# Research Statement

Han-Bom Moon

I am an algebraic geometer with broad interests. My main research area is the geometry, topology and combinatorics of *moduli spaces*. A moduli space is a family of mathematical objects of one particular type where in addition, the family enjoys a desired geometric structure. One of the simplest examples of a moduli space is the Grassmannian, which is the moduli space of sub-vector spaces of a fixed vector space.

I have been focused on the study of explicit geometric structure of many concrete examples of moduli spaces, using the framework of *birational geometry* and *geometric invariant theory (GIT)*. Examples of moduli spaces that I am interested in include moduli spaces of abstract and embedded varieties, vector bundles, and sheaves.

I have organized my past research into three themes (1) Topology of moduli spaces, (2) Mori's program of moduli spaces, and (3) Compactifications of moduli spaces. I briefly discuss these projects in Sections 1, 2, and 3 respectively, and present my future research plan in Section 4.

## 1. Topology of moduli spaces ([7, 8, 19, 20, 26])

A promising strategy, from the perspective of birational geometry, of the study of the geometric structure of a given variety $X$ is as follows: 1) construct a new variety $X'$ (a so-called *birational model*), which shares an open dense subset with $X$ but has simpler geometric properties, 2) study the structure of $X'$, and 3) measure the difference of $X$ and $X'$. Even for a highly nontrivial variety, by applying this strategy several times, sometimes we may reach a very simple variety.

When $X$ is a moduli space, often a birational model is also a moduli space parametrizing a slightly different collection of objects. This property enables us to keep track of the difference of two models, while for general higher dimensional varieties this is an extremely difficult task.

In some other cases, such a simpler birational model can be obtained by taking an algebraic quotient space (or *GIT quotient*) of an elementary variety. In this case its topological structure is able to be understood in-depth, and the structure is described in a combinatorial manner. Thus one can see and use interesting interactions among algebra, combinatorics, and geometry.

### 1.1. Moduli spaces of genus zero curves and algebraic quotients.

Here is a typical example showing the general idea of the strategy, in the case of moduli of curves of genus zero. One way to define a genus zero curve $C$ in $\mathbb{P}^r$ is to consider it as the image of a map $f = (f_0 : f_1 : \cdots : f_r) : \mathbb{P}^1 \to \mathbb{P}^r$. So a general $(r+1)$-tuple of homogeneous degree $d$ polynomials with two variables defines a degree $d$ genus zero curve $C = f(\mathbb{P}^1)$, and composing with a coordinate change of $\mathbb{P}^1$ does not affect $C \subset \mathbb{P}^r$. Thus roughly, we have a correspondence

$$C \subset \mathbb{P}^r \longleftrightarrow (f_0 : f_1 : \cdots : f_r)/_\sim$$

and the $\mathrm{SL}_2$-quotient of the space of $(r+1)$-tuples of degree $d$ homogeneous polynomials $\mathbb{P}(\mathrm{Sym}^d\mathbb{C}^2 \otimes \mathbb{C}^{r+1})//\mathrm{SL}_2$ is a birational model of the space of genus zero curves in $\mathbb{P}^r$.

**Results 1** ([19])**.** Let $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ be Kontsevich's compactification of the space of degree $d$ genus zero curves in $\mathbb{P}^r$ ([24]). When $d = 3$, we obtained $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ from $\mathbb{P}(\mathrm{Sym}^d\mathbb{C}^2 \otimes \mathbb{C}^{r+1})//\mathrm{SL}_2$ by five explicit algebro-geometric surgeries (so-called *blow-ups/downs*). As an application, we computed the Poincaré polynomial of $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ for $d \le 3$.

A recurring theme in my research is the moduli space $\overline{\mathrm{M}}_{0,n}$ of stable $n$-pointed genus zero curves, which is a canonical compactification of the moduli space of $n$-pointed $\mathbb{P}^1$'s. Since configurations of $n$-points on $\mathbb{P}^1$ (up to coordinate changes) can also be parametrized by the $\mathrm{SL}_2$-quotient $(\mathbb{P}^1)^n//\mathrm{SL}_2$, the spaces $\overline{\mathrm{M}}_{0,n}$ and $(\mathbb{P}^1)^n//\mathrm{SL}_2$ are birational.

