

Canonical singularity : appears in MMP.

$X$ : variety,  $D$ : Cartier divisor

$$R(X, D) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)).$$

$\text{Proj } R(X, D)$  is a birational model of  $X$  if

- ①  $D$  is big. ( $\dim = \dim X$ )
- ②  $R(X, D)$  is fin. gen.  $k$ -alg.

In general,  $R(X, D)$  is not fin. gen.

Thm. If  $\text{Im } \Phi_{nD}$  is bpf for some  $n > 0$ ,  
(i.e.  $D$  is semiample).  $R(X, D)$  is f.g.

Proof. ( $m=1$  case)

Serre's thm.<sup>[1]</sup>  $F$ : coh. sheaf on pwj var  $X$ .

$\mathcal{O}(1)$ : very ample

$$\Rightarrow \bigoplus_{n \geq 0} H^0(F(n)) \text{ is f.g. as } \bigoplus_{n \geq 0} H^0(\mathcal{O}(n))\text{-mod.}$$

$$\varphi = \varphi_D: X \xrightarrow{\sim} \varphi(X) \subset \mathbb{P}^n$$

$$\varphi^* \mathcal{O}(1) = \mathcal{O}_X(D)$$

$$\text{Set } A := \varphi_* \mathcal{O}_X. \Rightarrow \varphi_* \mathcal{O}(nD) = \varphi_* \varphi^* \mathcal{O}(n) = \varphi_* \mathcal{O}_X \otimes \mathcal{O}(n) = A(n).$$

$$\bigoplus_{n \geq 0} H^0(X, \mathcal{O}(nD)) = \bigoplus_{n \geq 0} H^0(\mathbb{P}^n, \varphi_* \mathcal{O}(nD)) = \bigoplus_{n \geq 0} H^0(\mathbb{P}^n, A(n)).$$

Why do we use  $\text{Proj } R(X, D)$  instead of  $\text{im } \varphi_{nD}$ ?

- ① it is intrinsic.
- ②  $\text{Proj } R(X, D)$  is normal.

(it is the normalization of  $\text{im } \varphi_{nD}$ .)

$$X \longrightarrow \text{Proj } R(X, D)$$

$\searrow$   $f \downarrow$  finite.

$\text{im } \varphi_{nD}$

③  $f^* \mathcal{O}(1)$  ample  $\Rightarrow f^* \mathcal{O}(n)$  very ample

$$\Rightarrow \text{Proj } R(X, D) = \text{im } \varphi_{nD} \text{ for } n \gg 0.$$

Def. ①  $X$  is a canonical var. if

$X$  has at worst can. sing. only and  
 $K_X$  is ample.

② For a variety  $Y$  of general type.

$X$  is a canonical model if

$X \xrightarrow{\text{bir.}} Y$  and  $X$  is a can. var.

Thm.  $Y$ : sm. proj. var of gen. type.

$Y$  has can. model  $X \Leftrightarrow R(Y, K_Y)$  is f.g.

Then  $X = \text{Proj } R(Y, K_Y)$ .

Thm. (BCHM).  $R(Y, K_Y)$  is f.g.

Sketch.  $X$ : canonical model.  $mK_X$  Cartier.

$R(X, mK_X)$  f.g.

$$H^0(X, mK_X) = H^0(\tilde{X}, mK_{\tilde{X}}) = H^0(Y, mK_Y)$$

↑  
resol. sing.      ↑  
                    bir. inv.

$$\begin{aligned} X &= \text{Proj } R(X, mK_X) = \text{Proj } R(Y, mK_Y) \\ &= \text{Proj } R(Y, K_Y). \end{aligned}$$

Converse. Suppose  $R(Y, K_Y)$  is f.g.

$$\exists \text{ m. s.t. } \text{Sym}^k H^0(mK_Y) \longrightarrow H^0(kmK_Y). \quad (*)$$

$Y' \rightarrow Y$  - resolution of  $\text{bs}(1mK_Y)$ .

may assume  $\text{bs}$  is a divisor. (replace  $Y$  by  $Y'$ )

$$1mK_Y = 1M + F$$

↑ bpf.

$$(*) \Rightarrow 1kmK_Y = 1kM + kF.$$

$$\begin{array}{ccc} \text{Consider } & \varphi_M : Y \rightarrow \text{im } \varphi_M \subset \mathbb{P}^N & \\ & \parallel & \text{im } \varphi_{M+F} \\ & \varphi & \end{array}$$

$$\text{Proj } R(Y, mK_Y) = X.$$

if  $m \gg 0$ .

Claim.  $T$ : irreduc. comp. of  $F \Rightarrow \dim \varphi(T) < \dim T \leftarrow$  asymptotic RR.

$$Y^0 = Y - \text{excep. of } \varphi$$

$$X^0 = \varphi(Y^0), \quad \mathcal{O}_{X^0} \cong \mathcal{O}_Y(mK_Y)|_{Y^0}$$

$$\mathcal{O}_{X^0}|_{X^0} \cong \mathcal{O}_Y(M)|_{Y^0} \cong \mathcal{O}_Y(mK_Y)|_{Y^0} \cong \mathcal{O}_X(mK_X)|_{X^0}$$

$$mK_X \text{ Cartier. } \quad mK_Y = \varphi^* mK_X + F.$$

$$\begin{array}{ccc} (H^0(X, mK_X)) & \xrightarrow{\quad \text{id.} \quad} & H^0(U, mK_{\tilde{X}}|_U) \\ & \downarrow & \uparrow \text{p.v.} \\ H^0(\tilde{X}, mK_{\tilde{X}}) & & H^0(V, mK_X|_V) \\ & \uparrow & \downarrow \text{non-sing.} \\ H^0(\tilde{X}, mK_{\tilde{X}} - \sum a_i E_i) & = & H^0(\tilde{X}, \varphi^*(mK_X)) \end{array}$$