear combination, \(\Delta_{\{i,j\}}\) with \(a_i + a_j \geq a/2\) does not appear. Because \(f(D_{\{i,j\}}) = \Delta_{\{i,j\}}, D\) is an effective linear combination of \(D_{\{i,j\}}\) with \(a_i + a_j < a/2\).

**Corollary 4.6.** Let \(L = \mathcal{O}(a_1, \cdots, a_n)\) be a \(\mathbb{Q}\)-linearization with a maximal stable locus. Then \(\text{Eff}((\mathbb{P}^1)^n/L SL_2)\) has precisely \(2n\) facets, namely,
\[
P_i := \text{Span}\{D_{\{i,j\}} | j \neq i\}, \quad 1 \leq i \leq n
\]
and

\[ N_i := \text{Span}\{ D_{j,k} \mid j, k \neq i\}, \quad 1 \leq i \leq n. \]

**Proof.** Take the hyperplane section \( \sum a_i = 2 \) in \( \text{Pic}( (\mathbb{P}^1)^n \sslash L \text{SL}_2)_Q \cong \text{Pic}( (\mathbb{P}^1)^n)_Q \). Then the intersection with the effective cone generated by \( \{ D_{i,j} \} \) is the hypersimplex

\[ \Delta(2, n) = \{(a_1, \ldots, a_n) \in \mathbb{Q}^n \mid \sum a_i = 2, 0 \leq a_i \leq 1\} \]

([Kap93, Section 1]). There is a one-to-one correspondence between the set of facets of \( \text{Eff}( (\mathbb{P}^1)^n \sslash L \text{SL}_2) \) and that of \( \Delta(2, n) \). Now the statement follows from [Kap93, Proposition 1.2.5]. \( \square \)

**Remark 4.7.** When \( L \) is a linearization with a maximal stable locus, the construction of the dual curve for each facet of \( \text{Eff}( (\mathbb{P}^1)^n \sslash L \text{SL}_2) \) is easy. Since \( (\mathbb{P}^1)^n \sslash L \text{SL}_2 \) is naturally a moduli space of \( n \)-pointed smooth rational curves ([Has03, Section 8]), it suffices to construct a one-dimensional family of \( n \)-pointed smooth rational curves with appropriate stability condition described by \( L \).

First of all, take \( n - 1 \) general lines \( \ell_2, \ldots, \ell_n \) on \( \mathbb{P}^2 \). Take a general point \( x \in \mathbb{P}^2 - \cup \ell_i \). Blow-up \( \mathbb{P}^2 \) at \( x \) and let \( \ell_1 \) be the exceptional divisor. Then \( \text{Bl}_x \mathbb{P}^2 \cong \mathbb{F}_1 \) is a \( \mathbb{P}^1 \)-bundle over \( \ell_1 \) and we can regard it as a family of \( n \)-pointed smooth rational curves on \( C_1 := \ell_1 \). Because \( a_i + a_j < a/2 \), any two marked points can collide. Thus all fibers are stable. So \( C_1 \) is a curve on \( (\mathbb{P}^1)^n \sslash L \text{SL}_2 \). Then \( C_1 \cdot D_{i,j} = 0 \) and \( C_1 \cdot D_{i,j} = 1 \) for \( 2 \leq i, j \leq n \). Therefore \( C_1 \) is a dual curve for \( P_1 \).

Now consider a trivial family \( \pi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) with \( (n-1) \) distinct constant sections \( \sigma_2, \ldots, \sigma_n \) and a diagonal section \( \sigma_1 \). Then \( (\pi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1, \sigma_1, \ldots, \sigma_n) \) is a family of pointed curves over \( B_1 := \mathbb{P}^1 \). So \( B_1 \) is a curve on \( (\mathbb{P}^1)^n \sslash L \text{SL}_2 \). Now \( B_1 \cdot D_{i,j} = 1 \) and \( B_1 \cdot D_{i,j} = 0 \) for \( 2 \leq i, j \leq n \). Therefore \( B_1 \) is the dual curve for \( N_1 \).

We close this section with a new notation for line bundles on \( (\mathbb{P}^1)^n \sslash L \text{SL}_2 \).

**Definition 4.8.** Suppose that \( n \geq 5 \). Let \( L \) be an effective linearization. We denote a \( \mathbb{Q} \)-line bundle \( E \) on \( (\mathbb{P}^1)^n \sslash L \text{SL}_2 \) by \( \mathcal{O}(b_1, \ldots, b_n) \) (or \( \mathcal{O}(\sum b_i e_i) \)) if \( E \) maps to the equivalent class of \( \mathcal{O}(b_1, \ldots, b_n) = \mathcal{O}(\sum b_i e_i) \) under the isomorphism

\[ \text{Pic}( (\mathbb{P}^1)^n \sslash L \text{SL}_2)_Q \cong \text{Pic}( (\mathbb{P}^1)^n)_Q / \langle \Delta_{i,j} \mid a_i + a_j \geq a/2 \rangle \]

in Proposition 4.1.

Note that if there are some unstable divisors, the expression is not unique. For instance, if \( \Delta_{(1,2)} \) is unstable, \( \mathcal{O}(b_1, \ldots, b_n) = \mathcal{O}(b_1 + c, b_2 + c, b_3, \ldots, b_n) \) for any \( c \in \mathbb{Q} \).