**Results 2** ([20, 26])**.** We described the birational map between $\overline{\mathrm{M}}_{0,n}$ and $(\mathbb{P}^1)^n//\mathrm{SL}_2$ in terms of explicit birational morphisms, and showed that all intermediate spaces are moduli spaces $\overline{\mathrm{M}}_{0,\mathcal{A}}$ of *weighted* pointed stable rational curves ([16]) with some weight data $\mathcal{A}$. As a byproduct, we gave a recursive formula for the Poincaré polynomial of $\overline{\mathrm{M}}_{0,\mathcal{A}}$ for arbitrary $\mathcal{A}$.

1.2. **Moduli spaces of sheaves.** Sometimes there is an unexpected connection among moduli spaces which seem to be unrelated. During a conversation with Chung, we discovered that three following moduli spaces are indeed birational. Note that they parametrize objects on different ambient spaces.

(1) Moduli space of sheaves on a quadric surface with $c_1 = (2, 2)$ and $\chi = 2$;
(2) Moduli space of sheaves on $\mathbb{P}^3$ with Hilbert polynomial $m^2 + 3m + 2$;
(3) Moduli space of conics in the Grassmannian $\mathrm{Gr}(2, 4)$.

**Results 3** ([8])**.** We described birational maps between them in theoretical ways. The map from (1) to (2) is a Fourier-Mukai transform, and the map from (3) to (2) is Kirwan's desingularization. As an application, we computed the virtual Poincaré polynomial of (1).

We have continued to study moduli spaces of sheaves on surfaces. Recently, the moduli spaces $\mathrm{M}_{\mathbb{P}^2}(dm + 1)$ of one-dimensional sheaves on $\mathbb{P}^2$ have attracted attention because they are used to define and evaluate Gopakumar-Vafa invariant of a local Calabi-Yau threefold. By using moduli spaces of pairs and Bridgeland stable objects, we were able to compute topological invariants of the first nontrivial case.

**Results 4** ([7])**.** We computed the cohomology ring and the Chow ring of $\mathrm{M}_{\mathbb{P}^2}(4m + 1)$.

## 2. MORI'S PROGRAM FOR MODULI SPACES ([9, 27, 28, 29, 30, 32, 33, 34])

When studying the birational geometry of a given variety $X$, one may aim to classify all birational models of $X$ - an ambitious goal. If $X$ is a Mori dream space ([18]) and if we restrict ourselves to birational models which are equivalent or less complicated than $X$, then at least theoretically a complete classification is possible. *Mori's program* provides such a theoretical framework to the classification. It consists of three steps:

(1) Study the space of all numerical classes of divisors (codimension one subvarieties) of $X$;
(2) For a given divisor $D$ of $X$, compute the associated model

$$X(D) := \mathrm{Proj} \bigoplus_{k \ge 0} \mathrm{H}^0(X, \mathcal{O}(kD)) \tag{1}$$

which is a projective variety;
(3) Study the difference of $X$ and $X(D)$.

One can apply this framework to moduli spaces, and it has been one of the most active directions in the study of moduli spaces in the last decade (for instance [4, 5, 17]). For moduli spaces, one may add an extra step to the program:

(4) Give a geometric or moduli theoretic interpretation of $X(D)$.

However, even in simple cases, the completion of Mori's program is very difficult and despite of many results, there are few completed examples. I have carried out several projects for moduli spaces of curves and sheaves.

2.1. **The space of divisors and the space of curves of $\overline{\mathrm{M}}_{0,n}$.** The first step toward Mori's program for a variety $X$ is to understand the space of divisors, or dually, the space of curves, which has a convex cone structure. In the case of $\overline{\mathrm{M}}_{0,n}$, the famous *F-conjecture* gives an explicit set of generators (so-called F-curves) of the cone of curves, but the conjecture is open for $n \ge 8$. Recently it was shown that $\overline{\mathrm{M}}_{0,n}$ is not a Mori dream space if $n$ is large ([6]). This implies that birational geometric properties of $\overline{\mathrm{M}}_{0,n}$ are too complicate to be fully analyzed if $n$ is large. However, it is believed that the $S_n$-invariant geometry of $\overline{\mathrm{M}}_{0,n}$ behaves well.

