Research Statement

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I am an algebraic geometer with broad interests. My main research area is geometry, topology and combinatorics of moduli spaces. A moduli space is a family of mathematical objects of one particular type where in addition, the family enjoys a desired geometric structure. My research has been focused on the application of the framework of birational geometry to the study of geometric/topological properties of various moduli spaces.

I have organized my research into four themes (1) Topology of moduli spaces, (2) Compactifications of moduli spaces of curves, (3) Mori's program of moduli spaces, and (4) Conformal blocks and their connection with birational geometry. I briefly discuss these projects in Sections 1, 2, 3, and 4 respectively.

1. Topology of moduli spaces ([KM10, KM11, Moo11, CM14, CM15])

A promising strategy of the computation of topological invariants of a given algebraic variety X is as follows: 1) construct a new algebraic variety X' (a so-called *birational model*), which shares an open dense subset with X but has simpler geometric properties, 2) study the topological structure of X', and 3) measure the difference of X and X'. Even for a highly nontrivial space, by applying this strategy several times, sometimes we may reach a very simple variety.

When X is a moduli space, in many cases, a birational model is also a moduli space parametrizing a slightly different collection of objects. In some other cases, such a variety can be obtained by taking an algebraic quotient (or *GIT quotient*) of an elementary variety. By analyzing the relation among moduli spaces and birational models from GIT, we can obtain geometric information of the initial moduli space.

1.1. Past research - Comparison of moduli spaces and algebraic quotients.

1.1.1. *Moduli spaces of genus-zero curves and algebraic quotients.* With my advisor Kiem, I studied moduli spaces of genus-zero curves in a projective space $\mathbb{P}^r$. One way to define a genus-zero curve in $\mathbb{P}^r$ is to consider it as the image of a map $f = (f_0 : f_1 : \cdots : f_r) : \mathbb{P}^1 \to \mathbb{P}^r$. So a general $(r+1)$-tuple of homogeneous degree d polynomials with two variables defines a degree d genus-zero curve, up to coordinate changes of $\mathbb{P}^1$. Thus roughly, we have a correspondence

$$C \subset \mathbb{P}^r \longleftrightarrow (f_0 : f_1 : \cdots : f_r).$$

Results 1 ([KM10]). Let $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ be Kontsevich's compactification of the space of degree d maps from $\mathbb{P}^1$ to $\mathbb{P}^r$. Let $\mathbb{P}^r_d := \mathbb{P}(\mathrm{Sym}^d\mathbb{C}^2 \otimes \mathbb{C}^{r+1})//\mathrm{SL}_2$ be the SL_2-quotient of the space of $(r+1)$-tuples of degree d polynomials. We show that when $d = 3$, $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ is obtained from $\mathbb{P}^r_d$ by taking five explicit algebro-geometric surgeries (so-called *blow-ups/downs*). As an application, we compute the Poincaré polynomial and (in some cases) the cohomology ring of $\overline{\mathrm{M}}_0(\mathbb{P}^r, d)$ for $d = 2, 3$.

A recurring theme in my research is the moduli space $\overline{\mathrm{M}}_{0,n}$ of stable n-pointed genus-zero curves without regarding embedding. It is a canonical compactification of the moduli space of smooth n-pointed $\mathbb{P}^1$'s. Since configurations of n-points on $\mathbb{P}^1$ (up to coordinate changes) can be parametrized by the SL_2-quotient $(\mathbb{P}^1)^n//\mathrm{SL}_2$, $\overline{\mathrm{M}}_{0,n}$ and $(\mathbb{P}^1)^n//\mathrm{SL}_2$ are birational. In this case, the GIT quotient depends on extra data, a so-called linearization L. Thus we have various birational models $(\mathbb{P}^1)^n//_L\mathrm{SL}_2$.

Results 2 ([KM11, Moo11]). We describe the birational map between $\overline{\mathrm{M}}_{0,n}$ and $(\mathbb{P}^1)^n//_L\mathrm{SL}_2$ in terms of blow-ups and Kirwan's desingularizations. All intermediate spaces between them are moduli spaces $\overline{\mathrm{M}}_{0,\mathcal{A}}$ of weighted pointed stable rational curves ([Has03]) with some weight data $\mathcal{A}$, and conversely all $\overline{\mathrm{M}}_{0,\mathcal{A}}$ are obtained in this way. As a byproduct, we give a recursive formula for the Poincaré polynomial of $\overline{\mathrm{M}}_{0,\mathcal{A}}$ for arbitrary weight data.

