Research Statement

Han-Bom Moon

I am an algebraic geometer with broad interests. My main research area is the geometry, topology and combinatorics of *moduli spaces*. A moduli space is a family of mathematical objects of one particular type where in addition, the family enjoys a desired geometric structure. One of the simplest examples is the Grassmannian, which is the moduli space of sub-vector spaces of a fixed vector space.

I have been focused on the study of explicit geometric structure of many concrete examples of moduli spaces, using the framework of *birational geometry* and *geometric invariant theory (GIT)*. Examples of moduli spaces that I am interested in include moduli spaces of abstract and embedded varieties, vector bundles, and sheaves.

I have organized my past research into three themes (1) Topology of moduli spaces, (2) Mori's program of moduli spaces, and (3) Moduli spaces and quantum invariant theory. I briefly discuss these projects in Sections 1, 2, and 3 respectively, and present my research plan in Section 4.

1. Topology of moduli spaces ([6–8, 17, 18, 23])

A promising strategy, from the perspective of birational geometry, of the study of the geometric structure of a given variety X is as follows: 1) construct a new variety X' (a so-called *birational model*), which shares an open dense subset with X but has simpler geometric properties, 2) study the structure of X', and 3) measure the difference of X and X'. Even for a highly nontrivial variety, by applying this strategy several times, sometimes we may reach a very simple variety.

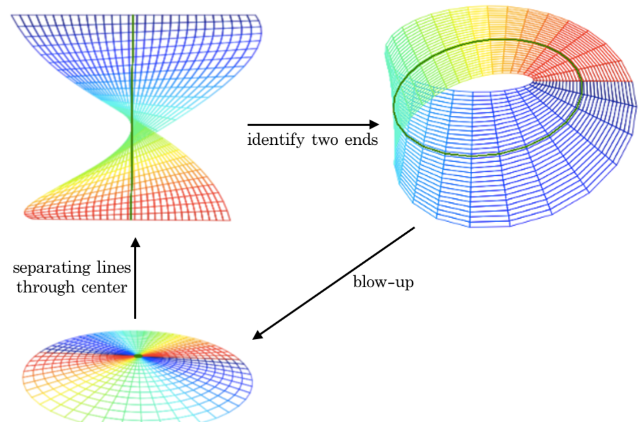

FIGURE 1. A circular disk is a birational model of the Möbius strip, because it is obtained by replacing a circle in the Möbius strip by a point.

When X is a moduli space, often a birational model is also a moduli space parametrizing a slightly different collection of objects. This property enables us to keep track of the difference of two models, while for general higher dimensional varieties this is an extremely difficult task.

In some other cases, such a simpler birational model can be obtained by taking an algebraic quotient space (or *GIT quotient*) of an elementary variety. In this case its topological structure is able to be understood in-depth, and the structure is described in a combinatorial manner. Thus one can see and use interesting interactions among algebra, combinatorics, and geometry.

1.1. **Moduli spaces of genus zero curves and algebraic quotients.**

Here is a typical example showing the general idea of the strategy, in the case of moduli of curves of genus zero. One way to define a genus zero curve C in $\mathbb{P}^r$ is to consider it as the image of a map $f = (f_0 : f_1 : \cdots : f_r) : \mathbb{P}^1 \to \mathbb{P}^r$. So a general $(r+1)$-tuple of homogeneous degree d polynomials with two variables defines a degree d genus zero curve $C = f(\mathbb{P}^1)$, and composing with a coordinate change of $\mathbb{P}^1$ does not affect $C \subset \mathbb{P}^r$. Thus roughly, we have a correspondence

$$C \subset \mathbb{P}^r \longleftrightarrow (f_0 : f_1 : \cdots : f_r)/_\sim$$

and the SL_2-quotient of the space of $(r+1)$-tuples of degree d homogeneous polynomials $\mathbb{P}(\mathrm{Sym}^d \mathbb{C}^2 \otimes \mathbb{C}^{r+1})//\mathrm{SL}_2$ is a birational model of the space of genus zero curves in $\mathbb{P}^r$.

With Kiem, we described the difference between various moduli spaces of genus zero curves and elementary GIT quotients.

Results 1. ([17, 18, 23]) We described the birational maps between the various moduli spaces $\overline{\mathrm{M}}_{0,\mathcal{A}}$ of (weighted) pointed abstract genus zero curves and various GIT quotients $(\mathbb{P}^1)^n//\mathrm{SL}_2$. We also studied the maps between moduli space $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ of embedded genus zero curves with $d \le 3$ and a GIT quotient $\mathbb{P}(\mathrm{Sym}^d \mathbb{C}^2 \otimes \mathbb{C}^{r+1})//\mathrm{SL}_2$. The birational maps are described explicit algebro-geometric surgeries (so-called *blow-ups/downs*). As an application, we gave a formula for the Poincaré polynomial of the moduli spaces.