5. BACKGROUND ON CONFORMAL BLOCKS

In the last three decades, the space of conformal blocks, which are fundamental objects in conformal field theory, have been studied intensively by many mathematicians and physicists. Although the original construction is using the representation theory of affine Lie algebra, in this section we give an elementary definition of the simplest case - \( sl_2 \) conformal blocks on \( \mathbb{P}^1 \) - and their algebraic/combinatorial realizations. Because we do not give the usual definition, we leave some references for the reader’s convenience. For the general definition of conformal blocks, see [Uen08]. The connection with the moduli space of parabolic vector bundles, see [Pau96].
5.1. A quick definition of $\mathfrak{sl}_2$ conformal blocks. In this section, we review an elementary definition of $\mathfrak{sl}_2$ conformal blocks on $\mathbb{P}^1$, described in [Loo09, Section 1]. For the equivalence of the following definition and the original one, consult [Bea96, Proposition 4.1].

We begin with some notational convention. In this section, we write a sequence $(k_1, \cdots, k_n)$ as $k$. $|k| = \sum_i k_i$ and $k! = \prod_i k_i!$.

For any nonnegative integer $k$, let $V_k = H^0(\mathbb{P}^1, \mathcal{O}(k))$ be an irreducible $\text{SL}_2$-representation with highest weight $k$. The vector space $V_k$ is identified with $\mathbb{C}[x, y]_k$, the space of homogeneous polynomials of degree $k$. The infinitesimal $\mathfrak{sl}_2$-action on $\mathbb{C}[x, y]_k$ is given by $e = x \partial_y, f = y \partial_x, h = x \partial_x - y \partial_y$ for the standard basis $e, f, h$ of $\mathfrak{sl}_2$. The highest weight vector of $V_k$ is $x^k$ and $f^jx^k = \frac{k!}{(k-j)!}x^{k-j}y^j$. We may dehomogenize it by taking $x = 1$. Then $V_k$ is identified $\mathbb{C}[y]_{\leq k}$ (the space of polynomials of degree at most $k$) and the action of $e$ is given by $\partial_y$.

For a sequence of nonnegative integers $k = (k_1, \cdots, k_n)$, let $V_k = V_{k_1} \otimes \cdots \otimes V_{k_n}$, with a natural diagonal $\text{SL}_2$-action. Set $2N = |k|$, for a half integer $N$. There is an isomorphism of $\text{SL}_2$-representations $\phi : V_k \to \mathbb{C}[y_1, \cdots, y_n]_{\leq k}$, where $\mathbb{C}[y_1, \cdots, y_n]_{\leq k}$ is the space of polynomials with degree $\leq k_i$ with respect to $y_i$. Then we can take a highest weight vector $v \in V_k$ such that $\phi(v) = 1$.

Let $e_i$ (resp. $f_i, h_i$) be the operator on $V_k$ which acts on the $i$-th factor $V_{k_i}$ as $e$ (resp. $f, h$) and trivially acts on the other factors. On $V_{k_i}$, $e = \sum_i e_i$ and so on. It is straightforward to check that $\phi(f^jv) = \frac{k!}{(k-j)!}y^j$.

By the action of $h \in \mathfrak{sl}_2$, we can decompose $V_k$ into eigenspaces $V_k(\lambda)$ with the eigenvalue $\lambda$. Note that a vector $w \in V_k$ is $\text{SL}_2$-invariant if and only if $w \in V_k(0)$ and $e \cdot w = 0$.

**Definition 5.1.** Let $\vec{p} = (p_1, \cdots, p_n)$ be a sequence of $n$ distinct points on $\mathbb{C} \subset \mathbb{P}^1$. Fix an integer $\ell \geq 0$. Let $k = (k_1, \cdots, k_n)$ be a sequence of nonnegative integers. The space of $\mathfrak{sl}_2$-conformal blocks of level $\ell$ relative to $\vec{p}$ in $V_k$ is the subspace of $\text{SL}_2$-invariants of $V_k$ which is annihilated by the operator $(\sum p_i e_i)^{\ell+1}$. We denote it by $V_{\ell}(k_1, \cdots, k_n)$.

**Remark 5.2.** (1) Note that there is a natural inclusion

$$V_{\ell}(k_1, \cdots, k_n) \subset V_{\ell+1}(k_1, \cdots, k_n).$$

Furthermore, if $\ell \geq N = |k|/2$, $V_{\ell}(k_1, \cdots, k_n) \cong V_{\ell}^{|\mathfrak{sl}_2|} \cong H^0(\mathbb{P}^1, \mathcal{O}(k_1, \cdots, k_n))_{\text{SL}_2}$ because the operator $(\sum p_i e_i)^{N+1}$ is trivial.

(2) For the natural $S_n$-action permuting $n$ irreducible factors of $V_{k_i}$,

$$V_{\ell}(k_1, \cdots, k_n) \cong V_{\ell}(k_{\sigma(1)}, \cdots, k_{\sigma(n)})$$

for every $\sigma \in S_n$.

(3) If $k_i > \ell$ for some $i$, $V_{\ell}(k_1, \cdots, k_n) = 0$.

(4) Since $V_0 \cong \mathbb{C}$, there is a natural isomorphism

$$V_{\ell}(k_1, \cdots, k_n, 0) \cong V_{\ell}(k_1, \cdots, k_n).$$

In the physics literature, this isomorphism is called the propagation of vacua.

The following lemma provides an elementary description of $\mathfrak{sl}_2$-conformal blocks.

**Lemma 5.3 ([Loo09, Lemma 1.3]).** An element $\beta \in V_k^{\text{SL}_2}$ is in $V_{\ell}(k_1, \cdots, k_n)$ (relative to $\vec{p} \in \mathbb{C}^n \subset (\mathbb{P}^1)^n$) if and only if $\phi(\beta)$ has zero of order at least $N - \ell$ at $\vec{p}$.