1.1.2. *Moduli spaces of sheaves.* Sometimes there is an unexpected connection among moduli spaces which seem to be unrelated. During a conversation with Chung, we discovered that three following moduli spaces are indeed birational ([CM14]).

(1) Moduli space of sheaves on a quadric surface with $c_1 = (2, 2)$ and $\chi = 2$;
(2) Moduli space of sheaves on $\mathbb{P}^3$ with Hilbert polynomial $m^2 + 3m + 2$;
(3) Moduli space of conics in the Grassmannian $\mathrm{Gr}(2, 4)$.

Results 3 ([CM14]). We describe birational maps between them in theoretical ways. The map from (1) to (2) is a Fourier-Mukai transform, and the map from (3) to (2) is Kirwan's desingularization. As an application, we compute the virtual Poincaré polynomial of (1).

We have continued to study moduli spaces of sheaves on surfaces. Recently, the moduli spaces $\mathrm{M}_{\mathbb{P}^2}(dm + 1)$ of one-dimensional sheaves on $\mathbb{P}^2$ have attracted attention because they can be used to define and evaluate Gopakumar-Vafa invariant of a local Calabi-Yau threefold. In general, these spaces have very complicate structure. But we were able to compute topological invariants of the first non-trivial case.

Results 4 ([CM15]). We computed the cohomology ring and the total Chern class of $\mathrm{M}_{\mathbb{P}^2}(4m + 1)$.

1.2. **Current and future research - Variations of moduli spaces of stable maps.** In past ten years, several alternative compactifications of the moduli space of embedded curves have been constructed. Examples include the moduli space of logarithmic stable maps ([Kim10]), of quasi-maps ([CFK10]), of stable quotients ([MOP11]), and of unramified stable maps ([KKO14]). They have played important roles in the virtual curve counting theory.

Problem 1. Compute topological invariants of various moduli spaces of maps.

With Chung, we will also continue to study moduli spaces of sheaves.

Problem 2. Compute topological invariants of moduli spaces of one-dimensional sheaves on surfaces.

2. Compactifications of moduli spaces ([GJM13, MSvAX15])

Many moduli spaces, such as the moduli space of smooth curves, are not compact. By compactifying them, we can use powerful tools from algebraic geometry, like intersection theory, to study them. Compactifications that are moduli spaces themselves, are the most desirable. It is often possible to construct many compactifications having different modular interpretations. Often there is a "best" or preferred compactification, but understanding the relationships between different compactifications helps us to obtain geometric information of compactifications. My research involves constructing different compactifications of a given moduli space in order to learn about its geometric/topological properties.

2.1. **Past research - alternative compactifications of $\mathrm{M}_{0,n}$.**

2.1.1. *GIT and compactifications of* $\mathrm{M}_{0,n}$ *([GJM13]).* One possible way to obtain a moduli space of abstract curves is to consider a moduli space of embedded curves in a fixed projective space $\mathbb{P}^d$ and taking the quotient by the automorphism group of $\mathbb{P}^d$. If $U_{d,n}$ is the incidence subvariety in $\mathrm{Chow}_{1,d}(\mathbb{P}^d) \times (\mathbb{P}^d)^n$ of (possibly singular) genus-zero curves of degree d in $\mathbb{P}^d$ and n points on them, by taking the SL_{d+1}-GIT quotient, we get a compactification of $\mathrm{M}_{0,n}$, the moduli space of smooth n pointed rational curves.

In a joint work with Giansiracusa and Jensen, we proved that:

Results 5 ([GJM13])**.** For each effective linearization L on $U_{d,n}$, the GIT quotient $U_{d,n}/\!/_L \mathrm{SL}_{d+1}$ has an explicit moduli theoretic meaning. All of previously known projective alternative modular compactifications of $\mathrm{M}_{0,n}$ are realized in this way, and there are many new examples.

2.1.2. *Combinatorics of extremal assignments.* Currently, there are two large families of alternative compactifications of $\mathrm{M}_{0,n}$. One family is obtained from GIT quotients as in Section 2.1.1. The other family is obtained from the stack theoretic viewpoint. In [Smy13], Smyth constructed a family of alternative compactifications $\overline{\mathrm{M}}_{0,n}(Z)$ indexed by *extremal assignments* Z, which are combinatorial data described in terms of labeled graphs. With my undergraduate students, we investigated combinatorics of extremal assignments and translated the result in terms of birational geometry of $\overline{\mathrm{M}}_{0,n}$.