1.2. **Moduli spaces of sheaves.** Chung and I have been working on birational maps between moduli spaces of sheaves on a surface and other seemingly unrelated moduli spaces such as moduli spaces of curves in the Grassmannian, moduli spaces of points on surfaces.

Results 2. ([6–8]) Let $\mathrm{M}_X(P(m))$ be the moduli space of sheaves with Hilbert polynomial $P(m)$ on a surface X. We studied birational maps from $\mathrm{M}_{\mathbb{P}^2}(4m+1)$, $\mathrm{M}_{\mathbb{P}^1\times\mathbb{P}^1}(4m+2)$, and $\mathrm{M}_{\mathbb{P}^1\times\mathbb{P}^1}(5m+1)$ to other moduli spaces such as the Grassmannian, the moduli space of points on surfaces, and the moduli space of conics in the Grassmannian. Those maps were described in terms of blow-ups/downs. We were obtained topological invariants including the Poincaré polynomial and Chow ring.

2. Mori's program for moduli spaces ([9, 11, 12, 24–30])

When studying the birational geometry of a given variety X, one may aim to classify all birational models of X - an ambitious goal. If X is a Mori dream space ([16]) and if we restrict ourselves to birational models which are equivalent or less complicated than X, then at least theoretically a complete classification is possible. *Mori's program* provides such a theoretical framework to the classification. It consists of three steps:

(1) Study the space of all numerical classes of divisors (codimension one subvarieties) of X;
(2) For a given divisor D of X, compute the associated model

$$X(D) := \mathrm{Proj} \bigoplus_{k \ge 0} \mathrm{H}^0(X, \mathcal{O}(kD))$$

which is a projective variety;
(3) Study the difference of X and $X(D)$.

One can apply this framework to moduli spaces, and it has been one of the most active directions in the study of moduli spaces in the last decade (for instance [4, 5, 15]). For moduli spaces, one may add an extra step to the program:

(4) Give a geometric or moduli theoretic interpretation of $X(D)$.

However, even in simple cases, the completion of Mori's program is very difficult and despite of many results, there are few completed examples. I have carried out several projects for moduli spaces of curves and sheaves.

2.1. **The space of divisors and curves of $\overline{\mathrm{M}}_{g,n}$.** The first step toward Mori's program for a variety X is to understand the space of divisors, or dually, the space of curves, which has a convex cone structure. In the case of $\overline{\mathrm{M}}_{g,n}$, the moduli space of ($n$-pointed) genus g curves, the famous *F-conjecture* gives an explicit set of generators (so-called F-curves) of the cone of curves. Furthermore, it was shown that genus zero case implies the general case ([13]). In general, birational geometric properties of $\overline{\mathrm{M}}_{0,n}$ are too complicated to be fully analyzed. However, it is believed that the S_n-invariant geometry of $\overline{\mathrm{M}}_{0,n}$ behaves well. With Swinarski, we have studied the space of curves and divisors on $\overline{\mathrm{M}}_{0,n}$.

Results 3. (1) ([30]) We found a combinatorial/computational statement which implies (and is strictly stronger than) the S_n-invariant F-conjecture and proved it for $n \le 19$.
(2) ([29]) We provided an effective algorithm to compute curve classes on $\overline{\mathrm{M}}_{0,n}$, by using a toric approximation of $\overline{\mathrm{M}}_{0,n}$.

2.2. **Mori's program for moduli spaces of curves.** The completion of Mori's program for $\overline{\mathrm{M}}_{g,n}$ is out of reach even for very small g and n. But for some geometric divisor classes there are several results.

Results 4. (1) ([24,25]) For a certain range of divisor classes generated by $K_{\overline{\mathrm{M}}_{g,n}}$, λ, and ψ_i (they are divisor classes defined using the moduli theoretic meaning of $\overline{\mathrm{M}}_{g,n}$), I computed the associated birational models of $\overline{\mathrm{M}}_{g,n}$.
(2) ([26,27]) For $n \le 7$ and for every S_n-invariant divisor D, I described $\overline{\mathrm{M}}_{0,n}(D)$.

On the other hand, for the moduli space of genus zero curves in a projective variety, by extending the result in [7], Chung and I obtained the following. This is one of few examples (see [31,32] for other examples) of complete Mori's program for a moduli space with high birational geometric complexity.