From the identification $V_k \cong \mathbb{C}[y]_{\leq k}$ and the description of $\mathfrak{sl}_2$-action as differential operators, it is straightforward to see that the map

$$V_k \otimes V_j \hookrightarrow V_{k+j}$$
given by \( f \otimes g \mapsto fg \) induces
\[
V_{k}^{\text{SL}_2} \otimes V_{j}^{\text{SL}_2} \hookrightarrow V_{k+j}^{\text{SL}_2}.
\]

Furthermore, by the identification of level \( \ell \) conformal blocks as polynomials vanishing at \( \bar{p} \) with multiplicity \( N - \ell \) in Lemma 5.3, we have the product map on the level of conformal blocks:
\[
V_{\ell}(a_1, \ldots, a_n) \otimes V_{m}(b_1, \ldots, b_n) \hookrightarrow V_{\ell+m}(a_1 + b_1, \ldots, a_n + b_n).
\]

5.2. Factorization and some combinatorial results on \( \mathfrak{sl}_2 \) conformal blocks. The rank of \( \mathfrak{sl}_2 \)-conformal blocks can be computed by the following inductive formula.

**Proposition 5.4** (Fusion rule and factorization rule, [Bea96, Section 4]). Let \( k_1, \ldots, k_n \) be \( n \) nonnegative integers such that \( k_i \leq \ell \).

1. The rank of \( V_{\ell}(k_1) \) is one when \( k_1 = 0 \). Otherwise the rank is zero.
2. The rank of \( V_{\ell}(k_1, k_2) \) is one when \( k_1 = k_2 \). Otherwise the rank is zero.
3. \( \text{rank } V_{\ell}(k_1, k_2, k_3) = 1 \) if and only if \( \sum k_i \) is even, \( \sum k_i \leq 2\ell \) and \( k_i \leq \ell \). Otherwise the rank is zero.
4. For any \( 1 \leq t \leq n \),
\[
\text{rank } V_{\ell}(k_1, \ldots, k_n) = \sum_{j=0}^{\ell} (\text{rank } V_{\ell}(k_1, \ldots, k_j)) (\text{rank } V_{\ell}(j, k_{i+1}, \ldots, k_n)).
\]

The rank of \( \mathfrak{sl}_2 \)-conformal blocks is indeed the number of certain combinatorial objects. Fix a positive integer \( \ell \) and let \( k = (k_1, \ldots, k_n) \) be a sequence of integers such that \( 0 \leq k_i \leq \ell \).

**Definition 5.5** (D. Swinarski). A **double sequence** of level \( \ell \) and shape \( k \) is a \( 2 \times n \) matrix
\[
DS = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}
\]
such that:

1. Each \( x_j \) and \( y_j \) is an integer between 0 and \( \ell \);
2. \( x_j + y_j = k_j \) for \( 1 \leq j \leq n \);
3. For each \( 1 \leq i \leq n \),
\[
x_i + \sum_{j=1}^{i-1} (x_j - y_j) \leq \ell;
\]
4. For each \( 1 \leq i \leq n \),
\[
-y_i + \sum_{j=1}^{i-1} (x_j - y_j) \geq 0;
\]
5. \( \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j \).

For a double sequence \( DS \), the **height** \( h(DS) \) is the maximum of \( x_i + \sum_{j=1}^{i-1} (x_j - y_j) \) for \( 1 \leq i \leq n - 1 \). Note that \( x_1 \leq h(DS) \leq \ell \).

**Remark 5.6.** Definition 5.5 implies several nontrivial implications. First of all, if \( \sum_{j=1}^{i-1} (x_j - y_j) = 0 \), then \( -y_i \geq 0 \) by (4). Since \( y_i \) is nonnegative by (1), \( y_i = 0 \). In particular, \( y_1 = 0 \). Also if \( \sum_{j=1}^{i} (x_j - y_j) = 0 \), then
\[
0 = \sum_{j=1}^{i} (x_j - y_j) = x_i - y_i + \sum_{j=1}^{i-1} (x_j - y_j) \geq x_i
\]
by (4). So \( x_i = 0 \) by (1) again. As a special case, \( x_n = 0 \).
This definition is equivalent to the definition of the boxed Catalan path, due to B. Alexeev. For a double sequence $DS$, we can draw a path in the first quadrant of $\mathbb{R}^2$ as the following. Start from the origin. For each $j$, draw $(1,1)$ vector $x_j$ times and draw $(1,-1)$ vector $y_j$ times. Then (3) and (4) imply that the path is lying on the region $0 \leq y \leq \ell$. By (5), the ending point is $(|k|,0)$. Remark 5.6 says that if there is a point $(x,0)$ at the end of $j$-th move, then the $j$-th move is a downward move and the $(j+1)$-th move is an upward move. The height $h(DS)$ is simply the height of the boxed Catalan path corresponding to $DS$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{A double sequence of level $\geq 3$ and shape $(2,2,3,1,1,1)$ and the corresponding boxed Catalan path}
\end{figure}

Let $S(\ell,k)$ be the set of double sequences of level $\ell$ and shape $k$. We have learned the following result of B. Alexeev from D. Swinarski.