Results 6 ([MSvAX15])**.** Every extremal assignment Z and the associated alternative compactification $\overline{\mathrm{M}}_{0,n}(Z)$ can be described in terms of a collection of set partitions with easily checked conditions. If we specialize in three important subfamilies of compactifications, we have three bijections:

(1) The set of smooth $\overline{\mathrm{M}}_{0,n}(Z)$'s and the set of simple intersecting families in hypergraph theory;
(2) The set of toric $\overline{\mathrm{M}}_{0,n}(Z)$'s and the set of complete multipartite graphs;
(3) The set of S_n-invariant $\overline{\mathrm{M}}_{0,n}(Z)$'s and the set of special families of integer partitions.

We also found some compactifications which are even not projective varieties.

2.2. **Current and future research.** For moduli spaces $\overline{\mathcal{M}}_g$ of stable curves with genus $g \le 6$, explicit simple birational models have been constructed as GIT quotients of elementary varieties. For $g = 7$, Mukai gave a beautiful idea to construct a birational model (the so-called *Mukai model*) of $\overline{\mathcal{M}}_7$ as a GIT quotient of a homogeneous variety. With Deopurkar, we are studying geometry of the GIT quotient.

Problem 3. Study the geometric/moduli theoretic properties of Mukai model.

3. Mori's program for moduli spaces ([Moo13, Moo15a, Moo15b, Moo14, MS15])

A central problem in the birational algebraic geometry when studying a variety X is to determine all birational models of X. If we restrict ourselves to birational models which are equivalent or less complicate than X in some sense, then under some assumptions (for instance being a Mori dream space ([HK00])) a complete classification is possible at least in simple cases. *Mori's program* provides such a theoretical framework to the classification. It consists of 1) Study the cone of divisors (or dually, that of curves) of X; 2) For a given divisor D of X, compute the associated model

$$X(D) := \mathrm{Proj} \bigoplus_{k \ge 0} \mathrm{H}^0(X, \mathcal{O}(kD)), \tag{1}$$

and study the difference of X and $X(D)$; 3) Give a geometric or modular interpretation of $X(D)$. Many results in modern birational geometry within the last several decades are concerned with overcoming technical problems along this line, and recently there have been staggeringly positive results including [BCHM10].

We can apply this tack to moduli spaces. For example, the *Hassett-Keel program* is a systematic approach to find the *canonical model* $\overline{\mathcal{M}}_g(K_{\overline{\mathcal{M}}_g})$ of $\overline{\mathcal{M}}_g$ ([HH09, HH13]). I have carried out a similar program for $\overline{\mathcal{M}}_{g,n}$, the moduli space of pointed stable curves.

3.1. Past research - Mori's program for moduli spaces of pointed curves.

3.1.1. *A universal formula for log canonical models of $\overline{\mathcal{M}}_{g,n}$.* I found and generalized a universal formula describing all of moduli spaces $\overline{\mathrm{M}}_{g,\mathcal{A}}$ of weighted pointed stable curves ([Has03]) as *log canonical models* of $\overline{\mathcal{M}}_{g,n}$. This result supplements and generalizes many results about log canonical models of $\overline{\mathcal{M}}_{g,n}$ for instance [Smy11, FS11, AS12].

Results 7 ([Moo13, Moo15a])**.** Let $\mathcal{A} = (a_1, \cdots, a_n)$ be a weight datum. Then

$$\overline{\mathcal{M}}_{g,n}(K_{\overline{\mathcal{M}}_{g,n}} + 11\lambda + \sum_{i=1}^{n} a_i\psi_i) \cong \overline{\mathrm{M}}_{g,\mathcal{A}}. \tag{2}$$

where $\overline{\mathrm{M}}_{g,\mathcal{A}}$ is the coarse moduli space of the moduli space of $\mathcal{A}$-stable curves ([Has03]). Here λ, ψ_i are tautological divisors defined using the moduli theoretic meaning of $\overline{\mathcal{M}}_{g,n}$.

