BIRATIONAL GEOMETRY OF THE MODULI SPACE OF PURE SHEAVES ON QUADRIC SURFACE

KIRYONG CHUNG AND HAN-BOM MOON

ABSTRACT. We study birational geometry of the moduli space of stable sheaves on a quadric surface with Hilbert polynomial $5m+1$ and $c_1 = (2,3)$. We describe a birational map between the moduli space and a projective bundle over a Grassmannian as a composition of smooth blow-ups/downs.

1. INTRODUCTION

The geometry of the moduli space of sheaves on a del Pezzo surface has been studied in various viewpoints, for instance curve counting, the strange duality conjecture, and birational geometry via Bridgeland stability. For a detailed description of the motivation, see [CM15] and references therein. In this paper we continue the study of birational geometry of the moduli space of torsion sheaves on a del Pezzo surface, which was initiated in [CM15]. More precisely, here we construct a flip between the moduli space of sheaves and a projective bundle, and show that their common blown-up space is the moduli space of stable pairs ([LP93]), in the case of a quadric surface.

Let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric surface in $\mathbb{P}^3$ with a very ample polarization $L := \mathcal{O}_Q(1,1)$. For the convenience of the reader, we start with a list of relevant moduli spaces.

Definition 1.1. (1) Let $\mathbf{M} := \mathbf{M}_L(Q,(2,3),5m+1)$ be the moduli space of stable sheaves F on Q with $c_1(F) = c_1(\mathcal{O}_Q(2,3))$ and $\chi(F(m)) = 5m+1$.

(2) Let $\mathbf{M}^\alpha := \mathbf{M}^\alpha_L(Q,(2,3),5m+1)$ be the moduli space of α-stable pairs (s,F) with $c_1(F) = c_1(\mathcal{O}_Q(2,3))$ and $\chi(F(m)) = 5m+1$ ([LP93] and [He98, Theorem 2.6]).

(3) Let $\mathbf{G} = \mathrm{Gr}(2,4)$ and let $\mathbf{G}_1$ be the blow-up of $\mathbf{G}$ along $\mathbb{P}^1$ (Section 2.1).

(4) Let $\mathbf{P} := \mathbb{P}(\mathcal{U})$ and $\mathbf{P}^- := \mathbb{P}(\mathcal{U}^-)$, where $\mathcal{U}$ (resp. $\mathcal{U}^-$) is a rank 10 vector bundle over $\mathbf{G}$ (resp. $\mathbf{G}_1$) defined in (2) in Section 2.1 (resp. Section 3.3).

The aim of this paper is to explain and justify the following commutative diagram between moduli spaces.

$$\begin{tikzcd}
\mathbf{M}^+ \arrow[r] \arrow[d, "r"'] & \mathbf{P}^- = \mathbb{P}(\mathcal{U}^-) \arrow[r, dashed, leftrightarrow] \arrow[dr] & \mathbb{P}(u^*\mathcal{U}) = \mathbf{G}_1 \times_{\mathbf{G}} \mathbf{P} \arrow[r] \arrow[d] & \mathbf{P} = \mathbb{P}(\mathcal{U}) \arrow[d] \\
\mathbf{M} \arrow[ur, dashed, leftrightarrow] & & \mathbf{G}_1 \arrow[r, "u"] & \mathbf{G}
\end{tikzcd}$$

We have to explain two flips (dashed arrows) on the diagram.

One of key ingredients is the *elementary modification* of vector bundles ([Mar73]), sheaves ([HL10, Section 2.B]), and pairs ([CC16, Section 2.2]). It has been widely used in the study of sheaves on

2010 *Mathematics Subject Classification.* 14E05, 14E30, 14D99.

Key words and phrases. Moduli space, Stable sheaf, Birational geometry, Elementary modification.

a smooth projective variety. Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X and $\mathcal{Q}$ be a vector bundle on a smooth divisor $Z \subset X$ with a surjective map $\mathcal{F}|_Z \twoheadrightarrow \mathcal{Q}$. The elementary modification of $\mathcal{F}$ along Z is the kernel of the composition

$$\mathrm{elm}_Z(\mathcal{F}) := \ker(\mathcal{F} \twoheadrightarrow \mathcal{F}|_Z \twoheadrightarrow \mathcal{Q}).$$

A similar definition is valid for sheaves and pairs, too.

