DEFINITIONS AND EASY EXAMPLES II

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We want to define ω_X and $\mathcal{O}(mK_X)$ for a singular variety X.

Let X be an n-dimensional variety. Consider $\Omega^n_{k(X)/k}$. $\Omega^n_{k(X)/k}$ is a 1 dimensional vector space over $k(X)$ and has basis $df_1 \wedge df_2 \wedge \cdots \wedge df_n$ if $f_1, \cdots, f_n$ is a separable transcendental basis for $k(X)/k$.

Let X^0 be the nonsingular locus of X and let $p \in X^0$. We can choose $\{x_1, \cdots, x_n\}$, local coordinates about p. For any $s \in \Omega^n_{k(X)/k}$ we can write

$$s = f dx_1 \wedge \cdots \wedge dx_n, \quad f \in k(X).$$

Definition 0.1. s is **regular** at p if f is regular at p.

Considering s as a section of differentials on X^0, s is regular at p if s is regular on some open U^0 containing p and $U^0 \subset X^0$. Now for any point $p \in X$, s is regular at p if there exists an open subset $U \subset X$ containing p such that s is regular on $U^0 = U \cap X^0$.

Define

$$\omega_X(U) = \{s \in \Omega^n_{k(X)/k} | s \text{ is regular on } U^0 = U \cap X^0\} = j_*(\Omega^n_{X^0})$$

where $j : X^0 \hookrightarrow X$.

Another description of ω_X is the double dual of Ω^n_X. One possible geometric reason is this: If X is singular, then Ω^n_X is not invertible and may have torsion at singular points. If we take double dual, then all torsions are removed. Moreover, if we take the double dual, then we can get a **reflexive sheaf**. It is known that a reflexive sheaf of rank 1 is corresponded to a Weil divisor. For more details, see for example [CLS11, Section 8.0].

0.1. The sheaves $\mathcal{O}_X(mK_X)$.

Definition 0.2.

$$\mathcal{O}_X(mK_X)(U) = \{s \in (\Omega^n_{k(X)/k})^m | s \text{ is regular on } U^0\} = j_*((\Omega_{k(X)/k})^m)(U).$$

Recall the definition of canonical singularities.

Definition 0.3. Let X be a variety. X has **canonical singularities** if

(1) rK_X is Cartier for some integer $r > 0$;

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(2) If $f : Y \to X$ is a resolution,

$$rK_Y = f^*(rK_X) + \sum a_i E_i, \quad \text{with } a_i \geq 0.$$

Example 0.4. Let $X = \{f = 0\} \subset \mathbb{A}^{n+1}$ be a hypersurface and $p \in X$ is a normal hypersurface singularity. Consider

$$s = \frac{dx_1 \wedge \cdots \wedge dx_n}{\partial f / \partial x_0} \in \Omega^n_{k(X)}$$

where $x_0, \cdots, x_n$ are local coordinates of $\mathbb{A}^{n+1}$. For any point $q \in X$ such that $\frac{\partial f}{\partial x_0}(q) \neq 0$, X is a manifold with local coordinates $x_1, \cdots, x_n$ and $s = (\text{unit}) dx_1 \wedge \cdots \wedge dx_n$ is a local basis on Ω^n_X.

Now since $f = 0$ on X,

$$\frac{\partial f}{\partial x_0} dx_0 + \cdots + \frac{\partial f}{\partial x_n} dx_n = 0.$$

So if $\frac{\partial f}{\partial x_1} \neq 0$, then

$$\begin{aligned} s &= \frac{dx_1 \wedge \cdots \wedge dx_n}{\partial f / \partial x_0} = \frac{\frac{\frac{\partial f}{\partial x_0} dx_0 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n}}{-\partial f / \partial x_1} \wedge dx_2 \wedge \cdots \wedge dx_n}{\partial f / \partial x_0} \\ &= \frac{\frac{\partial f}{\partial x_0} dx_0 \wedge dx_2 \wedge \cdots \wedge dx_n}{-\partial f / \partial x_1 \partial f / \partial x_0} = -\frac{dx_0 \wedge dx_2 \wedge \cdots \wedge dx_n}{\partial f / \partial x_1}. \end{aligned}$$

So we can extend the local regular section s to the open subset such that $\partial f / \partial x_1 \neq 0$. Since on the smooth part of X, at least one of $\partial f / \partial x_i$ is nonzero, s gives a section on X^0. By the definition of ω_X, s is a global section of ω_X. Since s is a basis of ω_X for every point $p \in X$, $\omega_X \cong \mathcal{O}_X$.

Example 0.5. Let $X = \mathbb{A}^2 / \mu_3$, where μ_3 is the group of cubic roots of 1. μ_3 acts by

$$e : (x, y) \mapsto (ex, ey).$$

Let X be the spec of ring of invariants and let $\pi : \mathbb{A}^2 \to X$ be the quotient map. Then $X = \operatorname{Spec} k[x^3, x^2 y, x y^2, y^3]$ which is isomorphic to

$$\operatorname{Spec} k[u_0, u_1, u_2, u_3] / (u_0 u_2 - u_1^2, u_1 u_3 - u_2^2, u_0 u_3 - u_1 u_2).$$

Under the action μ_3, we notice that

$$dx \wedge dy = e^2 (dx \wedge dy).$$

Thus $(dx \wedge dy)^3 \mapsto (dx \wedge dy)^3$. Therefore, $(dx \wedge dy)^3$ is a pull-back of a section on X.

Set

$$s = \frac{(du_0 \wedge du_1)^3}{u_0^4} \in (\Omega^2_{k(X)})^3.$$

Then

$$\pi^* s = \frac{(3x^2 dx \wedge (2xy dx + x^2 dy))^3}{x^{12}} = 27 (dx \wedge dy)^3.$$

We can check that

$$s = \frac{(du_0 \wedge du_1)^3}{u_0^4} = \frac{(du_2 \wedge du_3)^3}{u_3^4}.$$

So s is a basis of $\mathcal{O}_X(3K_X)$ and $3K_X$ is Cartier.

References

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011. 1