**Results 5** ([33])**.** With Swinarski, we found a combinatorial/computational statement which implies (and strictly stronger than) the $S_n$-invariant F-conjecture and proved it for $n \le 19$.

On the other hand, since the conjectural cone structure is extremely complicate, it is even not easy to determine whether a given curve class is indeed contained in the cone generated by F-curves. With Swinarski, we studied computational aspects of curve classes on $\overline{\mathrm{M}}_{0,n}$.

**Results 6** ([32])**.** We provided an effective algorithm to compute curve classes on $\overline{\mathrm{M}}_{0,n}$, by using a toric approximation of $\overline{\mathrm{M}}_{0,n}$. By applying this algorithm, we gave an effective computation of curve classes which are $G$-fixed loci for a subgroup $G \le S_n$.

2.2. **Mori's program for moduli spaces of curves.** I found and generalized a universal formula describing all moduli spaces $\overline{\mathrm{M}}_{g,\mathcal{A}}$ of weighted pointed stable curves ([16]) as birational models of $\overline{\mathcal{M}}_{g,n}$. This result supplements and generalizes many results about log canonical models of $\overline{\mathcal{M}}_{g,n}$ including [11, 37].

**Results 7** ([27, 29])**.** Let $\mathcal{A} = (a_1, \cdots, a_n)$ be a weight datum. Then

$$\overline{\mathcal{M}}_{g,n}(K_{\overline{\mathcal{M}}_{g,n}} + 11\lambda + \sum_{i=1}^{n} a_i\psi_i) \cong \overline{\mathrm{M}}_{g,\mathcal{A}}. \tag{2}$$

where $\overline{\mathrm{M}}_{g,\mathcal{A}}$ is the moduli space of $\mathcal{A}$-stable curves ([16]). Here $\lambda, \psi_i$ are divisors defined using the moduli theoretic meaning of $\overline{\mathcal{M}}_{g,n}$.

Result 7 and many results on Mori's program for $\overline{\mathcal{M}}_{g,n}$ describe birational models associated to divisors in a small region of the cone of divisors of $\overline{\mathcal{M}}_{g,n}$. However, outside this region, few results are known. For instance, there are many *flips* (roughly, a flip of $X$ is a modification of $X$ along a small subvariety) of $\overline{\mathrm{M}}_{0,n}$, but their moduli theoretic interpretations are not clear. As an initial step toward this direction, I studied birational models associated to $S_n$-invariant divisors on $\overline{\mathrm{M}}_{0,n}$.

**Results 8** ([28, 30])**.** For $n \le 7$ and for every $S_n$-invariant divisor $D$, we described $\overline{\mathrm{M}}_{0,n}(D)$.

On the other hand, for the moduli space of genus zero curves in a projective variety, by extending the result in [8], Chung and I obtained the following result.

**Results 9** ([9]). We completed Mori's program for the moduli space of conics in Grassmannian $\mathrm{Gr}(2, n)$.

2.3. **Mori's program for moduli spaces of parabolic bundles via conformal blocks.** Among algebraic geometers working on moduli spaces of (parabolic) principal $G$-bundles, conformal blocks are known as generalized theta functions. They are natural divisors on the moduli space of bundles. With Yoo, by studying GIT and the combinatorics of $\mathfrak{sl}_2$-conformal blocks, we obtain the following result.

**Results 10** ([34]). We completed Mori's program of the moduli space of rank 2, degree 0 parabolic bundles on $\mathbb{P}^1$. The cone of divisors is an $(n+1)$-dimensional polyhedral cone generated by $2^{n-1}$ extremal rays. For any divisor, the associated birational model is also a moduli space of parabolic bundles with some degree and some parabolic weights.

