TORIC METHODS FOR HYPERQUOTIENT SINGULARITIES II

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Recall the notation. Let μ_r denote the group of r-th roots of unity. For a choice of primitive root ϵ, we have an isomorphism $\mu_r \to \mathbb{Z}/r$ given by $\epsilon \mapsto 1$. We want to think of elements in $a \in \mathbb{Z}/r$ as characters of μ_r. For $a \in \mathbb{Z}/r$,

$$\begin{aligned} c_a : \mu &\to \mu \\ \epsilon &\mapsto \epsilon^a \end{aligned}$$

For a k-vector space V, the action of μ_r will break V into 1-dimensional eigenspaces. After finding them, we can say for standard basis b_i, $1 \le i \le n$,

$$\epsilon \cdot b_i = e^{a_i} b_i$$

for some $a_i \in \mathbb{Z}/r$.

Let's fix the notation for lattices. Let $\overline{M} = \mathbb{Z}^{n+1}$. Let $\overline{N} = \mathrm{Hom}(\overline{M}, \mathbb{Z})$ be the space of 1 parameter subgroups of $(\mathbb{C}^*)^{n+1}$.

We want to find a toric representation for $\mathbb{A}^{n+1}/\mu$. For the first quadrant $\sigma^* \subset \overline{M}$, $(\mathrm{Spec}\ k[\overline{M} \cap \sigma^*] = \mathbb{A}^{n+1})$, if we take the lattice $M \subset \overline{M}$ corresponding to the set of μ_r-invariant monomials, then

$$\mathrm{Spec}\ k[M \cap \sigma^*] \cong \mathbb{A}^{n+1}/\mu.$$

μ_r acts on $x^m := x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n}$ by $\epsilon \cdot x^m = \epsilon^{\alpha(m)} x^m$ where $\alpha(m) = \sum a_i m_i$. Thus μ_r acts on $\overline{M}$ by $\epsilon \cdot (m_0, \cdots, m_n) = (a_0 m_0, \cdots, a_n m_n)$. Then the μ_r-invariant sublattice M is given by

$$M = \{(m_0, \cdots, m_n) \in \overline{M} | \epsilon^{\alpha(m)} = 1\}.$$

Now $\epsilon^{\alpha(m)} = 1 \Leftrightarrow \alpha(m) \equiv 0 \mod r \Leftrightarrow r | \alpha(m) = \sum a_i m_i$. Therefore, the dual lattice N of M is given by

$$N = \overline{N} + \mathbb{Z}\frac{1}{r}(a_0, \cdots, a_n).$$

Indeed, this is a reason of the notation of cyclic quotient singularity $\frac{1}{r}(a_0, \cdots, a_n)$.

Example 0.1. If μ_3 acts on $\mathbb{A}^2$ by $\epsilon \cdot (x, y) = (\epsilon x, \epsilon^2 y)$, μ_3-invariant monomials are generated by x^3, xy, y^3. So M is generated by $(3,0), (1,1), (0,3)$ and N is generated by $(1,0), (0,1), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})$. It is easy to check that N is generated by $(1,0), (0,1)$ and $(\frac{1}{3}, \frac{2}{3})$, too. So $N = \overline{N} + \mathbb{Z}\frac{1}{3}(1,2)$.

Date: October 19, 2012.