**Proposition 5.7** (B. Alexeev).

$$\text{rank}\, V_\ell(k_1, \cdots, k_n) = |S(\ell,k)|.$$ 

**Proof.** It is straightforward to check the proposition for $n = 1,2,3$. Also the number of double sequences (or equivalently, boxed Catalan paths) satisfies the factorization rule. Indeed, consider a double sequence $DS = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ and the corresponding boxed Catalan path. For any $1 \leq t \leq n$, after $t$-th move, the $y$-coordinate is one of $0,1,\cdots,\ell$. If the coordinate is $h$, we can construct two double sequences $DS' = \begin{pmatrix} x_1 & x_2 & \cdots & x_t & 0 \\ y_1 & y_2 & \cdots & y_t & h \end{pmatrix}$, $DS'' = \begin{pmatrix} h & x_{t+1} & \cdots & x_n \\ 0 & y_{t+1} & \cdots & y_n \end{pmatrix}$. It is straightforward to check that $DS' \in S(\ell,(k_1,\cdots,k_t,h))$ and $DS'' \in S(\ell,(h,k_{t+1},\cdots,k_n))$. Conversely, for two double sequences $DS' \in S(\ell,(k_1,\cdots,k_t,h))$ and $DS'' \in S(\ell,(h,k_{t+1},\cdots,k_n))$, by removing the last column (resp. the first column) of $DS'$ (resp. $DS''$) and merging them, we obtain a double sequence in $S(\ell,k)$. Thus we have

$$|S(\ell,k)| = \sum_{h=1}^{\ell} |S(\ell,(k_1,\cdots,k_t,h))| \cdot |S(\ell,(h,k_{t+1},\cdots,k_n))|.$$ 

$\square$

6. THE EFFECTIVE CONE OF THE MODULI SPACE OF PARABOLIC VECTOR BUNDLES

In this section, we compute the effective cone of $M(\vec{a})$ with an arbitrary effective parabolic weight $\vec{a}$. The following is a direct consequence of Proposition 3.5.
Lemma 6.1. Let \( \bar{a} = (a_1, \ldots, a_n) \) be a parabolic weight such that \( \mathcal{M}(\bar{a}) \) has the maximal Picard number \( n + 1 \). Then \( \text{rank } \text{Pic}(\mathcal{M}(\bar{a}))_\mathbb{Q} = n + 1 \) and \( \text{Eff}(\mathcal{M}(\bar{a})) \) is identified with \( \text{Eff}(\text{Bl}_{\bar{p}}(\mathbb{P}^1)^n//_{L} L) \) for \( L \) with a maximal stable locus.

Thus for \( \mathcal{M}(\bar{a}) \) with Picard number \( n + 1 \), to compute \( \text{Eff}(\mathcal{M}(\bar{a})) \), it suffices to compute \( \text{Eff}(\mathcal{M}^+) \), where \( \mathcal{M}^+ := \text{Bl}_{\bar{p}}(\mathbb{P}^1)^n//_{L} L \) with \( L = O(a_1, \ldots, a_n) \). Since \( \text{Pic}(\mathcal{M}^+)_\mathbb{Q} \) is generated by \( O(D_{(i,j)}) = \overline{O}(e_i + e_j) \) and \( E \), we can uniquely write a \( \mathbb{Q} \)-line bundle on \( \mathcal{M}^+ \) (so on \( \mathcal{M}(\bar{a}) \)) as

\[
\overline{O}(b_1, \ldots, b_n) - tE
\]

for some \( b_i \) and \( t \).

The main result of this section is the following theorem.

Theorem 6.2. Let \( \bar{a} \) be a parabolic weight such that \( \mathcal{M}(\bar{a}) \) has the maximal Picard number \( n + 1 \). Then the effective cone \( \text{Eff}(\mathcal{M}(\bar{a})) \) is polyhedral and generated by \( \overline{O}(\sum_{j \in I} e_j) - (i - 1)E \) for every \( I \subset [n] \) with \( |I| = 2i \) for \( 0 \leq i \leq [n/2] \). All \( \overline{O}(\sum_{j \in I} e_j) - (i - 1)E \) are extremal, thus there are precisely \( 2^n - 1 \) extremal rays.

We give the proof of Theorem 6.2 after discussing several lemmas.

The followings are simple but important observations.

Lemma 6.3. For any \( t \geq 0 \), the linear system \( |\overline{O}(b_1, \ldots, b_n) - tE| \) is naturally identified with \( \text{V}_{N-t}(b_1, \ldots, b_n) \) where \( N = (\sum b_i)/2 \).

Proof. Since \( \mathcal{M}^+ \rightarrow (\mathbb{P}^1)^n//_{L} L \) is a blow-up at a smooth point, \( |\overline{O}(b_1, \ldots, b_n) - tE| \) is the sub linear system of \( |\overline{O}(b_1, \ldots, b_n)| \) consisting of the sections vanishing at \( \bar{p} \) with multiplicity \( \geq t \). By Lemma 5.3, it is identified with \( \text{V}_{N-t}(b_1, \ldots, b_n) \). \( \square \)

Lemma 6.4. For any \( I \subset [n] \) with \( |I| = 2i \) and \( 0 \leq i \leq [n/2] \), the linear system \( |\overline{O}(\sum_{j \in I} e_j) - (i - 1)E| \) is nonempty. For \( i \geq 1 \), it is a conformal block of level one.

Proof. When \( i = 0 \), we have \( E \), which is obviously effective. For \( i \geq 1 \), note that \( |\overline{O}(\sum_{j \in I} e_j) - (i - 1)E| = \text{V}_1(b_1, \ldots, b_n) \) where \( b_j = 1 \) if \( j \in I \) and \( b_j = 0 \) otherwise. By Proposition 5.4, we have rank \( \text{V}_1(b_1, \ldots, b_n) = 1 \). \( \square \)

The next lemma is a key combinatorial result for the computation of the effective cone.

