Research Statement

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I am an algebraic geometer with broad interests. My main research area is geometry, topology and combinatorics of moduli spaces with emphasis on their birational geometry.

In mathematics, one effective way to study an object is to consider it as a member of a family of objects and analyze possible degenerations within this family. A moduli space is a family of mathematical objects of one particular type where in addition, the family enjoys a desired geometric structure. By analyzing the geometry of moduli spaces, we can understand the nature of parameterized objects such as algebraic varieties or vector bundles.

Furthermore, these days moduli spaces play a central role in many other branches of mathematics such as topology and mathematical physics. For example, in the superstring theory, one of the key ingredients is Gromov-Witten invariant which virtually enumerates curves in a projective variety satisfying prescribed conditions. A successful mathematical definition of Gromov-Witten invariant, due to Kontsevich-Manin, Behrend-Fantechi and Li-Tian ([KM94, BF97, LT98]), was obtained by using the virtual intersection theory on the moduli space of stable maps, which is a compactification of the moduli space of curves of fixed numerical type in a projective variety. Also Gromov-Witten invariant is used to define a topological invariant the so-called quantum cohomology, which is a generalization of the ordinary cohomology ring of an algebraic variety.

On the other hand, the origin of birational geometry dates back to the beginning of algebraic geometry. From the late 19th century, it has been well-known that two birationally equivalent varieties (varieties share an isomorphic open dense subvariety) have many common properties. For each birational equivalence class, we can do more concrete work. If we can describe the relation between two birational varieties in terms of explicit blow-ups/downs, then we can catch many concrete geometric data of one of them from data of the other. Thus studying the birational map explicitly is a crucial step in understanding the geometry and topology of a given variety by using simpler or well-known varieties.

My research has been focused on the application of the framework of birational geometry to the study of various moduli spaces. I have organized my research into four themes (1) Topology of moduli spaces [KM10, KM11, Moo11, CM14, CM15], (2) Compactifications of moduli spaces of curves [GJM13, MSvAX15], (3) Mori's program of moduli spaces [Moo13, Moo15a, Moo15b, Moo14, MS15], and (4) Conformal blocks and their connection with birational geometry [GJMS13, MY15]. I briefly discuss these projects in Sections 1, 2, 3, and 4 respectively.

1. Topology of moduli spaces ([KM10, KM11, Moo11, CM14, CM15])

A promising strategy of the computation of topological invariants of a given algebraic variety X is follows: 1) construct a new algebraic variety X' (a so-called *birational model*), which is birational to X but has simpler geometric properties, 2) study the topological structure of X', and 3) measure the difference of X and X'. Even for highly nontrivial moduli spaces, by applying this strategy several times, sometimes we may reach a very simple variety.

When X is a moduli space, in many cases, a birational model is also a moduli space parametrizing a slightly different collection of objects. In some other cases, such a variety can be obtained by taking an algebraic quotient (or GIT quotient [MFK94]) of an elementary variety. By analyzing the relation among moduli spaces and birational models from GIT, we can obtain geometric information of the initial moduli space.

1.1. **Past research - Comparison of moduli spaces and algebraic quotients.**

1.1.1. *Moduli spaces of genus-zero curves and algebraic quotients.* With my advisor Kiem, I studied moduli spaces of rational curves (genus-zero curves).

Theorem 1. [KM10] Let $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ be the moduli space of genus-zero stable maps ([KM94]), a compactification of the moduli space of rational curves in $\mathbb{P}^r$ of degree d. Let

$$\psi : \overline{\mathrm{M}}_0(\mathbb{P}^r, d) \dashrightarrow \mathbb{P}(\mathrm{Sym}^d\mathbb{C}^2 \otimes \mathbb{C}^{r+1})/\!/\mathrm{SL}_2.$$

be the birational map between $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ and a GIT quotient.

(1) When $d = 3$, ψ is a composition of three blow-ups and two blow-downs with explicit centers.
(2) We compute topological invariants of $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$, for instance the Poincaré polynomial, the integral Picard group and (in some cases) the cohomology ring of $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ when $d = 2, 3$.

A recurring theme in my research is the moduli space $\overline{\mathrm{M}}_{0,n}$ of stable n-pointed rational curves.

