Research Statement

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I am an algebraic geometer with broad interests. My main research area is the geometry, topology and combinatorics of *moduli spaces*. A moduli space is a family of mathematical objects of one particular type where in addition, the family enjoys a desired geometric structure. One of the simplest examples of a moduli space is the Grassmannian, which is the moduli space of sub-vector spaces of a fixed vector space.

I have been focused on the study of explicit geometric structure of many concrete examples of moduli spaces, using the framework of *birational geometry* and *geometric invariant theory (GIT)*. Examples of moduli spaces that I am interested in include moduli spaces of abstract and embedded varieties, vector bundles, and sheaves.

I have organized my past research into three themes (1) Topology of moduli spaces, (2) Mori's program of moduli spaces, and (3) Compactifications of moduli spaces. I briefly discuss these projects in Sections 1, 2, and 3 respectively, and present my future research plan in Section 4.

1. Topology of moduli spaces ([4, 5, 6, 13, 14, 20])

A promising strategy, from the perspective of birational geometry, of the study of the geometric structure of a given variety X is as follows: 1) construct a new variety X' (a so-called *birational model*), which shares an open dense subset with X but has simpler geometric properties, 2) study the structure of X', and 3) measure the difference of X and X'. Even for a highly nontrivial variety, by applying this strategy several times, sometimes we may reach a very simple variety.

When X is a moduli space, often a birational model is also a moduli space parametrizing a slightly different collection of objects. This property enables us to keep track of the difference of two models, while for general higher dimensional varieties this is an extremely difficult task.

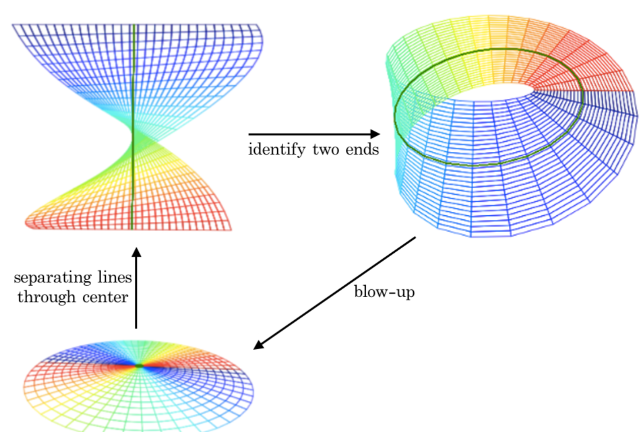

FIGURE 1. A circular disk is a birational model of the Möbius strip, because it is obtained by replacing a circle in the Möbius strip by a point.

In some other cases, such a simpler birational model can be obtained by taking an algebraic quotient space (or *GIT quotient*) of an elementary variety. In this case its topological structure is able to be understood in-depth, and the structure is described in a combinatorial manner. Thus one can see and use interesting interactions among algebra, combinatorics, and geometry.

Here is a typical example showing the general idea of the strategy, in the case of the moduli space of curves of genus zero. One way to define a genus zero curve C in a projective space $\mathbb{P}^r$ is to consider it as the image of a map $f = (f_0 : f_1 : \cdots : f_r) : \mathbb{P}^1 \to \mathbb{P}^r$. So a general $(r+1)$-tuple of homogeneous degree d polynomials with two variables defines a degree d genus zero curve $C = f(\mathbb{P}^1)$, and composing with a coordinate change of $\mathbb{P}^1$ does not affect $C \subset \mathbb{P}^r$. Thus roughly, we have a correspondence

$$C \subset \mathbb{P}^r \longleftrightarrow (f_0 : f_1 : \cdots : f_r)/_\sim$$

and the SL_2-GIT quotient of the space of $(r+1)$-tuples of degree d homogeneous polynomials $\mathbb{P}(\mathrm{Sym}^d\mathbb{C}^2 \otimes \mathbb{C}^{r+1})//\mathrm{SL}_2$ is a birational model of the space of genus zero curves in $\mathbb{P}^r$.