Lemma 6.5. Let \( DS \) be a double sequence of level \( \ell \), height \( h(DS) > 1 \), and of shape \( k = (k_1, \ldots, k_n) \) with \( k_1 \geq \cdots \geq k_n > 0 \). There is a nonempty even subset \( T \subset [n] \) such that there is a double sequence \( DS' \) with level \( \ell - 1 \), height \( h(DS') = h(DS) - 1 \), and shape \( k' = (k'_1, \ldots, k'_n) \) where

\[
k'_j = \begin{cases} k_j, & j \notin T \\ k_j - 1, & j \in T \end{cases}
\]

Proof. We will construct a new double sequence

\[
DS' = \left( \begin{array}{cccc}
  x'_1 & x'_2 & \cdots & x'_n \\
  y'_1 & y'_2 & \cdots & y'_n 
\end{array} \right)
\]

and \( T \) as the following. At the beginning, set \( DS' = DS \) and \( T = \emptyset \). The reader can understand the modification below by identifying \( DS \) with the corresponding boxed Catalan path.
Let $0 = b_1 < \cdots < b_t = |k|$ be the set of integers satisfying that the corresponding boxed Catalan path intersects the $x$-axis at $(b_i, 0)$. Then each $(b_i, 0)$ is the endpoint of the $j(i)$-th path corresponding to the $j(i)$-th column of $DS$. Note that $j(t) = n$. Set $j(1) = 0$.

We will modify $x_s$ and $y_s$ with $j(i) < s \leq j(i+1)$, for $1 \leq i \leq t - 1$.

Case 1. $k_{j(i)+1} = k_{j(i)+2} = h(DS)$.

By Remark 5.6, $x_{j(i)+1} = k_{j(i)+1} = h(DS)$. Since the path already reached the highest point, $x_{j(i)+2} = 0$ and $y_{j(i)+2} = h(DS) = k_{j(i)+2}$. So the y-coordinate of the endpoint of the $(j(i)+2)$-nd path is 0. Hence $j(i+1) = j(i) + 2$. In this case, set $x'_{j(i)+1} = y'_{j(i)+2} = h(DS) - 1$ and put $j(i)+1, j(i)+2$ in $T$. After this modification, the maximum height of the boxed Catalan path between $(j(i)+1)$-st and $(j(i)+1)$-th move is $h(DS) - 1$.

Case 2. $k_{j(i)+1} = h(DS), k_{j(i)+2} < h(DS)$.

In this case, $j(i+1) > j(i)+2$. Set $x'_{j(i)+1} = k'_{j(i)+1} = k_{j(i)+1} - 1 = h(DS) - 1$ and put $j(i)+1$ in $T$. If $k_{j(i)+1} > 1$, then set $y'_{j(i)+1} = k'_{j(i)+1} = k_{j(i)+1} - 1$ and put $j(i)+1$ in $T$. If $k_{j(i)+1} = 1$, then $y_{j(i)+1} > 0$ because if not, $x_{j(i)+1} - 1 = 1$ and after $(j(i)+1)$-nd move, the path meets the $x$-axis. Set $y'_{j(i)+1} = y_{j(i)+1} - 1, k'_{j(i)+1} = k_{j(i)+1} - 1$, and put $j(i)+1$ in $T$. Again, after this modification, the maximum height of the path between $(j(i)+1)$-st and $(j(i)+1)$-th move is $h(DS) - 1$. Also after the modification all $k_s$ for $j(i)+1 \leq s \leq j(i+1)$ is less than $h(DS) - 1 \leq \ell - 1$.

Case 3. $1 < k_{j(i)+1} < h(DS)$ and the path reaches the height $h(DS)$ between $(j(i)+1)$-st and $(j(i)+1)$-th move.

Note that $k_{j(i)+1} < h(DS) \leq \ell$. Thus $k_s \leq k_{j(i)+1} < \ell$ for all $s \geq j(i)+1$. We will apply the same modification rule as in Case 2.

Case 4. $k_{j(i)+1} = 1$ and the path reaches the height $h(DS)$ between $(j(i)+1)$-st and $(j(i)+1)$-th move.

In this case, $k_s = 1$ for every $s \geq j(i)+1$. Then there is $j(i) + 1 \leq r < j(i+1)$ such that after $r$-th move, the path reaches the highest point. Because $h(DS) > 1$, such $r$ is greater than $j(i)+1$ and less than $j(i)+1 - 1$. Now set $x'_r = k_r = 0$ and $y'_{r+1} = k_{r+1} = 0$ and put $r, r+1$ in $T$.

Case 5. $k_{j(i)+1} < h(DS)$ and the path does not reach the height $h(DS)$ between $(j(i)+1)$-st and $(j(i)+1)$-th move.

In this case, we will not modify entries.

After all of these modifications, $DS'$ is a double sequence of level $\ell$ and shape $k'$. $h(DS') = h(DS) - 1$. Note that every $k'_i$ such that $k_i = h(DS)$ become $k_i - 1$. Since $h(DS') \leq \ell - 1$ and every $k'_i$ is at most $h(DS) - 1 \leq \ell - 1$, $DS'$ is a double sequence of level $\ell - 1$, too. By the construction, $|T|$ is even. \hfill \square

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{An example of the modification of a double sequence}
\end{figure}
Proof of Theorem 6.2. We may assume that $\mathcal{M}(\vec{a}) = \mathcal{M}^+$, that is, a blow-up of $(\mathbb{P}^1)^n//_L\text{SL}_2$ at $[\vec{p}]$ where $L = \mathcal{O}(a_1, \ldots, a_n)$.

Take an effective divisor $D \in |\mathcal{O}(k_1, \ldots, k_n) - tE|$. Since $\mathcal{M}^+ \to (\mathbb{P}^1)^n//_L\text{SL}_2$ is a blow-up at a point and $E$ is the exceptional divisor, the intersection of $\text{Eff}(\mathcal{M}^+)$ with the half space $t \leq 0$ is generated by the extremal rays of $(\mathbb{P}^1)^n//_L\text{SL}_2$ and $E$.