Theorem 2. [KM11, Moo11]

(1) The birational morphism $\pi_L : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n/\!/_L\mathrm{SL}_2$ for any effective linearization L ([Kap93]) is decomposed into a sequence of smooth blow-ups and Kirwan's desingularizations ([Kir85]). All intermediate spaces are moduli spaces $\overline{\mathrm{M}}_{0,\mathcal{A}}$ of weighted pointed stable rational curves ([Has03]) with some weight data $\mathcal{A}$, and all $\overline{\mathrm{M}}_{0,\mathcal{A}}$ are obtained in this way.
(2) We give a recursive formula for the Poincaré polynomial of $\overline{\mathrm{M}}_{0,\mathcal{A}}$ for arbitrary weight data.

1.1.2. *Moduli spaces of sheaves.* Sometimes there is an unexpected connection among moduli spaces which seem to be unrelated. The following result, which was proved in a joint work with Chung, turned my attention to moduli spaces of sheaves.

Theorem 3. [CM14] Let $\mathrm{M}_Q((2,2),2)$ be the moduli space of semistable sheaves on a smooth quadric surface Q with $c_1 = (2,2)$ and $\chi = 2$. Let $\mathrm{M}_{\mathbb{P}^3}(m^2+3m+2)$ be the moduli space of semistable sheaves on $\mathbb{P}^3$ with Hilbert polynomial m^2+3m+2. Finally let $\overline{\mathrm{M}}_0(\mathrm{Gr}(2,4),2)$ be the moduli space of genus zero, degree two stable maps to the Grassmannian $\mathrm{Gr}(2,4)$. Then there are two birational regular morphisms

$$\mathrm{M}_Q((2,2),2) \stackrel{p}{\to} \mathrm{M}_{\mathbb{P}^3}(m^2+3m+2) \stackrel{q}{\leftarrow} \overline{\mathrm{M}}_0(\mathrm{Gr}(2,4),2),$$

where p is a Fourier-Mukai transform ([Huy06]), and q is Kirwan's desingularization ([Kir85]). In particular, $\mathrm{M}_Q((2,2),2)$ is a rational variety. We also compute the virtual Poincaré polynomial of $\mathrm{M}_Q((2,2),2)$.

We have continued to study moduli spaces of sheaves on surfaces. By using the Bridgeland wall-crossing ([BM14b, BM14a]) and the wall-crossing of pairs ([He98]), we obtained the following result.

Theorem 4. [CM15] Let $\mathrm{M}_{\mathbb{P}^2}(4m+1)$ be the moduli space of stable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $4m+1$. We compute its cohomology ring, Chow ring, and the total Chern class.

The moduli spaces $\mathrm{M}_{\mathbb{P}^2}(dm+1)$ are used to define Gopakumar-Vafa invariants of a local CY threefold ([Kat08]), and $d = 4$ is the first nontrivial case.

1.2. **Current and future research - Variations of moduli spaces of stable maps.** In past ten years, several alternative compactifications of the moduli space of embedded curves have been constructed. Examples include the moduli space of logarithmic stable maps ([Kim10]), of quasi-maps ([CFK10]), of stable quotients ([MOP11]), and of unramified stable maps ([KKO14]). They have played important roles in the virtual curve counting theory.

Problem 5. Compute topological invariants of various moduli spaces of maps.

With Chung, I will also continue to study topological properties of moduli spaces of sheaves.

2. COMPACTIFICATIONS OF MODULI SPACES ([GJM13, MSvAX15])

Many moduli spaces, such as the moduli space of smooth curves, are not compact. By compactifying them, we can use powerful tools from algebraic geometry, like intersection theory, to study them. Compactifications that are moduli spaces themselves, are the most desirable. It is often possible to construct many compactifications having different modular interpretations. Often there is a "best" or preferred compactification, but understanding the relationships between different compactifications helps us to obtain geometric information of compactifications. My research involves constructing different compactifications of a given moduli space in order to learn about its geometric/topological properties.

