

Grad AG seminar. (Nov. 2. Moon).

Toric method for hyperquotient singularities N.

Recall.  $Y = \{f=0\} \subset \mathbb{A}^{n+1}$ .

$$\begin{aligned} \Sigma \cdot (x_0, \dots, x_n) &\quad M_r \cong \mathbb{A}^{n+1}, \quad X := Y/M_r. \\ &= (\varepsilon^{a_0} x_0, \dots, \varepsilon^{a_n} x_n) \quad \cong Y. \end{aligned}$$

Ex)  $\odot r=1 \Rightarrow$  hypersurface singularities.

$\odot f=x_0 \Rightarrow$  cyclic quotient singularities.

Q. When is  $X$  terminal (canonical)?

Set up:  $0 \in Y$  is singular point.

$$\bar{N} \cong \mathbb{Z}^{n+1}, \quad \bar{M} = \text{Hom}_{\mathbb{Z}}(\bar{N}, \mathbb{Z}).$$

$$N := \bar{N} + \mathbb{Z} \frac{1}{r}(a_0, \dots, a_n), \quad M = \text{Hom}_{\mathbb{Z}}(\bar{M}, \mathbb{Z})$$

$\sigma$ : first quadrant.

$$\alpha \in N, \Rightarrow \alpha(x^m) := \langle \alpha, m \rangle, \quad (m \in M).$$

Thm.  $X$  is terminal (canonical) at 0.

$$\Rightarrow \alpha(x_0, \dots, x_n) \geq \alpha(f) + 1, \quad \forall \alpha \in N \cap \sigma^{\text{int}}, \text{ primitive.}$$

$$\text{where } \alpha(f) = \min\{\alpha(x^m) \mid x^m \in f\}.$$

Ex)  $r=1$  (hypersurface sing)

$$\begin{aligned} &Y \text{ is terminal (canonical)} \\ &\alpha = (1, \dots, 1) \\ &\Rightarrow n+1 \geq \min\{\alpha(x^m) \mid x^m \in f\} + 1 \Rightarrow \text{mult}_0 f \leq n. \end{aligned}$$

Ex)  $X = \mathbb{A}^2/M_r$ , quot. sing of dim 2.

of type  $\frac{1}{r}(1, k)$ , ( $1 \leq k \leq r-1$ ).

$X$  is terminal (canonical)

$$\Rightarrow \alpha(x_1 x_2) \geq 1, \quad (f = x_0).$$

$$\alpha = \frac{1}{r}(1, k) \Rightarrow \alpha(x_1 x_2) = \frac{1+k}{r} \leq 1, \quad \text{and } = 1 \text{ iff } k = r-1.$$

$\mathbb{A}^2/M_r, \frac{1}{r}(1, k)$  is not terminal & canonical iff  $\frac{1}{r}(1, r-1)$

Ex) Du Val's classification of

canonical hypersurface sing of dim 2.

$$\text{mult}_0 f \leq 2 \Rightarrow f = f_2 + \dots, \quad f_2 = \begin{cases} x^2 + y^2 + z^2 & \dots A_1 \\ xy \\ xz \end{cases}$$

$$f_2 = xy \Rightarrow \exists z^{n+1} \text{ term. } \xrightarrow{\substack{\text{analytic} \\ \text{locally}}} f = xy + z^{n+1} \dots a_n.$$

$$f_2 = xc^2 \Rightarrow \alpha = (2, 1, 1) \Rightarrow 4 = \alpha(xy) \geq \alpha(f) + 1. \\ \Rightarrow \exists \text{ one of } y^3, y^2z, yz^2, z^3. \\ \xrightarrow{\substack{\text{anal. locally}}} \exists y^3 + z^3, \text{ or } y^2z \text{ or } yz^2.$$

$$x^2 + y^3 + z^3 \dots D_4.$$

$$x^2 + y^2z + z^{n+1} \dots D_n$$

$$f = x^2 + y^3 + \dots \Rightarrow \alpha = (3, 2, 1) \Rightarrow 6 = \alpha(xy) \geq \alpha(f) + 1. \\ \Rightarrow \exists \text{ one of } \begin{matrix} z^4 & yz^3 & z^5 \\ \uparrow & \uparrow & \uparrow \\ \alpha=4 & \alpha=5 & \alpha=5 \end{matrix}$$

$$x^2 + y^3 + z^4 \dots E_6.$$

$$x^2 + y^3 + yz^3 \dots E_7.$$

$$x^2 + y^2 + z^5 \dots E_8$$

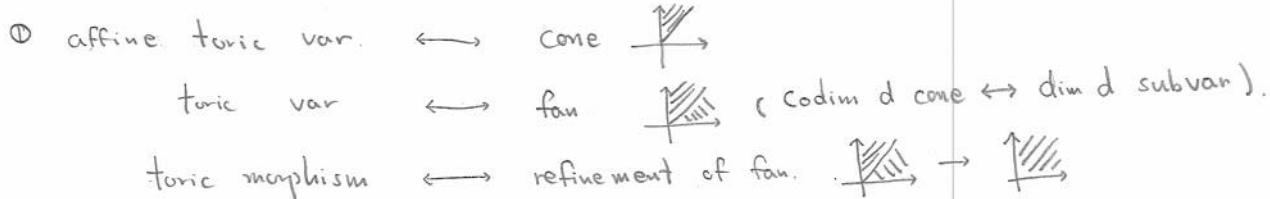
Ex) quotient sing. in general  $X = A^n / \langle u_1, \dots, u_r \rangle$