These two results [9, 34] are rare examples of a completed Mori's program with Picard number $\geq 3$.

## 3. Compactifications of moduli spaces ([12, 13, 31])

Many moduli spaces, such as the moduli space of smooth curves, are not compact. By compactifying them, we can use powerful tools from algebraic geometry, like intersection theory, to study them. One interesting feature in moduli theory is that it is often possible to construct many compactifications having different moduli theoretic interpretations. Understanding the relationships between different compactifications helps us to obtain geometric information of compactifications. My research involves constructing and classifying different compactifications of the moduli space of pointed curves to learn about its geometric/topological properties.

3.1. **Compactifications of $\mathrm{M}_{0,n}$, GIT, and conformal blocks ([12, 13]).** One standard way to obtain a moduli space of abstract curves is to consider a moduli space of embedded curves in a fixed projective space $\mathbb{P}^d$ and take the quotient by the automorphism group of $\mathbb{P}^d$. If $U_{d,n}$ is the incidence subvariety in $\mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n$ of (possibly singular) genus zero curves of degree $d$ in $\mathbb{P}^d$ and $n$ points on them, by taking the $\mathrm{SL}_{d+1}$-GIT quotient, we get a compactification of $\mathrm{M}_{0,n}$, the moduli space of smooth $n$-pointed genus zero curves. Here the GIT quotient depends on $(n+1)$ parameters a so-called linearization. In joint work with Giansiracusa, Gibney, Jensen and Swinarski, we proved that:

**Results 11** ([12, 13]). For each linearization $L$, the GIT quotient $U_{d,n} /\!/_L \mathrm{SL}_{d+1}$ has an explicit moduli theoretic meaning. All previously known projective alternative modular compactifications of $\mathrm{M}_{0,n}$ are realized in this way, and there are many new examples. We computed the numerical class of the canonical ample divisor for each quotient. As an application, we gave moduli theoretic descriptions of birational models naturally obtained from $\mathfrak{sl}_2$-conformal blocks with symmetric weight data.

3.2. **Combinatorics of extremal assignments.** The other known family of alternative compactifications of $\mathrm{M}_{0,n}$ is obtained from the stack theoretic viewpoint. In [38], Smyth constructed a family of alternative compactifications $\overline{\mathrm{M}}_{0,n}(Z)$ indexed by *extremal assignments* $Z$, which are combinatorial data described in terms of labeled graphs. With my undergraduate students, we investigated combinatorics of extremal assignments and translated the result in terms of birational geometry of $\overline{\mathrm{M}}_{0,n}$.

**Results 12** ([31])**.** Every extremal assignment $Z$ and the associated $\overline{\mathrm{M}}_{0,n}(Z)$ can be described in terms of a collection of set partitions. Furthermore we have three bijections between:

(1) The set of smooth $\overline{\mathrm{M}}_{0,n}(Z)$'s and the set of simple intersecting families in hypergraph theory;
(2) The set of toric $\overline{\mathrm{M}}_{0,n}(Z)$'s and the set of complete multipartite graphs;
(3) The set of $S_n$-invariant $\overline{\mathrm{M}}_{0,n}(Z)$'s and the set of special families of integer partitions.

## 4. Current and future research

In this section I describe my research plan.

4.1. **Topology of moduli spaces of embedded curves and sheaves.** Since the moduli space of stable maps was introduced by Kontsevich as a standard compactification of the moduli space of embedded curves in a projective variety, several alternative compactifications have been constructed. Examples include the moduli space of logarithmic stable maps ([21]), of quasi-maps ([10]), of stable quotients ([25]), and of unramified stable maps ([22]). They have played important roles in the virtual curve counting theory, but their global geometric properties are not well-known.

**Problem 1.** Compute topological invariants of various moduli spaces of embedded genus zero curves.

With Chung, we are investigating the geometry of moduli spaces of sheaves and moduli of curves on Fano varieties.

**Problem 2.** Compute topological invariants of moduli spaces of one-dimensional sheaves on surfaces. Study the Mori's program for moduli spaces of genus zero curves on projective spaces and Grassmannian.