Results 5 ([9])**.** We completed Mori's program for the moduli space of conics in the Grassmannian $\mathrm{Gr}(2,n)$.

2.3. **Compactifications of moduli spaces.** Many moduli spaces, such as moduli spaces of smooth curves, are not compact. By compactifying them, we can use powerful tools from algebraic geometry, like intersection theory, to study them. One interesting feature in moduli theory is that it is often possible to construct many compactifications having different moduli theoretic interpretations. This is an important step for Mori's program, because many approachable birational models are realized in this way. My research involves constructing and classifying different compactifications of the moduli space of pointed curves to learn about its geometric/topological properties.

One standard way to obtain a moduli space of (pointed) abstract curves is to consider a moduli space of (pointed) embedded curves in a fixed projective space $\mathbb{P}^d$ and take the quotient by the automorphism group of $\mathbb{P}^d$. This approach depends on the dimension d of the projective space and a continuous data the so-called linearization. In joint work with Giansiracusa, Gibney, Jensen and Swinarski, we proved that:

Results 6 ([11, 12])**.** For each dimension d and a linearization L, we described the moduli theoretic meaning of the quotient variety and studied their geometric properties. All previously known projective alternative modular compactifications of moduli space $\mathrm{M}_{0,n}$ of smooth pointed curves are realized in this way, and there are many new examples.

The other known family of alternative compactifications of $\mathrm{M}_{0,n}$ is obtained from the stack theoretic viewpoint. In [35], Smyth constructed a family of alternative compactifications $\overline{\mathrm{M}}_{0,n}(Z)$ indexed by *extremal assignments* Z, which are combinatorial data described in terms of labeled graphs. With my undergraduate students, we investigated combinatorics of extremal assignments and translated the result in terms of birational geometry of $\overline{\mathrm{M}}_{0,n}$.

Results 7 ([28])**.** Every extremal assignment Z, the associated $\overline{\mathrm{M}}_{0,n}(Z)$, and their several geometric properties can be described in terms of set partitions, intersecting families in hypergraph theory, complete multipartite graphs, and integer partitions. As application, we showed that the collections of birational models from [11,35] are insufficient to describe Mori's program for $\overline{\mathrm{M}}_{0,n}$.

3. Moduli spaces and quantum invariant theory ([31,32])

In representation theory of reductive groups, it is a classical problem to study invariant factors: For the tensor product $V_{\vec{\lambda}}$ of a collection of irreducible finite dimensional G-representations $V_{\lambda_1}, V_{\lambda_2}, \cdots, V_{\lambda_n}$, find the dimension of the trivial sub-representation $V_{\vec{\lambda}}^G$ in $V_{\vec{\lambda}}$. For GL_r and SL_r, this problem is known as Horn's conjecture, and it provides solutions for many classical problems such as the computation of Littlewood-Richardson coefficients. Now we have several complete proofs of Horn's conjecture and some generalizations.

Motivated by 2-dimensional conformal field theory, a quantum generalization of the invariant subspace $V_{\vec{\lambda}}^G$ was constructed. There is a natural finite increasing filtration

$$\cdots \subset \mathbb{V}_{\mathfrak{g},\ell,\vec{\lambda}} \subset \mathbb{V}_{\mathfrak{g},\ell+1,\vec{\lambda}} \subset \cdots \subset \mathbb{V}_{\mathfrak{g},\infty,\vec{\lambda}} = V_{\vec{\lambda}}^G,$$

which defends on a pointed curve in $\overline{\mathrm{M}}_{0,n}$. We call the subspace $\mathbb{V}_{\mathfrak{g},\ell,\vec{\lambda}}$ as the space of level ℓ *conformal blocks*. The sum of all conformal blocks $\mathbb{V}_{\mathfrak{g}} := \bigoplus_{\ell,\vec{\lambda}} \mathbb{V}_{\mathfrak{g},\ell,\vec{\lambda}}$ has a natural commutative algebra structure. There have been many attempts to understand the structure of the *algebra of conformal blocks* $\mathbb{V}_{\mathfrak{g}}$, which can be regarded as a quantum generalization of the algebra of invariants.

The original definition of conformal blocks employs representation theory of infinite dimensional affine Lie algebra. On the other hand, it was shown that conformal blocks can be identified with generalized theta functions on moduli spaces of (parabolic) principal G-bundles. They are natural ample divisors on the moduli space of (parabolic) bundles. Thus one may expect that geometric properties of the moduli spaces have a deep connection with algebraic properties of $\mathbb{V}_{\mathfrak{g}}$.