3.1.2. *Mori's program for $\overline{\mathrm{M}}_{0,n}$ with symmetric divisors.* Result 7 and many results on Mori's program for $\overline{\mathcal{M}}_{g,n}$ describe birational models associated to divisors in a small region of the cone of divisors of $\overline{\mathcal{M}}_{g,n}$. However, on the outside of the region, few results are known. For instance, there are many *flips* (roughly, a flip of X is a birational model of X with essentially equivalent data) of $\overline{\mathrm{M}}_{0,n}$, but their moduli theoretic interpretations are not clear. As an initial step toward this direction, I studied birational models associated to S_n-invariant divisors on $\overline{\mathrm{M}}_{0,n}$.

Results 8 ([Moo14, Moo15b])**.** For $n \le 7$ and for every S_n-invariant divisor D, $\overline{\mathrm{M}}_{0,n}(D)$ is described. We give a moduli theoretic interpretation of the first flip of $\overline{\mathrm{M}}_{0,7}$, which was not appeared in literature.

3.1.3. *Effective computation of curve classes on $\overline{\mathrm{M}}_{0,n}$.* The first step toward Mori's program for a variety X is to understand the cone of divisors, or dually, the cone of curves. The famous *F-conjecture* gives an explicit set of generators (so-called F-curves) of the cone of curves, but it is widely open for $n \ge 8$. Since the conjectural cone structure is extremely complicate, it is even not easy to determine whether a given curve class is indeed contained in the cone generated by F-curves. With Swinarski, we studied computational aspects of curve classes on $\overline{\mathrm{M}}_{0,n}$.

Results 9. [MS15] We provide an effective algorithm to compute curve classes on $\overline{\mathrm{M}}_{0,n}$, by using the geometry of a toric approximation of $\overline{\mathrm{M}}_{0,n}$. By applying this algorithm, we give an effective computation of curve classes which are G-fixed loci for a subgroup $G \le S_n$.

3.2. **Current and future research.**

3.2.1. *S_n-invariant F-conjecture.* Recently, it was shown that $\overline{\mathrm{M}}_{0,n}$ is not a Mori dream space if $n \ge 13$ ([CT15, GK14]). This implies that birational geometric properties of $\overline{\mathrm{M}}_{0,n}$ are too complicate to be fully analyzed if n is large. However, it is believed that S_n-invariant geometry of $\overline{\mathrm{M}}_{0,n}$ is well-behaved. I am trying to translate the S_n-invariant F-conjecture into purely combinatorial statements in terms of graphical algebras and finite graphs.

Problem 4 (S_n-invariant F-conjecture)**.** Show that the curve cone of $\overline{\mathrm{M}}_{0,n}/S_n$ is generated by F-curves.

Currently, it is known for $n \le 24$ due to Gibney ([Gib09]). The S_n-invariant F-conjecture for $\overline{\mathrm{M}}_{0,n}$ implies the F-conjecture for $\overline{\mathcal{M}}_g$ ([GKM02]).

3.2.2. *Modular interpretation of flips.* Except Result 8, there is no known moduli theoretic description of flips of $\overline{\mathrm{M}}_{0,n}$ in general. I am trying to understand a broader picture.

Problem 5. Find moduli theoretic interpretations of flips of $\overline{\mathrm{M}}_{0,n}$.

3.2.3. *Theoretical explanation of Result 7.* For any weight datum $\mathcal{A}$, there is a reduction morphism $\varphi_{\mathcal{A}}$: $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$. For a stable curve $(C, x_1, \cdots, x_n)$, its image $\varphi_{\mathcal{A}}(C)$ is given by the log canonical model

$$C(\omega_C + \sum_{i=1}^{n} a_i x_i) := \mathrm{Proj} \bigoplus_{k \ge 0} \mathrm{H}^0(C, k(\omega_C + \sum_{i=1}^{n} a_i x_i)). \tag{3}$$

Result 7 means the *same weight datum* determines the log canonical model of parameterized curves and that of the parameter space itself. So far, there is no known theoretical explanation of Result 7. I want to figure out the reason of this phenomenon and find some more supporting examples.

Problem 6. (1) Find the theoretical reason of the similarity between (2) and (3).
(2) Generalize Result 7 to moduli spaces of embedded genus-zero curves and Fulton-MacPherson spaces of configurations of points.