4.2. $S_n$**-invariant F-conjecture.** The $S_n$-invariant F-conjecture is one of the biggest open problems in the birational geometry of $\overline{\mathcal{M}}_g$, the moduli space of stable curves, since it was shown that assuming the $S_n$-invariant F-conjecture for $\overline{\mathrm{M}}_{0,n}$ we obtain the cone of curves of $\overline{\mathcal{M}}_g$ ([14]). With Swinarski we are working on this problem by refining a computational approach we developed in [33].

**Problem 3** ($S_n$-invariant F-conjecture)**.** Show that the curve cone of $\overline{\mathrm{M}}_{0,n}/S_n$ is generated by F-curves.

4.3. **Mori's program for moduli spaces of higher rank parabolic bundles.** With Yoo we continue to study moduli spaces of parabolic bundles.

**Problem 4.** Generalize the result in Section 2.3 to moduli spaces of higher rank parabolic bundles.

4.4. **Computational aspects of the GIT quotient.** In the last century, a standard approach to construct a moduli space was GIT. Whenever one studies the GIT quotient of an algebraic variety $X$ equipped with a reductive group $G$-action, the first inevitable step is the computation of *semi-stable locus* $X^{ss}$, or equivalently, a finite list of one-parameter subgroups of $G$, which is usually very delicate combinatorial computation. Algebraic geometers have spent a tremendous amount of time, energy, and journal pages to perform the same kind of numerical/combinatorial computations by bare hand. With Swinarski, we are studying an effective algorithm to compute the list of one-parameter subgroups, and have a plan to make a computer program which is open to public. It will let people skip the tedious stability computation and enjoy delightful investigation of geometry of quotient spaces.

**Problem 5.** Find an effective algorithm to find the semistable locus and implement the algorithm into a computer program.

The above result immediately can be applied to the study of many moduli problems. For moduli spaces $\overline{\mathcal{M}}_g$ of stable curves with genus $g \le 6$, explicit simple birational models have been constructed by GIT. For $7 \le g \le 9$, Mukai ([35, 36]) gave a beautiful idea to construct a birational model (the so-called *Mukai model*) of $\overline{\mathcal{M}}_g$ as a GIT quotient of a homogeneous variety. With Swinarski, by using a version of the algorithm, we are investigating the geometry of the GIT quotient.

**Problem 6.** Study the geometric/moduli theoretic properties of Mukai model.

4.5. **Moduli spaces of higher dimensional varieties.** In many cases, moduli spaces of (abstract or embedded) curves have good geometric properties such as smoothness and irreducibility. However, for higher dimensional varieties, their compactified moduli spaces have very complicated geometric structure. Even the definition of the moduli space has many subtle technical difficulties, so although the rigorous definition was suggested in late 80's ([1, 23]), the construction was completed very recently.

It seems that the study of geometric properties of general cases is out of reach. But in special cases of 1) varieties equipped with group actions ([2]); 2) varieties with combinatorial structures such as line configurations on del Pezzo surfaces or hyperplane arrangements ([3, 15]), the study of their geometric properties is approachable. One of my long-term research plans is to study moduli spaces of higher dimensional varieties.

**Problem 7.** Study the geometric/topological properties of moduli spaces of higher dimensional varieties with combinatorial structure.

4.6. **Finite dimensional description of conformal block bundles.** Most of the recent studies about connections between conformal blocks and birational geometry of $\overline{\mathrm{M}}_{0,n}$ have focused on numerical properties of them. In spite of many beautiful results, the geometric explanation of them along the boundary of $\overline{\mathrm{M}}_{0,n}$ is not fully satisfactory. On the locus of singular curves, the standard construction of conformal blocks is to use the representation theory of infinite dimensional affine Lie algebras, while on the locus of smooth curves, they can be defined as generalized theta functions on the moduli space of parabolic principal bundles.

**Problem 8.** Find a definition of conformal blocks vector *bundles* on $\overline{\mathrm{M}}_{0,n}$ in terms of finite dimensional algebraic geometry using relative moduli spaces of parabolic bundles and their appropriate degenerations.

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