TORIC METHODS FOR HYPERQUOTIENT SINGULARITIES III

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Recall the notation.

$Y \subset \mathbb{A}^{n+1}$ is a hypersurface $f = 0$ which is possibly singular. μ_r acts on $\mathbb{A}^{n+1}$ which preserves Y. For a generator $\epsilon \in \mu_r$,

$$\epsilon \cdot (x_0, \cdots, x_n) = (\epsilon^{a_0} x_0, \cdots, \epsilon^{a_n} x_n)$$

and $\epsilon \cdot f = \epsilon^e f$. $\frac{1}{r}(a_0, \cdots, a_n, e)$ is the type of the hyperquotient singularity. Set $X = Y/\mu_r \subset \mathbb{A}^{n+1}/\mu_r$.

Two special cases.

(1) $r = 1$ (no action): hypersurface singularities.
(2) Y smooth ($Y = \{x_0 = 0\}$): cyclic quotient, $X = \mathbb{A}^n/\mu_r$.

Question 0.1. When is $X = Y/\mu_r \subset \mathbb{A}^{n+1}/\mu_r$ terminal/canonical?

If Y is already not normal, then there is almost no hope about the normality of X. So assume that Y and μ_r-action are reasonable.

Let's fix the notation for lattices. Let $\overline{M} = \mathbb{Z}^{n+1}$ be the lattice of monomials. Let $\overline{N} = \mathrm{Hom}(\overline{M}, \mathbb{Z})$ be the space of 1 parameter subgroups of $(\mathbb{C}^*)^{n+1}$. Let $N = \overline{N} + \mathbb{Z}\frac{1}{r}(a_0, \cdots, a_r) \cong \mathbb{Z}^{n+1}$ and $M = \mathrm{Hom}(N, \mathbb{Z})$. Then $\overline{N} \subset N$ and $M \subset \overline{M}$.

Last time, we learned that

$$X \subset \mathbb{A}^{n+1}/\mu_r \cong \mathrm{Spec}\,(\mathbb{C}[M \cap \sigma^\vee])$$

Example 0.2. Consider a special case of (2), $X = \mathbb{A}^2/\mu_3$, an $\frac{1}{3}(1,2)$-singularity. $N = \overline{N} + \mathbb{Z}\frac{1}{3}(1,2)$ and generated by two elements $(1/3, 2/3), (0,1)$. $M = \{(m \in \overline{M} | m \cdot N \in \mathbb{Z}\} = \{(m_1, m_2) \in \overline{M} | m_1 + 2m_2 \equiv 0 \mod 3\}$. The lattice M is spanned by $(1,1), (0,3)$, and the semigroup $M \cap \sigma^\vee$ is generated by $(1,1), (0,3), (3,0)$.

$$\mathrm{Spec}\,\mathbb{C}[M \cap \sigma^\vee] = \mathrm{Spec}\,\mathbb{C}[x^3, y^3, xy] = \mathrm{Spec}\,\mathbb{C}[u,v,w]/ < uv - w^3 > .$$

Theorem 0.3. *A necessary condition for X to be terminal (canonical) is that*

$$\alpha(x_0 \cdots x_n) > (\geq)\ \alpha(f) + 1 \qquad (1)$$

for every primitive vector $\alpha \in N \cap \sigma$.

Let $\alpha \in N \cap \sigma$. $\alpha = (b_0, \cdots, b_n) \in N \cap \sigma$. Thank of α is a weighting on monomials.

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(1) $\alpha \in N$ means $\alpha \equiv \frac{1}{r}(ja_0, \cdots, ja_n) \mod \overline{N}$ for some j.
(2) $\alpha \in \sigma$ means $b_i \geq 0$.

We can define a weighting α by $\alpha(x_i) = b_i$ and $\alpha(x_0^{m_0} \cdots x_n^{x_n}) = \sum m_i b_i$. We say $x^m = x_0^{m_0} \cdots x_n^{m_n} \in f$ if x^m appears in f with non-zero coefficients. Define $\alpha(f) := \{\min x^m | x^m \in f\}$.

Example 0.4. $f = x + y + z$, $\alpha = (1, 1, 1)$. $\alpha(f) = 1$, $\alpha(xyz) = 3$.

But if $\alpha = (1, 0, 0)$. Then $\alpha(f) = 0, \alpha(xyz) = 1$. So it does not satisfy (1) in the theorem. Maybe we need to modify the statement as α is in the interior of $M \cap \sigma$?

Example 0.5. Consider $\mathbb{A}^2/\mu_r$, take $\alpha = \frac{1}{3}(1, 2) \in N \cap \sigma$. $\alpha(xy) = 1$ so $\mathbb{A}^2/\mu_3$ is not terminal. (In fact, it is canonical)

Example 0.6. Consider $\frac{1}{3}(1, 1)$ singularity. $N = \overline{N} + \mathbb{Z}\frac{1}{3}(1, 1)$. Take $\alpha = \frac{1}{3}(1, 1)$. $\alpha(xy) = \frac{2}{3}$. So X is not canonical. Indeed, $X = \mathrm{Spec}\ \mathbb{C}[x^3, x^2y, xy^2, y^3]$, so it is a cone over rational normal curve of degree 3. We know this singularity is not canonical.

Example 0.7. Consider $X = Y \subset \mathbb{A}^{n+1}$ a hypersurface singularity. Y is terminal (canonical) $\Rightarrow$

$$\alpha(x_0 \cdots x_n) > (\geq)\ \alpha(f) + 1 \overset{(\alpha=(1,\cdots,1))}{\Rightarrow} n + 1 > (\geq)\ \mathrm{mult}_p(f) + 1 \Rightarrow \mathrm{mult}_p f < n.$$

What happens if $n = 1$? Need to understand it.