4. Conformal blocks and their connection with birational geometry ([GJMS13, MY15])

Recent results on the birational geometry of $\overline{\mathrm{M}}_{0,n}$ have focused attention on vector bundles of conformal blocks ([Fak12, GG12, Gia13, AGS14, BGM15]). Each such vector bundle $\mathbb{V}(\mathfrak{g}, \ell, \underline{\lambda})$ depends on 1) a simple Lie algebra $\mathfrak{g}$, 2) a nonnegative integer ℓ, and 3) an n-tuple $\underline{\lambda} = (\lambda_1, \cdots, \lambda_n)$ of dominant integral weights of $\mathfrak{g}$ of length $\le \ell$, and can be constructed using the representation theory of affine Lie algebras. We call its first Chern class a *conformal block divisor*. It induces a morphism from $\overline{\mathrm{M}}_{0,n}$ to a projective variety. Indeed, many birational models of $\overline{\mathrm{M}}_{0,n}$ are able to be described as the images of such morphisms associated to conformal block divisors.

4.1. **Past research - birational geometry of moduli of curves and bundles via conformal blocks.**

4.1.1. *Birational models of $\overline{\mathrm{M}}_{0,n}$ come from conformal blocks divisors.* With a research group consisting of Gibney, Jensen, and Swinarski, I have studied $\mathfrak{sl}_2$-conformal block divisors on $\overline{\mathrm{M}}_{0,n}$.

Results 10 ([GJMS13])**.** We give a complete description of birational models in the case of $\mathfrak{sl}_2$, and symmetric weight data $(\omega_1, \cdots, \omega_1)$. All of the birational models are obtained by GIT compactifications in Section 2.1.1.

4.1.2. *Conformal blocks and birational geometry of moduli spaces of parabolic bundles.* Among algebraic geometers working on moduli spaces of (parabolic) principal G-bundles, conformal blocks are also known as generalized theta functions. They are natural divisors on the moduli space of bundles. With Yoo, by studying the combinatorics of $\mathfrak{sl}_2$-conformal blocks, we obtain the following result.

Results 11 ([MY15])**.** We complete Mori's program of the moduli space of rank 2, degree 0 parabolic bundles on $\mathbb{P}^1$. (For the three steps of Mori's program, see Section 3.) The cone of divisors is an $(n+1)$-dimensional polyhedral cone generated by 2^{n-1} level one conformal blocks. For any divisor D, the associated model is also a moduli space of parabolic bundles with some degree and parabolic points.

4.2. **Current and future research.**

4.2.1. *Generalization toward higher rank parabolic bundles.* We expect that we can obtain a similar result for moduli space of higher rank parabolic bundles.

Problem 7. Generalize the result in Section 4.1.2 to moduli spaces of higher rank parabolic bundles.

In [MY15], two key ingredients are 1) a concrete description of $\mathfrak{sl}_r$-conformal blocks and 2) an explicit study of elementary GIT quotients $(\mathbb{P}^1)^n /\!/_L \mathrm{SL}_2$ and linear systems on them. I expect that the right generalization of $(\mathbb{P}^1)^n$ in this context is a product of flag varieties.

4.2.2. *Finite dimensional description of conformal block bundles.* Most of the recent studies about conformal block divisors are focused on numerical properties of them. In spite of many beautiful results, there is no satisfactory geometric explanation of them so far. One main reason of this shortage is that there is no geometric definition of conformal block bundles. On the locus of singular curves, the only known construction of conformal blocks is to use the representation theory of infinite dimensional affine Lie algebras.

Problem 8. Find a definition of conformal blocks vector *bundles* on $\overline{\mathrm{M}}_{0,n}$ in terms of finite dimensional algebraic geometry.

As a first step, I am studying the $\mathfrak{sl}_2$ case. In [MY15], we showed that for a fixed pointed smooth curve $(\mathbb{P}^1, x_1, \cdots, x_n)$, the moduli space of rank 2 parabolic bundles is an explicit modification of $(\mathbb{P}^1)^n /\!/_L \mathrm{SL}_2$. The moduli space $\overline{\mathrm{M}}_{0,n}$ is also another explicit modification of $(\mathbb{P}^1)^n /\!/_L \mathrm{SL}_2$ (Result 2). We expect that we can construct a universal moduli space $\mathcal{M}$ of all parabolic bundles on arbitrary curves in $\overline{\mathrm{M}}_{0,n}$ as an $\mathrm{SL}_2 \times \mathrm{SL}_2$-quotient of an explicit modification of $(\mathbb{P}^1)^n \times (\mathbb{P}^1)^n$, by combining Results 2 and 11. Then the conformal block vector bundles will be able to be constructed by using line bundles on $\mathcal{M}$.

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