On $\mathbf{G}_1$, let $\mathcal{U}^- := \mathrm{elm}_{Y_{10}}(u^*\mathcal{U})$ be the elementary transformation of $u^*\mathcal{U}$ along a smooth divisor Y_{10} (Section 2.1).

Proposition 1.2. *Let $\mathbf{P}^- = \mathbb{P}(\mathcal{U}^-)$. The flip $\mathbf{P}^- \dashrightarrow \mathbb{P}(u^*\mathcal{U}) = \mathbf{G}_1 \times_{\mathbf{G}} \mathbb{P}(\mathcal{U})$ is a composition of a blow-up and a blow-down. The blow-up center in $\mathbf{P}^-$ (resp. $\mathbb{P}(u^*\mathcal{U})$) is a $\mathbb{P}^1$ (resp. $\mathbb{P}^7$)-bundle over Y_{10}.*

Theorem 1.3. *There is a flip between $\mathbf{M}$ and $\mathbf{P}^-$ which is a blow-up followed by a blow-down, and the master space is $\mathbf{M}^+$, the moduli space of $+$-stable pairs.*

As applications, we compute the Poincaré polynomial of $\mathbf{M}$ and show the rationality of $\mathbf{M}$ (Corollary 3.8) which were obtained by Maican by different methods ([Mai16]). Since each step of the birational transform is described in terms of blow-ups/downs along explicit subvarieties, in principle the cohomology ring and the Chow ring of $\mathbf{M}$ can be obtained from that of $\mathbf{G}$. Also one may aim for the completion of Mori's program for $\mathbf{M}$. We will carry on these projects in forthcoming papers.

2. Relevant moduli spaces

In this section we give definitions and basic properties of some relevant moduli spaces.

2.1. **Grassmannian as a moduli space of Kronecker quiver representations.** The moduli space of representations of a Kronecker quiver parametrizes the isomorphism classes of stable sheaf homomorphisms

$$\mathcal{O}_Q(0,1) \longrightarrow \mathcal{O}_Q(1,2)^{\oplus 2} \tag{1}$$

up to the natural action of the automorphism group $\mathbb{C}^* \times \mathrm{GL}_2/\mathbb{C}^* \cong \mathrm{GL}_2$. For two vector spaces E and F of dimension 1 and 2 respectively and $V^* := \mathrm{H}^0(Q, L)$, the moduli space is constructed as $\mathbf{G} := \mathrm{Hom}(F, V^* \otimes E)//\mathrm{GL}_2 \cong V^* \otimes E \otimes F^*//\mathrm{GL}_2$ with an appropriate linearization ([Kin94]). Note that the GL_2 acts as a row operation on the space of 2×4 matrices, $\mathbf{G} \cong \mathrm{Gr}(2, 4)$.

Let $\mathbf{H}(n) := \mathrm{Hilb}^n(Q)$, the Hilbert scheme of n points on Q. $\mathbf{H}(2)$ is birational to $\mathbf{G}$ because a general $Z \in \mathbf{H}(2)$, $I_Z(2,3)$ has a resolution of the form (1). For any $Z \in \mathbf{H}(2)$, let ℓ_Z be the unique line in $\mathbb{P}^3 \supset Q$ containing Z. Then either $\ell_Z \cap Q = Z$ or $\ell_Z \subset Q$. In the second case, the class of ℓ_Z is of the type $(1,0)$ or $(0,1)$. Let Y_{10} (resp. Y_{01}) be the locus of subschemes such that ℓ_Z is a line of the type $(1,0)$ (resp. $(0,1)$). Then Y_{10} and Y_{01} are two disjoint subvarieties which are isomorphic to a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$.