2.1. **Past research - alternative compactifications of $\mathrm{M}_{0,n}$.**

2.1.1. *GIT and compactifications of* $\mathrm{M}_{0,n}$ *([GJM13]).* One possible way to obtain a moduli space of abstract curves is to consider a moduli space of embedded curves in a fixed projective space $\mathbb{P}^d$ and taking the quotient by the automorphism group of $\mathbb{P}^d$. If $U_{d,n}$ is the incident subvariety in $\mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n$ of (possibly singular) rational curves of degree d in $\mathbb{P}^d$ and n points on them, by taking the SL_{d+1}-GIT quotient, we get a compactification of $\mathrm{M}_{0,n}$, the moduli space of smooth n pointed rational curves.

In a joint work with Giansiracusa and Jensen, we proved that:

Theorem 6. [GJM13] For each effective linearization L on $U_{d,n}$, the GIT quotient $U_{d,n}/\!/_L\mathrm{SL}_{d+1}$ has a modular meaning and there is a birational morphism $\pi_L : \overline{\mathrm{M}}_{0,n} \to U_{d,n}/\!/_L\mathrm{SL}_{d+1}$. Many of known alternative compactifications of $\mathrm{M}_{0,n}$ such as Hassett's spaces $\overline{\mathrm{M}}_{0,\mathcal{A}}$ ([Has03]) are realized in this way.

2.1.2. *Combinatorics of extremal assignments.* Currently, there are two large families of alternative compactifications of $\mathrm{M}_{0,n}$. One family is obtained from GIT quotients as in Section 2.1.1. The other family is obtained from the stack theoretic viewpoint. In [Smy13], Smyth constructed a family of alternative compactifications $\overline{\mathrm{M}}_{0,n}(Z)$ indexed by *extremal assignments* Z, which are combinatorial data described in terms of labeled graphs. With my undergraduate students, we investigated combinatorics of extremal assignments and translated the result in terms of birational geometry of $\overline{\mathrm{M}}_{0,n}$.

Theorem 7. [MSvAX15] Every extremal assignment Z and the associated alternative compactification $\overline{\mathrm{M}}_{0,n}(Z)$ can be described in terms of a collection of set partitions with easily checked conditions. If we specialize in three important subfamilies, we have three bijections:

(1) The set of smooth $\overline{\mathrm{M}}_{0,n}(Z)$ and the set of simple intersecting families in hypergraph theory;
(2) The set of toric $\overline{\mathrm{M}}_{0,n}(Z)$ and the set of complete multipartite graphs;
(3) The set of S_n-invariant $\overline{\mathrm{M}}_{0,n}(Z)$ and the set of special families of integer partitions.

2.2. **Current and future research.**

2.2.1. *Moduli theoretic interpretations of flips of* $\overline{\mathrm{M}}_{0,n}$. Recent studies on modular compactifications of $\mathrm{M}_{0,n}$ ([GJM13, Smy13, MSvAX15]) show that current descriptions of modular compactifications give only a small amount of information on the birational geometry of $\overline{\mathrm{M}}_{0,n}$. For instance, there are many *flips* (roughly, a flip of X is a birational model of X with essentially equivalent data) of $\overline{\mathrm{M}}_{0,n}$, but their moduli theoretic interpretations are not clear (see [Moo14] for an attempt when $n = 7$).

Problem 8. Find moduli theoretic interpretations of flips of $\overline{\mathrm{M}}_{0,n}$.

2.2.2. *GIT and birational models of* $\overline{\mathcal{M}}_7$. For moduli spaces $\overline{\mathcal{M}}_g$ of stable curves with genus $g \le 6$, explicit birational models have been constructed as GIT quotients of elementary varieties. For $g = 7$, Mukai gave a beautiful idea to construct a birational model (the so-called *Mukai model*) of $\overline{\mathcal{M}}_7$ as a GIT quotient of a homogeneous variety ([Muk95]). I am studying geometry of the GIT quotient.

Problem 9. Study the geometric/moduli theoretic properties of Mukai model.