With Yoo, we have studied birational geometry of the moduli space $\mathrm{M}(r)$ of rank r parabolic vector bundles over a projective line and obtained a fundamental finiteness result on $\mathbb{V}_{\mathfrak{sl}_r}$.

Results 8 ([31,32])**.**
(1) We completed Mori's program of $\mathrm{M}(r)$. In particular, we classified all birational models.
(2) The moduli space is a Mori dream space.
(3) The algebra $\mathbb{V}_{\mathfrak{sl}_r}$ is finitely generated.

4. Current and future research

In this section I describe my research plan.

4.1. **F-conjectures.** The F-conjecture and the S_n-invariant F-conjecture are two of the biggest open problems in the birational geometry of $\overline{\mathrm{M}}_{g,n}$, the moduli space of pointed curves, since it was shown that assuming the F-conjecture (resp. S_n-invariant F-conjecture) for $\overline{\mathrm{M}}_{0,n}$ we obtain the cone of curves of $\overline{\mathrm{M}}_{g,n}$ (resp. $\overline{\mathrm{M}}_{g}$) ([13]). I am working in two directions: With Huh at IAS, we are trying to disprove the F-conjecture, and with Swinarski we are working on the S_n-invariant F-conjecture.

Problem 1. (1) (F-conjecture) Prove that there is a curve class on $\overline{\mathrm{M}}_{0,n}$ which is not an effective linear combination of F-curves.
(2) (S_n-invariant F-conjecture) Show that the curve cone of $\overline{\mathrm{M}}_{0,n}/S_n$ is generated by F-curves.

4.2. **Computational aspects of the GIT quotient.** In the last century, a standard approach to construct a moduli space was GIT. Whenever one studies the GIT quotient of an algebraic variety X equipped with a G-action, the first inevitable step is the computation of the *semi-stable locus* X^{ss}, or equivalently, a finite list of one-parameter subgroups of G, which is usually very delicate combinatorial computation. Algebraic geometers have spent a tremendous amount of time, energy, and journal pages to perform the same kind of numerical/combinatorial computations with bare hands. With Gallardo, Martinez-Garcia, and Swinarski, we are studying an effective algorithm to compute the list of one-parameter subgroups, and have a plan to make a computer program which is open to public. It will let people skip the tedious stability computation and enjoy delightful investigation of geometry of quotient spaces.

Problem 2. Find an effective algorithm to find the semistable locus and implement the algorithm into a computer program.

The above result immediately can be applied to the study of many moduli problems. For instance, explicit simple birational models of $\overline{\mathrm{M}}_g$ with $g \le 6$ have been constructed by GIT. With Swinarski, by using a version of the algorithm, we are investigating the geometry of Mukai's birational models ([[33,34]) for moduli spaces of curves of genus $g \le 9$.

Problem 3. Study moduli theoretic properties of Mukai models and other birational models coming from GIT.

4.3. **Topology of moduli spaces of curves and sheaves.** Since the moduli space of stable maps was introduced by Kontsevich as a standard compactification of the moduli space of embedded curves in a projective variety, several alternative compactifications have been constructed ([10,19,20,22]). They have played important roles in the virtual curve counting theory, but their global geometric properties are not well-known. With Chung and Yoo, we are investigating the geometry of various moduli spaces of genus zero curves as well as that of moduli spaces of bundles and sheaves. We are also studying their application to quantum invariants.

Problem 4. Compute topological invariants of various moduli spaces of genus zero curves, bundles, and one-dimensional sheaves. Run Mori's program for moduli spaces for these moduli spaces and apply it to understand the algebra of conformal blocks.

4.4. **Moduli spaces of higher dimensional varieties.** In many cases, moduli spaces of (abstract or embedded) curves have good geometric properties such as smoothness and irreducibility. However, for higher dimensional varieties, their compactified moduli spaces have very complicated geometric structure. Even the definition of the moduli space has many subtle technical difficulties, so although the rigorous definition was suggested in late 80's ([1, 21]), the construction was completed very recently.

It seems that the study of geometric properties of general cases is out of reach. But in special cases of 1) varieties equipped with group actions ([2]); 2) varieties with combinatorial structures such as line configurations on del Pezzo surfaces or hyperplane arrangements ([3, 14]), the study of their geometric properties is approachable. One of my long-term research plans is to study moduli spaces of higher dimensional varieties.

Problem 5. Study the geometric/topological properties of moduli spaces of higher dimensional varieties with combinatorial structure.

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