Proposition 2.1 ([BC13, Example 6.1])**.** *There exists a morphism $t : \mathbf{H}(2) \longrightarrow \mathbf{G}_1 \stackrel{u}{\longrightarrow} \mathbf{G}$. The first (resp. the second) map contracts the divisor Y_{01} (resp. Y_{10}) to $\mathbb{P}^1$. If $\ell_Z \cap Q = Z$, then $t(Z) = I_Z(2,3)$. If $Z \in Y_{10}$, then $t(Z) = E_{10} \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_Q(1,3), \mathcal{O}_{\ell_Z}(1))) = \{\mathrm{pt}\}$. If $Z \in Y_{01}$, then $t(Z) = E_{01} \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_Q(2,2), \mathcal{O}_{\ell_Z})) = \{\mathrm{pt}\}$.*

There is a *universal morphism* $\phi : p_1^*\mathcal{F} \otimes p_2^*\mathcal{O}_Q(0,1) \to p_1^*\mathcal{E} \otimes p_2^*\mathcal{O}_Q(1,2)$ where $p_1 : \mathbf{G} \times Q \to \mathbf{G}$ and $p_2 : \mathbf{G} \times Q \to Q$ are two projections ([Kin94]). Let $\mathcal{U}$ be the cokernel of $p_{1*}\phi$. On the stable locus, $p_{1*}\phi$ is injective. Thus we have an exact sequence

$$0 \to \mathcal{F} \otimes \mathrm{H}^0(\mathcal{O}_Q(0,1)) \to \mathcal{E} \otimes \mathrm{H}^0(\mathcal{O}_Q(1,2)) \to \mathcal{U} \to 0 \tag{2}$$

and $\mathcal{U}$ is a rank 10 vector bundle. Let $\mathbf{P} := \mathbb{P}(\mathcal{U})$.

2.2. **Moduli space M of stable sheaves.** Recall that $\mathbf{M} := \mathbf{M}_L(Q,(2,3),5m+1)$ is the moduli space of stable sheaves F on Q with $c_1(F) = c_1(\mathcal{O}_Q(2,3))$ and $\chi(F(m)) = 5m+1$. There are four types of points in $\mathbf{M}$ ([Mai16, Theorem 1.1]). Let $C \in |\mathcal{O}_Q(2,3)|$.

(0) $F = \mathcal{O}_C(p+q)$, where the line $\langle p,q\rangle$ is not contained in Q;
(1) $F = \mathcal{O}_C(p+q)$, where the line $\langle p,q\rangle$ in Q is of type $(1,0)$;
(2) $F = \mathcal{O}_C(0,1)$;
(3) F fits into a non-split extension $0 \to \mathcal{O}_E \to F \to \mathcal{O}_\ell \to 0$ where E is a $(2,2)$-curve and ℓ is a $(0,1)$-line.

Let $\mathbf{M}_i$ be the locus of sheaves of the form (i). Then $\mathbf{M}_i$ is a subvariety of codimension i. $\mathbf{M}_1$ is a $\mathbb{P}^9$-bundle over $\mathbb{P}^2 \times \mathbb{P}^1$. $\mathbf{M}_2$ is isomorphic to $|\mathcal{O}_Q(2,3)|$. Finally, $\mathbf{M}_3$ is a $\mathbb{P}^1$-bundle over $|\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)|$. $\mathbf{M}_1 \cap \mathbf{M}_2 = \mathbf{M}_1 \cap \mathbf{M}_3 = \emptyset$, but $\mathbf{M}_{23} := \mathbf{M}_2 \cap \mathbf{M}_3 \cong |\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)|$ ([Mai16, Theorem 1.1]). Note that $\dim \mathrm{H}^0(F) = 1$ in general, but $\mathbf{M}_2$ parametrizes sheaves that $\dim \mathrm{H}^0(F) = 2$.

2.3. **Moduli spaces of stable pairs.** A pair (s,F) consists of $F \in \mathsf{Coh}(Q)$ and a section $\mathcal{O}_Q \xrightarrow{s} F$. Fix $\alpha \in \mathbb{Q}_{>0}$. A pair (s,F) is called *α-semistable* (resp. *α-stable*) if F is pure and for any proper subsheaf $F' \subset F$, the inequality

$$\frac{P(F')(m) + \delta \cdot \alpha}{r(F')} \le (<) \frac{P(F)(m)) + \alpha}{r(F)}$$

holds for $m \gg 0$. Here $\delta = 1$ if the section s factors through F' and $\delta = 0$ otherwise. Let $\mathbf{M}^\alpha := \mathbf{M}_L^\alpha(Q,(2,3),5m+1)$ be the moduli space of S-equivalence classes of α-semistable pairs whose support have a class $c_1(\mathcal{O}_Q(2,3))$ ([LP93, Theorem 4.12] and [He98, Theorem 2.6]). The extremal case that α is sufficiently large (resp. small) is denoted by $\alpha = \infty$ (resp. $\alpha = +$). The deformation theory of pairs is studied in [He98, Corollary 1.6 and Corollary 3.6].