3. MORI'S PROGRAM FOR MODULI SPACES ([Moo13, Moo15a, Moo15b, Moo14, MS15])

A central problem in the birational algebraic geometry when studying a variety X is to determine all birational models of X. If we restrict ourselves to birational models which are equivalent or less complicate than X in some sense, then under some assumptions (for instance being a Mori dream space ([HK00])) a complete classification is possible. *Mori's program* provides such a theoretical framework to the classification. It consists of 1) Study the cone of divisors (or dually, that of curves) of X; 2) For a given divisor D of X, compute the associated model

$$X(D) := \mathrm{Proj} \bigoplus_{k \ge 0} \mathrm{H}^0(X, \mathcal{O}(kD)), \tag{1}$$

3) Study the difference of X and $X(D)$. Many results in modern birational geometry within the last several decades are concerned with overcoming technical problems along this line, and recently there have been staggeringly positive results [BCHM10].

We can apply this tack to moduli spaces. For example, the *Hassett-Keel program* is a systematic approach to find the *canonical model* $\overline{\mathcal{M}}_g(K_{\overline{\mathcal{M}}_g})$ of $\overline{\mathcal{M}}_g$ ([HH09, HH13]). I have carried out a similar program for $\overline{\mathcal{M}}_{g,n}$, the moduli space of pointed stable curves.

3.1. **Past research - Mori's program for moduli spaces of pointed curves.**

3.1.1. *A universal formula for log canonical models of $\overline{\mathcal{M}}_{g,n}$ ([Moo13, Moo15a]).* I found and generalized a universal formula describing all of moduli spaces $\overline{\mathrm{M}}_{g,\mathcal{A}}$ of weighted pointed stable curves ([Has03]) as *log canonical models* of $\overline{\mathcal{M}}_{g,n}$. This theorem supplements and generalizes many results about log canonical models of $\overline{\mathcal{M}}_{g,n}$ for instance [Smy11, FS11, AS12].

Theorem 10. [Moo13, Moo15a] Let $\mathcal{A} = (a_1, \cdots, a_n)$ be a weight datum. Then

$$\overline{\mathcal{M}}_{g,n}(K_{\overline{\mathcal{M}}_{g,n}} + 11\lambda + \sum_{i=1}^{n} a_i\psi_i) \cong \overline{\mathrm{M}}_{g,\mathcal{A}}. \tag{2}$$

where $\overline{\mathrm{M}}_{g,\mathcal{A}}$ is the coarse moduli space of the moduli space of $\mathcal{A}$-stable curves ([Has03]). Here λ, ψ_i are tautological divisors defined using the modular meaning of $\overline{\mathcal{M}}_{g,n}$.

3.1.2. *Mori's program for $\overline{\mathrm{M}}_{0,n}$ with symmetric divisors.* Theorem 10 and many results on Mori's program for $\overline{\mathcal{M}}_{g,n}$ describe birational models associated to divisors in a small region of the cone of divisors of $\overline{\mathcal{M}}_{g,n}$ well. However, on the outside of the region, few results are known. As an initial step toward this direction, I studied birational models associated to S_n-invariant divisors on $\overline{\mathrm{M}}_{0,n}$.

Theorem 11. [Moo14, Moo15b] For $n \le 7$ and for every S_n-invariant effective divisor D, $\overline{\mathrm{M}}_{0,n}(D)$ is described. $\overline{\mathrm{M}}_{0,n}/S_n$ is a Mori dream space when $n = 7$.

3.1.3. *Effective computation of curve classes on $\overline{\mathrm{M}}_{0,n}$.* The first step toward Mori's program for a variety X is to understand the cone of divisors, or dually, the cone of curves. The famous *F-conjecture* gives an explicit set of generators (so-called F-curves) of the cone of curves, but it is widely open for $n \ge 8$. Since the conjectural cone structure is extremely complicate, it is even not easy to determine whether a given curve class is indeed contained in the cone generated by F-curves.

Theorem 12. [MS15] We provide an effective algorithm to compute curve classes on $\overline{\mathrm{M}}_{0,n}$, by using the geometry of a toric approximation of $\overline{\mathrm{M}}_{0,n}$. By applying this algorithm, we give an effective computation of curve classes which are G-fixed loci for a subgroup $G \le S_n$.