With my collaborators, in [4, 5, 6, 13, 14, 20], we described the difference between various moduli spaces and elementary GIT quotients. In some cases the moduli spaces are obtained from GIT quotients by several explicit algebro-geometric surgeries (so-called *blow-ups/downs*, [13, 14, 20]). In some other cases, the relation between birational models can be described in terms of theoretical gadgets, for instance Fourier-Mukai transform and Kirwan's partial desingularization ([4, 5, 6]). As a consequence of these descriptions, we obtain several topological invariants such as the Poincaré polynomial and the cohomology ring.

2. Mori's program for moduli spaces ([6, 21, 22, 23, 24, 26, 27, 28])

When studying the birational geometry of a given variety X, one may aim to classify all birational models of X - an ambitious goal. If X satisfies a certain technical condition (being a Mori dream space ([12])) and if we restrict ourselves to birational models which are equivalent or less complicated than X, then at least theoretically a complete classification is possible. *Mori's program* provides such a theoretical framework to the classification. It consists of three steps:

(1) Study the space of all numerical classes of divisors (codimension one subvarieties) of X;
(2) For a given divisor D of X, compute the associated model

$$X(D) := \mathrm{Proj} \bigoplus_{k \geq 0} \mathrm{H}^0(X, \mathcal{O}(kD)) \tag{1}$$

which is a projective variety;
(3) Study the difference of X and $X(D)$.

Therefore one may think that the space of divisors is an atlas of all birational models of X.

One can apply this framework to moduli spaces, and it has been one of the most active directions in the study of moduli spaces in the last decade (for instance [3, 11]). For moduli spaces, one may add an extra step to the program:

(4) Give a geometric or moduli theoretic interpretation of $X(D)$.

However, even in simple cases, the completion of Mori's program is very difficult and despite of many results, there are few completed examples. I have carried out several projects for moduli spaces of curves and sheaves.

With Chung and Yoo, we completed Mori's programs for the moduli space of conics in Grassmannian $\mathrm{Gr}(2, n)$ and for the moduli space of rank 2 parabolic bundles on $\mathbb{P}^1$ ([6, 28]). These two results are rare examples of a completed Mori's program. For $\overline{\mathrm{M}}_{g,n}$, the moduli space of pointed stable curves, I have described associated birational models to a certain range of divisors ([21, 22, 23, 24]). Many of well-known birational models of $\overline{\mathrm{M}}_{g,n}$ appeared in this program, and several more moduli spaces were also obtained.

The first step toward Mori's program for a variety X is to understand the space of divisors, or dually, the space of curves, which has a convex cone structure. In the case of $\overline{\mathrm{M}}_{0,n}$, the famous *F-conjecture* gives an explicit set of generators of the cone of curves, but the conjecture is open for $n \geq 8$. In joint work with Swinarski, we recognized that the problem studying the curve cone structure can be investigated via computational mathematics using symbolic calculation ([26, 27]) in several ways. In particular, by using classical invariant theory, we found a purely computational statement which implies (and is strictly stronger than) the S_n-invariant version of the F-conjecture, and proved it for $n \leq 19$.

3. Compactifications of moduli spaces ([8, 9, 25])

Many moduli spaces, such as the moduli space of smooth curves, are not compact. By compactifying them, we can use powerful tools from algebraic geometry, like intersection theory, to study them. One interesting feature in moduli theory is that it is often possible to construct many compactifications having different moduli theoretic interpretations. Often there is a "best" or preferred compactification (the moduli space $\overline{\mathrm{M}}_{g,n}$ is such a compactification of the moduli space of smooth pointed curves), but understanding the relationships between different compactifications helps us to obtain geometric information of compactifications. My research involves constructing and classifying different compactifications of the moduli space of pointed smooth genus zero curves $\mathrm{M}_{0,n}$ to learn about its geometric/topological properties.