Suppose that $t \geq 0$. Then $D$ is the zero set of a section $s \in \mathbb{V}_{N-t}(k_1, \ldots, k_n)$ where $2N = \sum k_i$. By rearranging the indices, we may assume that $k_1 \geq \cdots \geq k_n$. Also by the propagation of vacua, we may assume that $k_n > 0$. Since $\mathbb{V}_{N-t}(k_1, \ldots, k_n) \neq 0$, by Proposition 5.7, there is a double sequence $D S$ of level $N - t$ and shape $k$.

By Lemma 5.5, we can construct a set $T \subset [n]$ and a double sequence $D S'$ of level $N - t - 1$ and shape $k'$ (see Lemma 5.5 for notations). By Proposition 5.7 again, $\mathbb{V}_{N-t-1}(k'_1, \ldots, k'_n) > 0$. Furthermore, if we set

$$b_i = \begin{cases} 1, & i \in T, \\ 0, & i \notin T, \end{cases}$$

then $\mathbb{V}_1(b_1, \ldots, b_n) \neq 0$ by the factorization rule and we have a morphism

$$\mathbb{V}_1(b_1, \ldots, b_n) \otimes \mathbb{V}_{N-t-1}(k'_1, \ldots, k'_n) \to \mathbb{V}_{N-t}(k_1, \ldots, k_n),$$

which is given by the multiplication of sections (see Section 5.1). If we set $|k'| = \sum k'_i$, $N' = |k'|/2$, then $N - t - 1 = N' - (t + 1 - |T|)/2$. Therefore the divisor $D$ is numerically equivalent to the sum of a divisor corresponding to a level one conformal block and an effective divisor whose class is $\mathcal{O}(a'_1, \ldots, a'_n) - t'E$ where $t' := t + 1 - |T|/2 \leq t$.

By induction on $|k|$, we can see that $D$ is numerically equivalent to an effective sum of level one conformal blocks, $E$ and a divisor corresponding to $\mathbb{V}_r(c_1, \ldots, c_n)$ where $c_i$ is either 0 or 1 and $r \geq 1$. The very last divisor is an effective sum of a level one conformal block and the divisor $E$. In summary, $D$ is in the cone generated by level one conformal blocks and $E$.

It remains to show that all of the generators are indeed extremal rays. It is shown in Proposition 6.6 below.

**Proposition 6.6.** For $0 \leq i \leq \lfloor n/2 \rfloor$ and $I \subset [n]$ with $|I| = 2i$, a divisor in $|\mathcal{O}(\sum_{j \in I} e_j) - (i - 1)E|$ is an extremal ray of $\text{Eff}(\mathcal{M}(\vec{a}))$.

**Lemma 6.7.** For each $n \geq 3$, there are $n$ subsets $J_1, \ldots, J_n \subset \lfloor n \rfloor$ such that

1. $|J_k| = n - 2$;
2. $\{\sum_{j \in J_k} e_j\}$ form a basis of $\mathbb{Q}^n$.

**Proof.** For $1 \leq k \leq n - 1$, let $J_k = [n - 1] - \{k\}$. Let $A$ be an $(n - 1) \times (n - 1)$ matrix whose $k$-th row is $\sum_{j \in J_k} e_j$. Then $A = J - I$ where $J$ is the matrix that all entries are 1, and $I$ is the identity matrix. It is straightforward to check that the characteristic polynomial of $J$ is $P(t) = (-1)^{n-1}t^{n-2}(t - n + 1)$. Now $\det(A - I) = P(1) \neq 0$, thus $\sum_{j \in J_k} e_j, \ldots, \sum_{j \in J_{n-1}} e_j$ are linearly independent. Finally, take any $J' \subset [n - 1]$ where $|J'| = n - 3$ and define $J_n = J' \cup \{n\}$. Then $\sum_{j \in J_1} e_j, \ldots, \sum_{j \in J_n} e_j$ are linearly independent. □

**Proof of Proposition 6.6.** Let $S = \{\mathcal{O}(\sum_{j \in I} e_j) - (i - 1)E \mid 0 \leq i \leq \lfloor n/2 \rfloor, I \subset [n], |I| = 2i\}$,
the set of generators of \( \text{Eff}(\mathcal{M}(\vec{a})) \). For \( i = 0, E \) is the exceptional divisor of a blow-up, so it is extremal. For each \( i \geq 1 \) and \( I \subset [n] \), we will construct \( n \) linearly independent functionals \( \ell_1, \cdots, \ell_n \in \text{Pic}(\mathcal{M}(\vec{a}))_\mathbb{Q} \) such that

1. \( \ell_k(\overline{\mathcal{O}}(\sum_{j \in I} e_j) - (i - 1)E) = 0; \)
2. \( \ell_k(D) \geq 0 \) for all \( D \in S; \)

for \( 1 \leq k \leq n \). Since \( \text{Pic}(\mathcal{M}(\vec{a}))_\mathbb{Q} \) has rank \( n + 1 \), we can conclude that all elements of \( S \) are extremal rays of \( \text{Pic}(\mathcal{M}(\vec{a}))_\mathbb{Q} \). By symmetry, it is enough to show for \( I = \{1, 2, \cdots, 2i\} \), i.e., \( \overline{\mathcal{O}}(\sum_{j=1}^{2i} e_j) - (i - 1)E \). When \( i \geq 2 \), let \( J_1, \cdots, J_{2i} \) be \( 2i \) subsets of \( [2i] \) constructed in Lemma 6.7. Define \( \ell_k \) as:

1. \( \ell_k(\overline{\mathcal{O}}(\sum_{j \in I} a_j e_j) - tE) = \sum_{j \in I} a_j + \sum_{j > 2i} a_j - 2t \) for \( 1 \leq k \leq 2i; \)
2. \( \ell_k(\overline{\mathcal{O}}(\sum_{j \in I} a_j e_j) - tE) = a_k \) for \( 2i < k \leq n. \)

