# BRIEF INTRODUCTION TO GLOBAL CANONICAL VARIETIES II

JIE WANG, HAN-BOM MOON

Let's review weighted projective spaces.

Let $Q = (q_0, \cdots, q_r)$ be a sequence of positive integers. Consider a graded ring $S(Q) = k[T_0, \cdots T_r]$, where $\deg T_i = q_i$.

**Definition 0.1.**

$$\mathbb{P}(Q) = \mathrm{Proj}\ S(Q).$$

**Example 0.2.** (1) $\mathbb{P}(1, \cdots, 1) = \mathbb{P}^r$.

(2) $\mathbb{P}(1,1,2,2) = \mathrm{Proj}\ (k \oplus < T_0, T_1 > \oplus < T_0^1, T_0T_1, T_1^2, T_2, T_3 > \oplus \cdots)$ It is not generated by degree 1 part $S_1$. For $S = S(1,1,2,2)$,

$$S_{(T_0)} = k[\frac{T_1}{T_0}, \frac{T_2}{T_0^2}, \frac{T_3}{T_0^2}].$$

Thus $\mathrm{Spec}\ S_{(T_0)} = \mathbb{A}^3$. But

$$S_{(T_2)} = k[\frac{T_0^2}{T_2}, \frac{T_0T_1}{T_2}, \frac{T_1^2}{T_2}, \frac{T_3}{T_2}] \cong k[X,Y,Z,W]/ < XZ - Y^2 > .$$

Therefore $\mathrm{Spec}\ S_{(T_2)}$ is an affine cone.

Geometrically, $\mathbb{P}(Q) = \mathbb{A}^{r+1} - \{0\}/\mathbb{C}^*$ where $\mathbb{C}^*$-action is given by

$$\lambda \cdot (x_0, \cdots, x_r) = (\lambda^{q_0} x_0, \cdots, \lambda^{q_r} x_r).$$

$\mathbb{P}(Q)$ has finite quotient singularities. Consider $\mu_Q := \mu_{q_0} \times \cdots \times \mu_{q_r}$. $\mu_Q$ acts on $\mathbb{P}^r$. We can get a map of graded ring $S(Q) \to S = k[X_0, \cdots, X_r]$ which sends $T_i \mapsto X_i^{q_i}$, then the image of $S^{\mu_Q}$. Then $\mathbb{P}(Q) = \mathbb{P}^r/\mu_Q$.

As a consequence, $\mathbb{P}(Q)$ is normal irreducible variety. For more information, see [Dol82].

**Definition 0.3.**

$$\mathcal{O}_X(n) = \widetilde{S(Q)(n)}.$$

**Example 0.4.** For $X = \mathbb{P}(1,1,2,2)$, $\mathcal{O}(1)$ is NOT locally free.

$$S(Q)_{(T_2)} = k[\frac{T_0^2}{T_2}, \frac{T_0T_1}{T_2}, \frac{T_1^2}{T_2}, \frac{T_3}{T_2}]].$$

$$S(Q)(1)_{(T_2)} = k[\frac{T_0^3}{T_2}, \frac{T_0^2T_1}{T_2}, \frac{T_0T_1^2}{T_2}, \frac{T_1^3}{T_2}, T_0, T_1, \frac{T_3T_0}{T_2}, \frac{T_3T_1}{T_2}]$$

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and it is generated by $T_0, T_1$ as a $S(Q)_{(T_2)}$ but not generated by single element. Also it is not free. It is a reflexive sheaf.

But $\mathcal{O}(2)$ is a line bundle. $S(Q)(2)_{(T_2)}$ is generated by $T_2$. For instance, $T_2 = T_2 \cdot T_3/T_2$. $H^0(\mathcal{O}(2)) = \mathrm{span}\{T_0^2, T_0T_1, T_1^2, T_2, T_3\}$.

**Theorem 0.5.**

$$H^0(\mathcal{O}(n)) = S(Q)_n$$

*for every* $n \in \mathbb{Z}$.

In fact, for $X = \mathbb{P}(1,1,2,2)$, we have a map

$$\phi = \phi_{|\mathcal{O}_X(2)|} : X \hookrightarrow \mathbb{P}^4$$

and the image of $\phi$ is a cone of rank 3 in $\mathbb{P}^4$. The image is the join of a line and a conic.

**Example 0.6.** Example of canonical threefolds. ($X$ has canonical singularities and $K_X$ is ample).

Take $Y = \mathbb{P}(1,1,2,2,7) = \mathbb{P}(x_1, x_2, y_1, y_2, w)$. $X$ is a hypersurface defined by $w^2 = f(x_1, x_2, y_1, y_2)$ where $f$ is a general homogeneous polynomial of degree 14.

Consider the generically $2:1$ map $X \to \mathbb{P}(1,1,2,2)$ obtained by forgetting $w$. $X$ is a ramified cover of $\mathbb{P}(1,1,2,2)$ branched along $f = 0$. If we think $\mathbb{P}(1,1,2,2)$ as a cone in$\mathbb{P}^4$, then the branch locus is a $Q \cap F$ where $F$ is a hypersurface of degree 7. $F$ intersects the line (note that $\mathbb{P}(1,1,2,2)$ is the join of a line and a conic) at 7 points and these 7 points are singular points of $X$.

**Question 0.7.** Why these points have a canonical singularities?

By adjunction formula,

$$K_X = (K_{\mathbb{P}(1,1,2,2,7)} + X)|_X = \mathcal{O}_X(-1-1-2-2-7+14) = \mathcal{O}_X(1)$$

so $K_X$ is $\mathbb{Q}$-ample.

There is a degree 28 map

$$f : \mathbb{P}^4 \to \mathbb{P}(1,1,2,2,7).$$

$Y = f^{-1}(X)$ is a degree 14 hypersurface in $\mathbb{P}^4$.

$$K_X^3 = \mathcal{O}_Y(1)^3/28 = 14 \cdot 1 \cdot 1 \cdot 1/28 = \frac{1}{2}.$$

By applying asymptotic Riemann-Roch,

$$P_m = h^0(X, mK_X) \sim \frac{1}{3!}(K_X)^3 m^3 + \text{ lower terms} = \frac{1}{3!} \cdot \frac{1}{2} m^3 + \cdots$$

Note that $P_m$ is a birational invariant and for smooth $X$ $K_X^3$ must be an integer. Thus the above computation implies that $X$ has no smooth minimal model.

## References

[Dol82] Igor Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982. 1