BRIEF INTRODUCTION TO GLOBAL CANONICAL VARIETIES III

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Recall: In the last week, we discussed a degree 14 hypersurface

$$X_{14} \subset \mathbb{P}(1,1,2,2,7)$$

in a weighted projective space.

Example 0.1. Let

$$X_{6,6,6} \subset \mathbb{P}(2^4, 3^3)$$

be a general complete intersection of three degree 6 weighted homogeneous hypersurfaces. Fix a homogeneous coordinates $x_0, \cdots, x_3, y_0, \cdots, y_2$. A monomial of weighted degree 6 in $x_0, \cdots, x_3, y_0, \cdots, y_2$ is either in x_i's or y_i's. Thus if we consider an embedding

$$\mathbb{P}(2^4, 3^3) \stackrel{|\mathcal{O}(6)|}{\hookrightarrow} \mathbb{P}^{25},$$

the image is a join of degree 3 Veronese embedding of $\mathbb{P}^3$ and degree 2 Veronese embedding of $\mathbb{P}^2$. In this embedding, the pullback of a hyperplane is a degree 6 hypersurface in $\mathbb{P}(2^4, 3^3)$.

Observation 1. X has 27 Veronese cone singularities at its intersection with the 3-dimensional strata $v_3(\mathbb{P}^3)$.

There are two singular locus on $X_{6,6,6}$: Two Veronese embeddings. One is three dimensional, and the other is two dimensional. Thus if we take three general hyperplanes, then the intersection with $v_2(\mathbb{P}^2)$ is empty by dimension reason, the intersection with $v_3(\mathbb{P}^3)$ is 27 points, because $\deg(v_3(\mathbb{P}^3)) = 27$. They are Veronese cone singularities... Why?

Observation 2.

$$K_{X_{6,6,6}} = \mathcal{O}(6+6+6-2-2-2-2-3-3-3) = \mathcal{O}(1).$$

Observation 3.

$$K^3_{X_{6,6,6}} = \frac{6 \cdot 6 \cdot 6}{2^4 \cdot 3^3} = \frac{1}{2}.$$

$$H^0(K_{X_{6,6,6}}) = H^0(\mathcal{O}(1)) = 0 \Rightarrow p_g = 0.$$

So it is another canonical variety without smooth canonical minimal model.

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Example 0.2. We don't fully understand the following.

For the same example, look at 2-dimensional linear system

$$F_\lambda = \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3$$

where $\lambda \in \mathbb{P}^2$, F_i are degree 6 polynomials vanishing on $X_{6,6,6}$. Note that

$$F_\lambda = c_\lambda(x_0, \cdots, x_3) + q_\lambda(y_0, \cdots, y_2)$$

for a cubic c_λ and a quadric q_λ. Here c_λ is a net of cubics in $\mathbb{P}^3$ and q_λ is net of conics in $\mathbb{P}^2$ parametrized by the same $\lambda \in \mathbb{P}^2$.

The conic $Q_\lambda = \{q_\lambda = 0\}$ is union of two lines when $\lambda \in E$ where E is a discriminant curve. Let $S_\lambda = \{c_\lambda = 0$. For general λ, S_λ is a smooth cubic surface. Thus it contains 27 lines. Define

$$B = \{(\ell, m) | \ell, m \text{ lines}, \ell \subset Q_\lambda, m \subset S_\lambda\}.$$

Then there is 54:1 map $B \to E$. For each $(\ell, n) \in B$, we can construct their weighted join $\mathbb{P}(2, 2, 3, 3)$ in F_λ.

It is written that this example is related to intermediate Jacobian and Abel-Jacobi map, but we don't understand it fully.

There is a (maybe not complete) list of canonical hypersurfaces in weighted projective spaces in [Rei87].

References

[Rei87] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987. 2