TORIC METHODS FOR HYPERQUOTIENT SINGULARITIES

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Hyperquotient singularities.

Let μ_r denote the group of r-th roots of unity. For a choice of primitive root ϵ, we have an isomorphism $\mu_r \to \mathbb{Z}/r$ given by $\epsilon \mapsto 1$. We want to think of elements in $a \in \mathbb{Z}/r$ as characters of μ_r. For $a \in \mathbb{Z}/r$,

$$\begin{aligned} c_a : \mu &\to \mu \\ \epsilon &\mapsto \epsilon^a \end{aligned}$$

For a k-vector space V, the action of μ_r will break V into 1-dimensional eigenspaces. After finding them, we can say for standard basis b_i, $1 \leq i \leq n$,

$$\epsilon \cdot b_i = e^{a_i} b_i$$

for some $a_i \in \mathbb{Z}/r$.

Let $Y : \{f = 0\} \subset \mathbb{A}^{n+1}$ be a hypersurface and $Q \in Y$ be a hypersurface singularity. And assume Y has a μ_r-action. Let $P \in X = Y/\mu_r$ be the image of Q.

Question 0.1. When is P terminal? canonical?

We want to answer in terms of μ_r-action and the Newton polyhedron of f.

Definition 0.2. Let $f = \sum a_u x^u$ (sum taken over finitely many points in $\mathbb{Z}^{n+1}$).

$$\mathrm{Newt}(f) = \mathrm{Conv}\{u \in \mathbb{Z}^{n+1} | a_u \neq 0\}$$

Example 0.3.

$$f(x, y, z) = axy + bx^5 + cy^2$$
$$\mathrm{Newt}(f) = \mathrm{Conv}\{(1, 1), (5, 0), (0, 2)\}$$

The convex hull in this situation is not usual one. The correct picture is the following. Pick given integral points on $\mathbb{R}^{n+1}$ and plot all exponents of monomials with nonzero coefficient. At each integral point p, we can draw a first quadrant with origin at p. $\mathrm{Newt}(f)$ is the convex hull of the union of such first quadrants.

Any cyclic quotient singularity is of the form $X = \mathbb{A}^n/\mu_r$. μ_r-action breaks up k^n into 1-dimensional eigenspaces

$$\epsilon \in \mu_r, \quad \epsilon : (X_1, \cdots, X_n) \mapsto (\epsilon^{a_i} X_1, \cdots, \epsilon^{a_n} X_n), a_i \in \mathbb{Z}/r$$

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Singularity is determined by this information.

Definition 0.4. X as above. Then the type of X is denoted by

$$\frac{1}{r}(a_1, \cdots, a_n).$$

Back to $Y : \{f = 0\}$. Choose analytic coordinates, then we can extend μ_r-action to $\mathbb{A}^{n+1}$ (since μ_r acts on $T_{Y,Q}$).

$$\epsilon : (X_0, \cdots, X_n) \mapsto (\epsilon^{a_0} X_0, \cdots, \epsilon^{a_n} X_n)$$

Y is fixed by μ_r-action. So $\epsilon : f \mapsto \epsilon^e f$ for some e. Then we say

$$\frac{1}{r}(a_0, \cdots, a_n; e)$$

is the type of hyperquotient singularity $P \in X$.

We want to see the μ_r-action on the generator of ω_Y

$$s = \frac{dx_1 \wedge \cdots \wedge dx_n}{\partial f/\partial x_0} = \mathrm{Res}_Y \frac{dx_0 \wedge \cdots \wedge dx_n}{f}.$$

The weight on s is $(a_1 + \cdots, a_n) - (e - a_0) = \sum a_i - e$.

Remark 0.5. Need explanation about residue. Locally we can take the coordinate $f, x_1, \cdots, x_n$ because $\partial f/\partial x_0 \neq 0$. Formally, the residue have a property

$$\mathrm{Res}_Y(\frac{f}{df} \wedge \beta) = \beta.$$

Because $df = \sum \partial f/\partial x_i dx_i$,

$$\begin{aligned}\mathrm{Res}_Y(\frac{dx_0 \wedge \cdots \wedge x_n}{f}) &= \mathrm{Res}_Y(\frac{\frac{df - \partial f/\partial x_1 dx_1 - \cdots - \partial f/\partial x_n dx_n}{\partial f/\partial x_0} \wedge dx_1 \wedge \cdots \wedge dx_n}{f}) \\ &= \mathrm{Res}_Y(\frac{df \wedge dx_1 \wedge \cdots \wedge dx_n}{f \partial f/\partial x_0}) = \frac{dx_1 \wedge \cdots \wedge dx_n}{\partial f/\partial x_0}.\end{aligned}$$