Proposition 2.2. *(1) There exists a natural forgetful map $r : \mathbf{M}^+ \longrightarrow \mathbf{M}$ which maps (s,F) to F.*
(2) ([He98, Section 4.4]) The moduli space $\mathbf{M}^\infty$ of ∞-stable pairs is isomorphic to the relative Hilbert scheme of two points on the complete linear system $|\mathcal{O}_Q(2,3)|$.

The birational map $\mathbf{M}^\infty \dashrightarrow \mathbf{M}^+$ is analyzed in [Mai16, Theorem 5.7]. It turns out that this is a single flip over $\mathbf{M}^4$ and is a composition of a smooth blow-up and a smooth blow-down. The blow-up center $\mathbf{M}_3^\infty$ is isomorphic to a $\mathbb{P}^2$-bundle over $|\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)|$ where a fiber $\mathbb{P}^2$ parameterizes two points lying on a $(0,1)$-line. After the flip, the flipped locus on $\mathbf{M}^+$ is $\mathbf{M}_3^+$.

For the forgetful map $r : \mathbf{M}^+ \to \mathbf{M}$, we define $\mathbf{M}_i^+ := r^{-1}(\mathbf{M}_i)$ if $i \ne 3$ and $\mathbf{M}_3^+$ is the proper transform of $\mathbf{M}_3$. It contracts $\mathbf{M}_2^+$, which is a $\mathbb{P}^1$-bundle over $\mathbf{M}_2$ and $\mathbf{M}^+ \setminus \mathbf{M}_2^+ \cong \mathbf{M} \setminus \mathbf{M}_2$. Maican proved that r is a smooth blow-up along the Brill-Noether locus $\mathbf{M}_2$ ([Mai16, Proposition 5.8]).

3. Decomposition of the birational map between $\mathbf{M}$ and $\mathbf{P}$

In this section we prove Proposition 1.2 and Theorem 1.3 by describing the birational map between $\mathbf{M}$ and $\mathbf{P}$.

3.1. Construction of a birational map $\mathbf{M}^+ \dashrightarrow \mathbf{P}$.

Lemma 3.1. *There exists a surjective morphism $w : \mathbf{M}^+ \longrightarrow \mathbf{G}$ which maps $(s, \mathcal{O}_C(p+q)) \in \mathbf{M}_0^+$ to $I_{\{p,q\}}(2,3)$, maps $(s, \mathcal{O}_C(p+q)) \in \mathbf{M}_1^+$ to the line $\langle p, q\rangle$ of the type $(1,0)$, maps $(s, F) \in \mathbf{M}_2^+$ to a $(0,1)$-line determined by a section, and maps $(s, F) \in \mathbf{M}_3^+$ to ℓ (see Section 2.2 for the notation), a $(0,1)$-line.*

Proof. By Proposition 2.2, $\mathbf{M}^\infty$ is the relative Hilbert scheme of 2 points on the universal $(2,3)$-curves, which is a $\mathbb{P}^9$-bundle over $\mathbf{H}(2)$ ([CC16, Lemma 2.3]). By composing with $t : \mathbf{H}(2) \to \mathbf{G}$ in Proposition 2.1, we have a morphism $\mathbf{M}^\infty \to \mathbf{G}$. On the other hand, since the flip $\mathbf{M}^\infty \to \mathbf{M}^+$ is the composition of a single blow-up/down, the blown-up space $\widetilde{\mathbf{M}}^\infty$ admits two morphisms to $\mathbf{M}^\infty$ and $\mathbf{M}^+$, and the flipped locus is $\mathbf{M}_3^+$. Note that each point in $\mathbf{M}_3^+$ can be regarded as a collection of data (E, ℓ, e) where E is a $(2,2)$-curve, ℓ is a $(0,1)$-line, and $e \in \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_\ell, \mathcal{O}_E)$. The fiber $\widetilde{\mathbf{M}}^\infty \to \mathbf{M}^+$ over the point in the blow-up center $\mathbf{M}_3^+$ is a $\mathbb{P}^2$ which parameterizes two points on ℓ. The composition map $\widetilde{\mathbf{M}}^\infty \to \mathbf{M}^\infty \to \mathbf{G}$ is constant along the $\mathbb{P}^2$, because $\mathbf{G}$ does not remember points on the line $\ell \subset Q$. By the rigidity lemma, $\widetilde{\mathbf{M}}^\infty \to \mathbf{G}$ factors through $\mathbf{M}^+$ and we obtain a map $w : \mathbf{M}^+ \to \mathbf{G}$. $\square$