If \( i = 1 \) (thus \( I = \{1, 2\} \)), define \( \ell_k \) as:

1. \( \ell_k(\overline{\mathcal{O}}(\sum_{j \in I} a_j e_j) - tE) = \sum_{j=1}^{n} a_j - a_k - t \) for \( k = 1, 2; \)
2. \( \ell_k(\overline{\mathcal{O}}(\sum_{j \in I} a_j e_j) - tE) = a_k \) for \( 2 < k \leq n. \)

It is a routine computation to check that those linear functionals are linearly independent,

\[
\ell_k(\overline{\mathcal{O}}(\sum_{j=1}^{2i} e_j) - (i - 1)E) = 0,
\]

and \( \ell_k(D) \geq 0 \) for every \( D \in S. \)

\( \square \)

Remark 6.8. If \( \mathcal{M}(\vec{a}) \) is general (i.e. stability coincides with semistability) but does not have the maximal Picard number, then it is a rational contraction (a composition of several flips and divisorial contractions) of \( \mathcal{M}(\vec{a}') \) with Picard number \( n + 1 \). If we denote the rational contraction by \( \phi : \mathcal{M}(\vec{a}') \rightarrow \mathcal{M}(\vec{a}) \), then there is a well-defined push-forward

\[
\phi_* : \text{Pic}(\mathcal{M}(\vec{a}'))_\mathbb{Q} \rightarrow \text{Pic}(\mathcal{M}(\vec{a}))_\mathbb{Q}
\]

and \( \text{Eff}(\mathcal{M}(\vec{a})) = \text{im} \, \phi_* (\text{Eff}(\mathcal{M}(\vec{a}')) \) since all divisors on \( \mathcal{M}(\vec{a}) \) are Cartier. So \( \text{Eff}(\mathcal{M}(\vec{a})) \) is generated by \( \{\phi_* (\overline{\mathcal{O}}(\sum_{j \in I} e_j) - (i - 1)E)\}. \) Therefore essentially Theorem 6.2 gives \( \text{Eff}(\mathcal{M}(\vec{a})) \) for a general parabolic weight \( \vec{a}. \)

7. Theta divisors and birational models

Theorem 6.2 tells us that any effective divisor on \( \mathcal{M}(\vec{a}) \) can be described as a nonnegative linear combination of conformal blocks and the exceptional divisor \( E. \) This result has an interesting consequence (Theorem 7.3).

Lemma 7.1. For a general parabolic weight \( \vec{a} \in W^0, \mathcal{M}(\vec{a}) \) is a Mori dream space.

**Proof.** Abe showed that when \( \vec{b} = (1/2, \cdots, 1/2), \mathcal{M}(\vec{b}) \) is a Fano variety ([Abe04, Proposition 2.7]). It is straightforward to see that \( \vec{b} \) is on a stability wall only if \( n \) is even. Thus by [BCHM10, Corollary 1.3.2], \( \mathcal{M}(\vec{b}) \) is a Mori dream space if \( n \) is odd. Set \( \mathcal{M}(\vec{b}') := \mathcal{M}(\vec{b}). \)

When \( n \) is even, \( \vec{b} \) is lying on a stability wall so in this case the Picard number of \( \mathcal{M}(\vec{b}) \) is not maximal. But if we perturb the parabolic weight slightly, then the anticanonical divisor is on the boundary of the nef cone and if we subtract a boundary divisor with small coefficient, then it becomes ample. Thus for the perturbed parabolic weight \( \vec{b}', \mathcal{M}(\vec{b}') \) has the maximal Picard number and it is log Fano. By [BCHM10, Corollary 1.3.2] again, \( \mathcal{M}(\vec{b}') \) is a Mori dream space, too.
Therefore in any case, $\mathcal{M}(\vec{b})$ is a Mori dream space and has the maximal Picard number. Because $\mathcal{M}(\vec{b})$ and $\mathcal{M}(\vec{a})$ are connected by finitely many flips, if one is a Mori dream space then so is the other.

Finally, for a general parabolic weight $\vec{a}$, it is a smooth contraction of certain $\mathcal{M}(\vec{a})$ with the maximal Picard number. Thus it is a Mori dream space, too. □

By above lemma and [HK00, Proposition 1.11], we know that for any effective divisor we can construct a projective model

$$\mathcal{M}(\vec{a})(D) := \text{Proj} \bigoplus_{m \geq 0} H^0(\mathcal{M}(\vec{a}), [\mathcal{O}(mD)])$$

and there are only finitely many of them.

In [Pau96], Pauly described a generalization of the theta divisor on the Jacobian of a curve, to the moduli space of parabolic vector bundles.

**Definition 7.2** ([Pau96, Theorem 3.3]). In $\text{Pic}(\mathcal{M}(\vec{a}))_\mathbb{Q}$, the **theta divisor** $\Theta_{\vec{a}}$ is a divisor such that for any family $(E, \{V_i\})$ over $\pi : S \to \mathcal{M}(\vec{a})$,

$$\pi^*(\Theta_{\vec{a}}) = (\det R\pi_*E)^{-k} \otimes \bigotimes_{i=1}^n \det Q_i^{ka_i} \otimes (\det E|_{S \times \{y\}})^e$$

where $k$ is the smallest positive integer such that $ka_i$ are all integers, $y$ is a point of $\mathbb{P}^1$ and $e$ is determined by $e = k(1 - (\sum a_i)/2)$.

