# DEFINITIONS AND EASY EXAMPLES

JOE TENINI, HAN-BOM MOON

The basic assumption of this paper is

- All varieties are quasi-projective, normal and over algebraically closed field of characteristic 0.
- 'Normal' means singularities are in codimension greater than 1 and a pole of regular function has codimension 1.

**Definition 0.1.** Let $X$ be a variety. $X$ has **canonical singularities** if

(1) $rK_X$ is Cartier for some integer $r > 0$;
(2) If $f : Y \to X$ is a resolution,

$$rK_Y = f^*(rK_X) + \sum a_i E_i, \quad \text{with } a_i \geq 0.$$

Why do we care about canonical singularities? The canonical model of a variety of general type will have canonical singularities. The canonical model is $\mathrm{Proj}\ R(X)$, where

$$R(X) := \sum_{r=0}^{\infty} H^0(X, \omega_X^r).$$

In recent paper [BCHM10], it was proved that for a variety of general type with mild singularities, $R(X)$ is always finitely generated so $\mathrm{Proj}\ R(X)$ is a projective variety.

**Definition 0.2.** For a singular point $p \in X$, the smallest $r$ for which $rK_X$ is Cartier in a neighborhood of $X$ is called the **index of singularity**.

**Definition 0.3.** Set $\Delta = \frac{1}{r} \sum a_i E_i$. Then we can write formally

$$K_Y = f^*(K_X) + \Delta.$$

$\Delta$ is called the **discrepancy** of $f$.

Let $V$ be a smooth variety of dimension $n$. $\omega_V = \mathcal{O}_V(K_V) = \Omega_V^n$. For a point $p \in V$, take local coordinate $x_1, \cdots, x_n$. Then $\omega_V$ has a local basis $dx_1 \wedge \cdots \wedge dx_n$. So we say $s \in H^0(V, \omega_V^n)$ is called a **global** $n$**-differentials**.

Here are several important properties of canonical divisors.

- If $\omega_V$ is (very) ample, we have a canonical embedding $V \hookrightarrow \mathbb{P}^N$.

*Date*: August 24, 2012.

- Serre duality.
  If $X$ is Cohen-Macaulay, there is a sheaf
  $$H^i(X, F) = H^{n-i}(X, F^* \otimes \omega^o)^*$$
  when $F$ is locally free. If $X$ is projective and nonsingular, $\omega^0 = \omega_X$.
- Birational nature of $\omega_X$.
  If $V$ and $W$ are birational non-singular varieties, then
  $$H^0(V, \omega_V^n) \cong H^0(W, \omega_W^n).$$

**Example 0.4.** Consider a quadratic cone $V(xz-y^2) \subset \mathbb{A}^3$. The origin is the unique singular point. Take the blow-up $Y$ of the origin. Then the exceptional divisor $E$ is a quadric in $\mathbb{P}^2$. The canonical divisor formula is

$$rK_Y = f^*(rK_X) + aE \tag{1}$$

for some $r$ and $a$.

$E$ is isomorphic to $\mathbb{P}^1$ because it is a quadric on $\mathbb{P}^2$. The self intersection $E^2$, which is the degree of the normal bundle in $Y$ is isomorphic to $\mathcal{O}(-2)$. Indeed, the normal bundle is the restriction of $\mathcal{O}_{\mathbb{P}^2}(-1)$ to $\mathbb{P}^2$. because the normal bundle of the exceptional divisor $\mathbb{P}^2$ in $bl_0\mathbb{A}^3$ is $\mathcal{O}_{\mathbb{P}^2}(-1)$ and the blow-up of $X$ is the restriction of $bl_0\mathbb{A}^3$. Since $E$ is a degree two curve in $\mathbb{P}^2$, the restriction of $\mathcal{O}_{\mathbb{P}^2}(-1)$ is degree $-2$ and it is $\mathcal{O}(-2)$.

Now by adjunction formula,
$$-2 = \deg K_E = (K_Y + E) \cdot E = K_Y \cdot E + E^2 = K_Y \cdot E - 2.$$
Thus $K_Y \cdot E = 0$. From the canonical divisor formula (1), after taking intersection with $E$, we obtain
$$0 = rK_Y \cdot E = (f^*(rK_X) + aE) \cdot E = f^*(rK_X) \cdot E - 2a = -2a$$
and $a = 0$. Therefore $X$ has a canonical singularities.

**Remark 0.5.** By a similar computation, it can be shown that a degree $d$ cone $X$ over a smooth curve has a non-canonical singularity if $d \geq 3$.

## References

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010. 1