Note that $\mathbf{M}_1^+ \cong \mathbf{M}_1$ is a $\mathbb{P}^9$-bundle over $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbf{M}_2^+$ is a $\mathbb{P}^1$-bundle over $|\mathcal{O}_Q(2,3)| \cong \mathbb{P}^{11}$. They are disjoint divisors on $\mathbf{M}^+$.

Proposition 3.2. *There is a birational morphism $q : \mathbf{M}^+ \backslash \mathbf{M}_1^+ \to \mathbf{P} = \mathbb{P}(\mathcal{U})$ such that $p \circ q : \mathbf{M}^+ \backslash \mathbf{M}_1^+ \to \mathbf{P} \to \mathbf{G}$ coincides with $w|_{\mathbf{M}^+ \backslash \mathbf{M}_1^+}$ in Lemma 3.1. Furthermore, q is the smooth blow-down along $\mathbf{M}_2^+$.*

The proof consists of several steps. Since $\mathbf{P} = \mathbb{P}(\mathcal{U})$ is a projective bundle over $\mathbf{G}$, it is sufficient to construct a surjective homomorphism $w^*\mathcal{U}^* \to \mathcal{L} \to 0$ over $\mathbf{M}^+ \setminus \mathbf{M}_1^+$ for some $\mathcal{L} \in \mathrm{Pic}(\mathbf{M}^+ \setminus \mathbf{M}_1^+)$, or equivalently, a *bundle* morphism $0 \to \mathcal{L}^* \to w^*\mathcal{U}$.

Recall that a family $(\mathcal{L}, \mathcal{F})$ of pairs on a scheme S is a collection of data $\mathcal{L} \in \mathrm{Pic}(S)$, $\mathcal{F} \in \mathsf{Coh}(S \times Q)$, which is a flat family of pure sheaves, and a surjective morphism $\mathcal{E}xt^2_\pi(\mathcal{F}, \omega_\pi) \twoheadrightarrow \mathcal{L}$ where $\pi : S \times Q \to S$ is the projection and ω_π is the relatively dualizing sheaf (See [LP93, Section 4.3] for the explanation why we take the dual.). Now let $(\mathcal{L}, \mathcal{F})$ be the universal pair ([He98, Theorem 4.8]) on $\mathbf{M}^+ \times Q$. By applying $\mathcal{H}om(-, \mathcal{O})$ to $\mathcal{E}xt^2_\pi(\mathcal{F}, \omega_\pi) \twoheadrightarrow \mathcal{L}$, we obtain $0 \to \mathcal{L}^* \to \mathcal{H}om(\mathcal{E}xt^2_\pi(\mathcal{F}, \omega_\pi), \mathcal{O})$. It can be shown that $\mathcal{H}om(\mathcal{E}xt^2_\pi(\mathcal{F}, \omega_\pi), \mathcal{O}) \cong \mathcal{E}xt^1_\pi(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}), \mathcal{O})$ (see [CM15, Section 3.2]). So we have a non-zero element $e \in \mathrm{Hom}(\mathcal{L}^*, \mathcal{E}xt^1_\pi(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}), \mathcal{O})) \cong \mathrm{Ext}^1(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}), \pi^*\mathcal{L})$ ([CM15, Section 3.2]), which provides $0 \to \pi^*\mathcal{L} \to \mathcal{E} \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}) \to 0$ on $\mathbf{M}^+ \times Q$. By taking $\mathcal{H}om_\pi(-, \omega_\pi)$, we have $\mathcal{E}xt^2_\pi(\mathcal{E}, \omega_\pi) \to \mathcal{E}xt^2_\pi(\pi^*\mathcal{L}, \omega_\pi) \cong \mathcal{L}^* \to 0$ because $\mathcal{L}$ is a line bundle. This implies the existence of a flat family of pairs $(\mathcal{L}^*, \mathcal{E})$ on $\mathbf{M}^+ \times Q$. We may explicitly describe this construction fiberwisely in the following way. Let $(s, F) \in \mathbf{M}^+$. Let $F^D := \mathcal{E}xt^1(F, \omega_Q)$. For a non-zero section $s \in \mathrm{H}^0(F) \cong \mathrm{H}^1(F^D)^* \cong \mathrm{Ext}^1(F^D(2,2), (s^*) \otimes \mathcal{O}_Q)$, we have a pair (s^*, G) given by