Pauly showed that $\Theta_{\vec{a}}$ is ample ([Pau96, Theorem 3.3]) and

$$H^0(\mathcal{M}(\vec{a}), \Theta_{\vec{a}}) \cong \bigwedge_k(ka_1, \ldots, ka_n)$$

([Pau96, Corollary 6.7]) when $0 < a_i < 1$, or equivalently, $0 < ka_i < k$ for every $1 \leq i \leq n$.

Now we can prove the second main theorem of this paper.

**Theorem 7.3.** For any $\mathbb{Q}$-divisor $D \in \text{int} \text{ Eff}(\mathcal{M}(\vec{a}))$, the birational model $\mathcal{M}(\vec{a})(D)$ is isomorphic to $\mathcal{M}(\vec{b})$ for some parabolic weight $\vec{b}$.

Proof. When rank $\text{Pic}(\mathcal{M}(\vec{a}))_\mathbb{Q}$ is not $n+1$, $\mathcal{M}(\vec{a})$ is a rational contraction of $\mathcal{M}(\vec{a}')$ with the maximal Picard number. Then $\text{Eff}(\mathcal{M}(\vec{a}))$ is embedded into $\text{Eff}(\mathcal{M}(\vec{a}'))$ naturally. So it suffices to show for $\mathcal{M}(\vec{a})$ with the maximal Picard number. Write $D$ as $\mathcal{O}(b_1, \ldots, b_n) - tE$.

First of all, suppose that $t > 0$. We may replace $D$ by its integral multiple and assume that $D$ is sufficiently divisible integral divisor. Then $|D| = \mathcal{V}_{N-t}(b_1, \ldots, b_n)$ where $N = \sum b_i/2$. If $D$ is in the interior of the effective cone, $mD - E$ is effective for $m \gg 0$. This implies that $\mathcal{V}_{m(N-t)-1}(mb_1, \ldots, mb_n) \neq 0$, so $mb_i \leq m(N-t) - 1$ by Item (3) of Remark 5.2. Therefore $b_i < N-t$. Then $D$ is a theta divisor on $\mathcal{M}(1/(N-t))$. Since $D$ is ample on $\mathcal{M}(1/(N-t))$, $\mathcal{M}(\vec{a})(D) \cong \mathcal{M}(1/(N-t))$.

If $t \leq 0$, define $D' := \mathcal{O}(b_1, \ldots, b_n) = D + tE$. Then $\mathcal{M}(\vec{a})(D') \cong (\mathbb{P}^1)^n /_{\mathbb{P}} \text{SL}_2$, where $L = \mathcal{O}(b_1, \ldots, b_n)$. Since $E$ is the exceptional divisor of the rational contraction $\mathcal{M}(\vec{a}) \dashrightarrow (\mathbb{P}^1)^n /_{\mathbb{P}} \text{SL}_2$, $\mathcal{M}(\vec{a})(D) = \mathcal{M}(\vec{a})(D')$. By Proposition 3.3, $(\mathbb{P}^1)^n /_{\mathbb{P}} \text{SL}_2 \cong \mathcal{M}(\vec{b})$ for $0 < c < 2/(\sum b_i)$. □

**Remark 7.4.** The projective models for $D \in \partial \text{Eff}(\mathcal{M}(\vec{a}))$ are also described by moduli spaces of parabolic vector bundles.

There are two different types of degenerations of a theta divisor $\mathcal{V}_k(k_1, \ldots, k_n)$. One is the case that $k_i = 0$ for some $i$. The other one is that $k_i = k$ (Of course, both cases can arise together). If $k_1, \ldots, k_{r+1} > 0$ and $k_{r+1} = \cdots = k_n = 0$, then from the propagation of vacua, $\mathcal{V}_k(k_1, \ldots, k_n) \cong \mathcal{V}_k(k_1, \ldots, k_r)$, thus the projective model is a moduli space of parabolic vector bundles with fewer parabolic points.
The projective model corresponding to the second degeneration is described by Bertram in [Ber94, Section 3]. We may assume that \( k_1 = \cdots = k_r = k \) and \( k_1 < k \) for \( i > r \). For a family of parabolic bundles \((\mathcal{E}, \{\mathcal{V}_i\}, \vec{\alpha})\) of degree \( d \) over \( S \), we can construct a new family of parabolic bundles of degree \( d - r \) and \((n - r)\) marked points as taking the kernel of

\[
\mathcal{E} \rightarrow \bigoplus_{i=1}^{r} \mathcal{E}|_{p_i}/\mathcal{V}_i,
\]

and taking \( n - r \) subspaces \( \mathcal{V}_i \) for \( r < i \leq n \).

Thus we have a rational map

\[
p : \mathcal{M}(\vec{\alpha}, d) \dashrightarrow \mathcal{M}(\vec{\alpha}', d - r),
\]

where \( \vec{\alpha}' = (a_{r+1}, \cdots, a_n) \). In general, \( p \) is not regular because it does not guarantee the stability of the induced family. But when \( a_i \rightarrow 1 \) for \( 1 \leq i \leq r \) (equivalently, \( k_i \) is very close to \( k \) for every \( 1 \leq i \leq r \)), \( p \) is a regular morphism. Bertram showed that the pull-back of the canonical polarization from GIT on \( \mathcal{M}(\vec{\alpha}', -r) \) to \( \mathcal{M}(\vec{\alpha}) \) is precisely \( \mathcal{V}_k(k, \cdots, k, k+r+1, \cdots, k_n) \), where \( \vec{\alpha} = \frac{1}{k}(k_1, \cdots, k_n) \) and \( k_i \) is very close to \( k \) for every \( 1 \leq i \leq r \).

REFERENCES


DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NY 10458

E-mail address: hmoon8@fordham.edu

DEPARTMENT OF MATHEMATICS, POSTECH, POHANG 790-784, KOREA

E-mail address: sangbunyoo@postech.ac.kr