3.2. Current and future research.

3.2.1. *S_n-invariant F-conjecture.* Recently, it was shown that $\overline{\mathrm{M}}_{0,n}$ is not a Mori dream space if $n \ge 134$ ([CT15]). This implies that birational geometric properties of $\overline{\mathrm{M}}_{0,n}$ are too complicate to be fully analyzed. However, it is believed that S_n-invariant geometry of $\overline{\mathrm{M}}_{0,n}$ is well-behaved. I am trying to translate the S_n-invariant F-conjecture into purely combinatorial statements in terms of graphical algebras and finite graphs.

Problem 13 (S_n-invariant F-conjecture)**.** Show that the curve cone of $\overline{\mathrm{M}}_{0,n}/S_n$ is generated by F-curves.

Currently, it is known for $n \le 24$ due to Gibney ([Gib09]). The S_n-invariant F-conjecture for $\overline{\mathrm{M}}_{0,n}$ implies the F-conjecture for $\overline{\mathcal{M}}_g$ ([GKM02]).

3.2.2. *Theoretical explanation of Theorem 10.* It is well-known that for any weight datum $\mathcal{A}$, there is a reduction morphism $\varphi_{\mathcal{A}} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$. For a stable curve $(C, x_1, \cdots, x_n)$, its image $\varphi_{\mathcal{A}}(C)$ is given by the log canonical model

$$C(\omega_C + \sum_{i=1}^n a_i x_i) := \mathrm{Proj} \bigoplus_{k \ge 0} \mathrm{H}^0(C, k(\omega_C + \sum_{i=1}^n a_i x_i)). \tag{3}$$

Theorem 10 means the *same weight datum* determines the log canonical model of parameterized curves and that of the parameter space itself. So far, there is no known theoretical explanation of Theorem 10. I want to figure out the reason of this phenomenon and find some more supporting examples.

Problem 14. (1) Find the theoretical reason of the similarity between (2) and (3).
(2) Generalize Theorem 10 to moduli spaces of stable maps ([KM94]) and Fulton-MacPherson spaces ([FM94]).

4. Conformal blocks and their connection with birational geometry

Recent results on the birational geometry of $\overline{\mathrm{M}}_{0,n}$ have focused attention on vector bundles of conformal blocks ([Fak12, GG12, Gia13, AGS14, BGM15]). Each such vector bundle $\mathbb{V}(\mathfrak{g}, \ell, \underline{\lambda})$ depends on 1) a simple Lie algebra $\mathfrak{g}$, 2) a nonnegative integer ℓ, and 3) an n-tuple $\underline{\lambda} = (\lambda_1, \cdots, \lambda_n)$ of dominant integral weights of $\mathfrak{g}$ of length $\le \ell$, and can be constructed using the representation theory of affine Lie algebras. We call its first Chern class $\mathbb{D}(\mathfrak{g}, \ell, \underline{\lambda}) := c_1(\mathbb{V}(\mathfrak{g}, \ell, \underline{\lambda}))$ a *conformal block divisor*. The divisors $\mathbb{D}(\mathfrak{g}, \ell, \underline{\lambda})$ are base-point-free for any choice of data and thus induce a morphism from $\overline{\mathrm{M}}_{0,n}$ to a projective variety. Many birational models of $\overline{\mathrm{M}}_{0,n}$ arise in this way.

4.1. Past research - birational geometry of moduli of curves and bundles via conformal blocks.

4.1.1. *Birational models come from conformal blocks divisors.* With a research group consisting of Gibney, Jensen, and Swinarski, I have studied the birational models for $\mathfrak{sl}_2$-conformal block divisors on $\overline{\mathrm{M}}_{0,n}$. We gave a complete description of birational models in the case of $\mathfrak{sl}_2$, and symmetric weight data $\omega_1^n = (\omega_1, \cdots, \omega_1)$. It generalizes a partial result in [AGS14].

Theorem 15. [GJMS13] If $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^n)$ is nontrivial, the birational model $\overline{\mathrm{M}}_{0,n}(\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^n))$ (see (1) in Section 3 for the definition) is isomorphic to a GIT compactification $U_{g+1-\ell,n}/\!/_L \mathrm{SL}_{g+2-\ell}$ with an explicit linearization L (Section 2.1.1).