$$0 \to (s^*) \otimes \mathcal{O}_Q \to G \to F^D(2,2) \to 0. \tag{3}$$

Lemma 3.3. *The map $(s, F) \mapsto (s^*, G)$ defines a dominant rational map $\mathbf{M}^+ \dashrightarrow \mathbf{P} = \mathbb{P}(\mathcal{U})$, which is regular on $\mathbf{M}^+ \setminus (\mathbf{M}_1^+ \sqcup \mathbf{M}_2^+)$.*

Proof. Since we have a relative construction of pairs, it suffices to describe the extension (s^*, G) set theoretically. If $(s, F) \in \mathbf{M}_0^+ \sqcup \mathbf{M}_1^+$, then $F \cong \mathcal{O}_C(p+q) \cong I_{Z,C}^D(0,-1)$ for some curve C and $Z = \{p, q\} \in \mathbf{H}(2)$ such that the line ℓ_Z containing Z is not in Q ([He98, Section 4.4]). Then $F^D(2,2) \cong I_{Z,C}(2,3)$. Since $\mathrm{Ext}^1(F^D(2,2), \mathcal{O}_Q) \cong \mathrm{H}^1(F^D)^* \cong \mathrm{H}^0(F) \cong \mathbb{C}$, from $0 \to \mathcal{O}_Q(-2,-3) \cong I_{C,Q} \to I_{Z,Q} \to I_{Z,C} \to 0$, we obtain $G = I_{Z,Q}(2,3)$. If $(s, F) \in \mathbf{M}_0^+$, then we have an element $(s^*, G) \in \mathbf{P}$ because G has a resolution of the form $\mathcal{O}_Q(0,1) \to \mathcal{O}_Q(1,2)^{\oplus 2}$. However, if $(s, F) \in \mathbf{M}_1^+$, then we have $0 \to I_{\ell_Z,Q}(2,3) \to G = I_{Z,Q}(2,3) \to I_{Z,\ell_Z}(2,3) \to 0$ and $I_{\ell_Z,Q}(2,3) = \mathcal{O}_Q(1,3)$, $I_{Z,\ell_Z}(2,3) = \mathcal{O}_{\ell_Z}(1)$. In particular, $\mathrm{Hom}(\mathcal{O}_Q(1,3), G) \neq 0$ and G does not admit a resolution $\mathcal{O}_Q(0,1) \to \mathcal{O}_Q(1,2)^{\oplus 2}$. So $G \notin \mathbf{G}$.

Suppose that $(s, F) \in \mathbf{M}_3^+ \setminus \mathbf{M}_2^+$. Then F fits into a non-split extension $0 \to \mathcal{O}_E \to F \to \mathcal{O}_\ell \to 0$. Apply $\mathcal{H}om(-, \omega_Q)$, then we have $0 \to \mathcal{O}_\ell(0,1) \to F^D(2,2) \to \mathcal{O}_E(2,2) \to 0$. Since $\mathrm{Ext}^1(\mathcal{O}_E(2,2), \mathcal{O}_Q) \cong \mathrm{Ext}^1(F^D(2,2), \mathcal{O}_Q) \cong \mathbb{C}$, the sheaf G is given by the pull-back:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Q \arrow[r] & \mathcal{O}_Q(2,2) \arrow[r] & \mathcal{O}_E(2,2) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_Q \arrow[r] \arrow[u, equal] & G \arrow[r] \arrow[u, dashed] & F^D(2,2) \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{4}$$

By applying the snake lemma to (4), we conclude that the unique non-split extension G lies on $0 \to \mathcal{O}_\ell(0,1) \to G \to \mathcal{O}_Q(2,2) \to 0$. Hence $G \in \mathbf{G}$ (Proposition 2.1) and we have an element $(s^*, G) \in \mathbf{P}$.