$$1 \leq k \leq r-1, \alpha_k := \frac{1}{r}(\overline{k}a_1, \dots, \overline{k}a_n) \in \mathbb{N}. \\ \uparrow \\ 0 \leq \overline{k}a_i \leq r-1, \overline{k}a_i \equiv k a_i \pmod{r}$$

$\alpha \in X$  is terminal (canonical)

$$\Leftrightarrow \alpha_k(x_1, \dots, x_n) = \frac{1}{r} \sum_i \overline{k}a_i > 1, \forall k. \\ \uparrow \text{ Reid-Tai Criterion.}$$

preparation for proof. (Toric blow-up)



Ex) toric blow-up

Cone gen. by  $e_1, \dots, e_n$ . take barycentric subdivision.



weighted blow-up: take barycentric subdivision

at  $\sum a_i e_i$ .



except. div  $\Gamma \cong \mathbb{P}(a_1, \dots, a_n)$ .

② For any toric var.  $A$ ,  $K_A = -D_A$ . ( $K_A + D_A = G_A$ ).

$D_A$ : sum of all torus-inv. codim 1. subvar.

(comes from  $0 \rightarrow \Omega_{\mathbb{C}[G^\vee]} \rightarrow M \otimes \mathbb{C}[G^\vee] \rightarrow \bigoplus_i \Gamma(G_{D_{A,i}}) \rightarrow 0$ )

$$d(x^m) \mapsto m \otimes x^m \mapsto \langle m, p_{A,i} \rangle \overline{x^m}$$

Ex)  $A = \text{Spec}(\mathbb{C}[G^\vee]) \cong \mathbb{A}^{n+1}$ .

$$\frac{dx_0 \wedge \dots \wedge dx_n}{x_0 \cdots x_n} \text{ is unique (upto scalar)}$$

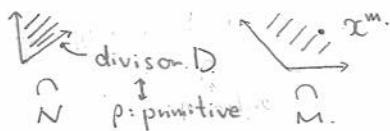
↑ vanishing order of  $x^m$   
along  $D_{A,i}$

section of  $K_A + D_A \Rightarrow K_A + D_A = G_A$

In particular, for a toric blow-up  $B \xrightarrow{\pi} A$ ,

$$K_B + D_B = \pi^*(K_A + D_A)$$

③ Affine toric var:



Vanishing order of  $x^m$  along  $D = \langle \rho, m \rangle$ .

Proof.  $\pi: B \xrightarrow{\alpha=(a_0, \dots, a_n)} A = \mathbb{A}^{n+1}$  weighted blow-up.  $\Gamma$ : except. div.

$X'$ : proper transform of  $X \subset A$ .

$$X' = \pi^* X - (\text{mult}_0 X) \Gamma = \pi^* X - \alpha(f) \Gamma$$

$$\pi^*(D_A) = D_B + (\alpha(x_0) + \dots + \alpha(x_n) - 1) \Gamma$$

$$= D_B + (\alpha(x_0 \cdots x_n) - 1) \Gamma$$

$$K_B = \pi^*(K_A) + \pi^*(D_A) - D_B = \pi^*(K_A) + (\alpha(x_0 \cdots x_n) - 1) \Gamma.$$

$$\begin{aligned} K_{X'} &= (K_B + X')|_{X'} = (\pi^* K_A + (\alpha(x_0 \cdots x_n) - 1) \Gamma + \pi^* X - \alpha(f) \Gamma)|_{X'} \\ &\stackrel{(*)}{=} (\pi^*(K_A + X) + (\alpha(x_0 \cdots x_n) - \alpha(f) - 1) \Gamma)|_{X'} \\ &= \pi^* K_X + (\alpha(x_0 \cdots x_n) - \alpha(f) - 1)(\Gamma \cap X'). \end{aligned}$$

If  $\alpha(x_0 \cdots x_n) \leq \alpha(f) + 1$ , we have a divisor

with non positive coefficient. ( $\Gamma \cap X'$ ).

$Y \xrightarrow{\varphi} X'$  any resol. of sing.

on  $Y$ , the pull-back of  $\Gamma \cap X'$  still has  
non positive coefficient.

$\Rightarrow$  Not terminal.

Recall.  $X$  has terminal sing.

$\Rightarrow Y \xrightarrow{\varphi} X$  res. sing

$$K_Y = \varphi^*(K_X) + \sum a_i E_i, a_i > 0.$$