4.1.2. *Conformal blocks and birational geometry of moduli spaces of parabolic bundles.* Among algebraic geometers working on moduli spaces of (parabolic) principal G-bundles, conformal blocks also known as generalized theta functions ([Pau96, LS97]). They are natural effective divisors on the moduli space of bundles.

By studying the combinatorics of $\mathfrak{sl}_2$-conformal blocks, we are able to obtain the following result on the birational geometry of the moduli space of parabolic bundles.

Theorem 16. [MY15] Let $\mathcal{M}(2, 0, \vec{a})$ be the moduli space of rank 2, degree 0 parabolic bundles on $\mathbb{P}^1$ with n parabolic points $x_1, \cdots, x_n$ with weight $\vec{a}$. Suppose that $\mathcal{M}(2, 0, \vec{a})$ has the maximal Picard number $n+1$.

(1) The cone of divisors of $\mathcal{M}(2, 0, \vec{a})$ is generated by level 1 conformal blocks $\mathbb{V}(\mathfrak{sl}_2, 1, \underline{\lambda})|_{(\mathbb{P}^1, x_1, \cdots, x_n)}$. Thus it is a polyhedral cone generated by 2^{n-1} extremal rays.

(2) Every model of $\mathcal{M}(2,0,\vec{a})$ appearing in Mori's program is $\mathcal{M}(2,d,\vec{b})$ for some weight $\vec{b}$ and some degree d.

4.2. **Current and future research.**

4.2.1. *Generalization toward higher rank parabolic bundles.* In the proof of Theorem 16, two key ingredients are 1) a concrete description of $\mathfrak{sl}_r$-conformal blocks [FSV95] and 2) an explicit study of elementary GIT quotients $(\mathbb{P}^1)^n//_L\mathrm{SL}_2$ and linear systems on them.

Problem 17. Generalize Theorem 16 for moduli spaces of higher rank parabolic bundles.

I expect that the right generalization of $(\mathbb{P}^1)^n$ in this context is a product of flag varieties.

4.2.2. *Finite dimensional description of conformal block bundles.* Most of the recent studies ([GG12, GJMS13, Gia13, AGS14, BGM15]) about conformal block divisors are focused on numerical properties of them. In spite of many beautiful results, there is no satisfactory geometric explanation of them so far. One main reason of this shortage is that there is no geometric definition of conformal block bundles. On the locus of singular curves, the only known construction of conformal blocks is to use the representation theory of infinite dimensional affine Lie algebras.

As we discussed before, for a smooth pointed curve, a conformal block is a generalized theta function on the moduli space of parabolic bundles. But still it is not completely understood how to extend this interpretation to the boundary of $\overline{\mathrm{M}}_{0,n}$.

Problem 18. Find a definition of conformal blocks vector *bundles* on $\overline{\mathrm{M}}_{0,n}$ in terms of finite dimensional algebraic geometry.

A natural approach is to relativize the problem. If $\pi : \mathcal{M}(r,d,\underline{\lambda}) \to \overline{\mathrm{M}}_{0,n}$ is the *relative* moduli space of parabolic vector bundles and their degenerations, and if $\mathcal{L}$ is the correct extension of the generalized theta function, then the conformal block vector bundle $\mathbb{V}(\mathfrak{sl}_r,\ell,\underline{\lambda})$ must be $\pi_*(\mathcal{L})$. However, it is very difficult to construct a degeneration of a given moduli space with desired properties.

As a first step, I am studying the $\mathfrak{sl}_2$ case. From the proof of Theorem 16, we know that for a pointed smooth curve $(\mathbb{P}^1, x_1,\cdots,x_n)$, the moduli space $\mathcal{M}(2,0,\vec{a})$ is an explicit modification of $(\mathbb{P}^1)^n//_L\mathrm{SL}_2$. The moduli space $\overline{\mathrm{M}}_{0,n}$ is also another explicit modification of $(\mathbb{P}^1)^n//_L\mathrm{SL}_2$ (Theorem 2). We expect that the relative moduli space $\mathcal{M}(\vec{a})$ can be constructed as an $\mathrm{SL}_2\times\mathrm{SL}_2$-quotient of an explicit modification of $(\mathbb{P}^1)^n\times(\mathbb{P}^1)^n$, by combining Theorems 2 and 16.

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