Now suppose that $(s, F) \in \mathbf{M}_2^+$, so $F = \mathcal{O}_C(0,1)$. Then $F^D(2,2) = \mathcal{O}_C(2,2)$. So we have $0 \to (s^*) \otimes \mathcal{O}_Q \to G \to \mathcal{O}_C(2,2) \to 0$. By the snake lemma (Consult the proof of [CM15, Lemma 3.7].), G fits into $0 \to \mathcal{O}_Q(2,2) \to G \to \mathcal{O}_\ell \to 0$ where ℓ is the line of type $(0,1)$ determined by the section s. So $\mathrm{Hom}(\mathcal{O}_Q(2,2), G) \neq 0$ and this implies G does not admit a resolution $\mathcal{O}_Q(0,1) \to \mathcal{O}_Q(1,2)^{\oplus 2}$. Thus the correspondence is not well-defined on $\mathbf{M}_2^+$. □

3.2. **The first elementary modification and the extension of the domain.** We can extend the morphism in Lemma 3.3 by applying an elementary modification of pairs ([CC16, Section 2.2]) on $\mathbf{M}_2^+$.

Lemma 3.4. *There exists an exact sequence of pairs $0 \to (0, K) \to (\mathcal{L}^*|_{\mathbf{M}_2^+}, \mathcal{E}|_{\mathbf{M}_2^+ \times Q}) \to (\mathcal{L}'', \mathcal{O}_{\mathcal{Z}}) \to 0$ where $\mathcal{Z}$ is the pull-back of the universal family of $(0,1)$-lines to $\mathbf{M}_2^+ \times Q$ and $K_{\{m\} \times Q} \cong \mathcal{O}_Q(2,2)$ for $m = [(s, F)] \in \mathbf{M}_2^+$.*

Proof. The last part of the proof of Lemma 3.3 tells us that there is an exact sequence of *sheaves* $0 \to K \to \mathcal{E}|_{\mathbf{M}_2^+ \times Q} \to \mathcal{O}_{\mathcal{Z}} \to 0$. Now it is sufficient to show that for each fiber $G = \mathcal{E}|_{\{(s,F)\} \times Q}$, the section s^* of G does not come from $\mathrm{H}^0(\mathcal{O}_Q(2,2))$. If it is, we have an injection $\mathcal{O}_Q \subset \mathcal{O}_Q(2,2)$ whose cokernel is $\mathcal{O}_E(2,2)$ for some elliptic curve E. By the snake lemma once again, we obtain $0 \to \mathcal{O}_E(2,2) \to F^D(2,2) = \mathcal{O}_C(2,2) \to \mathcal{O}_\ell \to 0$. It violates the stability of $F^D(2,2)$. □

Let $(\mathcal{L}', \mathcal{E}')$ be the elementary modification of $(\mathcal{L}^*, \mathcal{E})$ along $\mathbf{M}_2^+$, that is,

$$\mathrm{Ker}((\mathcal{L}^*, \mathcal{E}) \twoheadrightarrow (\mathcal{L}^*|_{\mathbf{M}_2^+}, \mathcal{E}|_{\mathbf{M}_2^+ \times Q}) \twoheadrightarrow (\mathcal{L}^{''}, \mathcal{O}_{\mathcal{Z}})).$$

Lemma 3.5. *For a point $m = [(s, F = O_C(0,1))] \in \mathbf{M}_2^+$, the modified pair $(\mathcal{L}', \mathcal{E}')|_{\{m\}\times Q}$ fits into a non-split exact sequence $0 \to (s', \mathcal{O}_\ell) \to (s', \mathcal{E}'|_{\{m\}\times Q}) \to (0, \mathcal{O}_Q(2,2)) \to 0$ where ℓ is a $(0,1)$-line.*

Proof. An elementary modification of pairs interchanges the sub pair with the quotient pair ([He98, Lemma 4.24]). Thus we obtain the sequence. It remains to show that the sequence is non-split. We will show that the normal bundle $\mathcal{N}_{\mathbf{M}_2^+/\mathbf{M}^+}$ at m is canonically isomorphic to $\mathrm{H}^0(\mathcal{O}_\ell)^*$. Then the element m corresponds to the projective equivalent class of nonzero elements in $\mathrm{H}^0(\mathcal{O}_\ell)^* \cong \mathrm{Ext}^1((0, \mathcal{O}_Q(2,2)), (s', \mathcal{O}_\ell))$, so it is non-split.