In modern algebraic geometry, two standard ways to construct a moduli space of abstract curves are using GIT and applying stack theory. In [8, 9], by applying GIT, we obtained a large family of modular compactifications of $\mathrm{M}_{0,n}$. All previously known projective alternative modular compactifications of $\mathrm{M}_{0,n}$ are realized in this way, and there are many new examples.

On the other hand, in [31], via stack theory, Smyth constructed a family of alternative compactifications $\overline{\mathrm{M}}_{0,n}(Z)$ indexed by *extremal assignments* Z, which are combinatorial data described in terms of labeled graphs. With my undergraduate students, in [25], we investigated combinatorics of extremal assignments and translated the result in terms of birational geometry of $\overline{\mathrm{M}}_{0,n}$.

4. Current and future research

In this section I describe my research plan.

4.1. **Topology of moduli spaces of embedded curves and sheaves.** Since the moduli space of stable maps was introduced by Kontsevich ([18]) as a standard compactification of the moduli space of embedded curves in a projective variety, several alternative compactifications have been constructed ([7, 15, 16, 19]). They have played important roles in the virtual curve counting theory, but their global geometric properties are not well-known. With Chung and Yoo, we are investigating the geometry of various moduli spaces of genus zero curves as well as that of moduli spaces of bundles and sheaves.

Problem 1. Compute topological invariants of various moduli spaces of genus zero curves, bundles, and one-dimensional sheaves. Run Mori's program for these moduli spaces.

4.2. S_n**-invariant F-conjecture.** The S_n-invariant F-conjecture is one of the biggest open problems in the birational geometry of $\overline{\mathrm{M}}_g$, the moduli space of stable curves, since it was shown that assuming the S_n-invariant F-conjecture for $\overline{\mathrm{M}}_{0,n}$ we obtain the cone of curves of $\overline{\mathrm{M}}_g$ ([10]). With Swinarski we are working on this problem by refining a computational approach that we developed in [27].

Problem 2 (S_n-invariant F-conjecture)**.** Show that the curve cone of $\overline{\mathrm{M}}_{0,n}/S_n$ is generated by F-curves.

4.3. **Computational aspects of the GIT quotient.** In the last century, a standard approach to construct a moduli space was GIT. Whenever one studies the GIT quotient of an algebraic variety X equipped with a group G-action, the first inevitable step is the computation of *semi-stable locus* X^{ss}, or equivalently, a finite list of one-parameter subgroups of G, which is usually very delicate combinatorial computation. Algebraic geometers have spent a tremendous amount of time, energy, and journal pages to perform the same kind of numerical/combinatorial computations with bare hands. With Gallardo, Martinez-Garcia, and Swinarski, we are studying an effective algorithm to compute the list of one-parameter subgroups, and have a plan to make a computer program which is open to public. It will let people skip the tedious stability computation and enjoy delightful investigation of geometry of quotient spaces.

Problem 3. Find an effective algorithm to find the semistable locus and implement the algorithm into a computer program.

The above result immediately can be applied to the study of many moduli problems. With Swinarski, by using a version of the algorithm, we are investigating the geometry of Mukai's birational models ([29, 30]) for moduli spaces of low genus curves.

Problem 4. Study the geometric/moduli theoretic properties of Mukai models.

4.4. **Moduli spaces of higher dimensional varieties.** In many cases, moduli spaces of curves have good geometric properties such as smoothness and irreducibility. However, for higher dimensional varieties, their compactified moduli spaces have very complicated geometric structure. Even the definition of the moduli space has many subtle technical difficulties, so although the rigorous definition was suggested in late 80's ([17]), the construction of the moduli space was completed very recently.

It seems that the study of geometric properties of general cases is out of reach. But in special cases of 1) varieties equipped with group actions ([1]); 2) varieties with combinatorial structures such as hyperplane arrangements ([2]), the study of their geometric properties is approachable. One of my long-term research plans is to study moduli spaces of higher dimensional varieties.

Problem 5. Study the geometric/topological properties of moduli spaces of higher dimensional varieties with combinatorial structure.

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