DEFINITIONS AND EASY EXAMPLES III

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Let's begin with the definition of canonical singularities.

Definition 0.1. Let X be a variety. X has **canonical singularities** if

(1) rK_X is Cartier for some integer $r > 0$;
(2) If $f : Y \to X$ is a resolution,

$$rK_Y = f^*(rK_X) + \sum a_i E_i, \quad \text{with } a_i \ge 0.$$

In the following example, we will compute the discrepancy.

Example 0.2. Let $X = \{f = 0\} \subset \mathbb{A}^{n+1}$ be a hypersurface with an isolated singularity at $p \in X$. Assume that the projectivized normal cone at p is nonsingular. Maybe $f = x_0^k + \cdots + x_n^k$ is a nice example. Then the single blow-up at p gives a resolution of singularities.

We've seen that $\omega_X \cong \mathcal{O}_X$ and ω_X is generated by a differential form locally represented by

$$s = \frac{dx_1 \wedge \cdots dx_n}{\partial f / \partial x_0}.$$

To compute the discrepancy, let's compute the blow-up concretely.

On an affine chart of $bl_p\mathbb{A}^{n+1}$, we can find a coordinate system $\{y_0, \cdots, y_n\}$ and

$$x_i = y_i y_n \text{ for } i \le n-1 \text{ and } x_n = y_n.$$

Then for $\sigma : Y = bl_p X \to X$,

$$\sigma^* f = f(y_0 y_n, \cdots, y_n) = y_n^k (y_0^k + \cdots + y_{n-1}^k + 1).$$

Now

$$\begin{aligned} \sigma^* s &= \frac{d(y_n y_1) \wedge \cdots \wedge d(y_n y_{n-1}) \wedge dy_n}{k x_0^{k-1}} \\ &= \frac{y_n^{n-1} dy_1 \wedge \cdots \wedge dy_n}{k y_0^{k-1} y_n^{k-1}} = y_n^{n-k} \frac{dy_1 \wedge \cdots dy_n}{\partial g / \partial y_0} \end{aligned}$$

where g is the equation for the proper transform.

Therefore in this case, the index is 1 and $K_Y = \sigma^*(K_X) + (n-k)E$. So if $n - k \ge 0$, it is a canonical singularity and if $n - k < 0$ it is not.

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We can compute this example with adjunction formulas. Let $S = \mathbb{A}^{n+1}$ and $\hat{S} = bl_p\mathbb{A}^{n+1}$. Then

$$\begin{aligned} K_Y &= (K_{\hat{S}} + Y)|_Y = (\sigma^*(K_S) + nE + \sigma^*(X) - kE)|_Y = \sigma^*(K_S + X)|_Y + (n-k)E|_Y \\ &= K_Y + (n-k)E|_Y. \end{aligned}$$

Remark 0.3. We can translate the canonical singularity in the following way: If we take a regular form s on X (that means, it is regular on the smooth locus) and take the pull-back to Y, then it has no pole along exceptional divisors.

Example 0.4. Consider $X = \mathbb{A}^2/\mu_3$ where μ_3 is the group of cubic roots of 1. μ_3 acts as

$$e \cdot (x, y) = (ex, ey).$$

We have the quotient map $\pi : \mathbb{A}^2 \to X$ and

$$X = \mathrm{Spec}\ k[x^3, x^2y, xy^2, y^3] = \mathrm{Spec}\ k[u_0, u_1, u_2, u_3]/(u_0u_3 - u_1u_2, u_0u_2 - u_1^2, u_1u_3 - u_2^2).$$

In the last seminar, we've seen $3K_X$ is Cartier and

$$s = \frac{(du_0 \wedge du_1)^3}{u_0^4} \in (\Omega^2_{k(X)})^3$$

gives a section.

Let's compute the discrepancy using local computation. If $\sigma : Y \to X$ is the blow-up, for an affine chart, the functions $u_0, \cdots, u_3$ is pulled back to $v_0, v_0v_1, v_0v_2, v_0v_3$ respectively.

So the equations of X is pulled back to the equations

$$u_0u_3 - u_1u_2 = v_0^2(v_3 - v_1v_2), u_0u_2 - u_1^2 = v_0^2(v_2 - v_1^2), u_1u_3 - u_2^2 = v_0^2(v_1v_3 - v_2^2),$$

thus the proper transform of X is defined by

$$v_3 - v_1v_2, v_2 - v_1^2, v_1v_3 - v_2^2.$$

Therefore it is easy to see that locally $\{v_0, v_1\}$ can be a coordinate system of Y.

Now

$$\sigma^*(s) = \frac{(dv_0 \wedge d(v_0v_1))^3}{v_0^4} = \frac{(v_0dv_0 \wedge dv_1)^3}{v_0^4} = \frac{(dv_0 \wedge dv_1)^3}{v_0},$$

so it has a poll on the exceptional divisor. Therefore X has non canonical singularity.

REFERENCES