The pair (s, F) fits into $0 \to (0, \mathcal{O}_Q(-2,-2)) \to (s, \mathcal{O}_Q(0,1)) \to (s, F) \to 0$. Thus we have

$$0 \to \mathrm{Ext}^0((0, \mathcal{O}_Q(-2,-2)), (s, F)) \to \mathrm{Ext}^1((s, F), (s, F)) \to \mathrm{Ext}^1((s, \mathcal{O}_Q(0,1)), (s, F)) \to \cdots.$$

The first term $\mathrm{Ext}^0((0, \mathcal{O}_Q(-2,-2)), (s, F)) \cong \mathrm{H}^0(\mathcal{O}_C(2,3)) \cong \mathbb{C}^{11}$ is the deformation space of curves C on Q. The second term $\mathrm{Ext}^1((s, F), (s, F))$ is $\mathcal{T}_m\mathbf{M}^+$ ([He98, Theorem 3.12]). For the third term, by [He98, Theorem 3.12], we have

$$0 \to \mathrm{Hom}(s, \mathrm{H}^0(F)/\langle s\rangle) \to \mathrm{Ext}^1((s, \mathcal{O}_Q(0,1)), (s, F)) \to \mathrm{Ext}^1(\mathcal{O}_Q(0,1), F) \xrightarrow{\phi} \mathrm{Hom}(s, \mathrm{H}^1(F)).$$

The first term $\mathrm{Hom}(s, \mathrm{H}^0(F)/\langle s\rangle) = \mathbb{C}$ is the deformation space of the line ℓ in Q determined by the section s. By Serre duality, $\phi : \mathrm{H}^0(\mathcal{O}_Q(0,1))^* \to \mathrm{H}^0(\mathcal{O}_Q)^*$ and the kernel is $\mathrm{H}^0(\mathcal{O}_\ell(0,1))^* \cong \mathrm{H}^0(\mathcal{O}_\ell)^*$. This proves our assertion. □

Recall that the modified pair $(\mathcal{L}', \mathcal{E}')$ provides a natural surjection $\mathcal{E}xt^2_\pi(\mathcal{E}', \omega_\pi) \twoheadrightarrow \mathcal{L}'$ on $\mathbf{M}^+ \times Q$. It is straightforward to check that $\mathcal{E}xt^2_\pi(\mathcal{E}', \omega_\pi)$ has rank 10 at each fiber, thus it is locally free.

Proof of Proposition 3.2. We claim that there exists a surjection $w^*\mathcal{U}^* \to \mathcal{L}' \to 0$ up to a twisting by a line bundle on $\mathbf{M}^+ \setminus \mathbf{M}_1^+$. Then there is a morphism $\mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P}$.

Consider the following commutative diagram

$$\begin{tikzcd} (\mathbf{M}^+ \setminus \mathbf{M}_1^+) \times Q \arrow[r, "w' := w\times \mathrm{id}"'] \arrow[d, "\pi"'] & \mathbf{G} \times Q \arrow[d, "\pi"] \\ \mathbf{M}^+ \setminus \mathbf{M}_1^+ \arrow[r, "w"] & \mathbf{G}. \end{tikzcd}$$

Note that $\mathcal{U} = \pi_*(\mathcal{W})$ where $\mathcal{W} = \mathrm{coker}(\phi)$ is the universal quotient on $\mathbf{G} \times Q$ (Section 2.1). One can check that $\mathcal{W}$ is flat over $\mathbf{G}$. By its construction of w, $\mathcal{E}'|_{\{m\}\times Q} \cong w'^*\mathcal{W}|_{\{m\}\times Q}$ restricted to each point $m \in \mathbf{M}^+ \setminus \mathbf{M}_1^+$. The universal property of $\mathbf{G}$ (as a quiver representation space [Kin94, Proposition 5.6]) tells us that $w'^*\mathcal{W} \cong \mathcal{E}'$ up to a twisting by a line bundle on $\mathbf{M}^+ \setminus \mathbf{M}_1^+$. The base change property implies that there exists a natural isomorphism (up to a twisting by a line bundle) $w^*\mathcal{U} = w^*(\pi_*\mathcal{W}) \cong \pi_*(w'^*\mathcal{W}) = \pi_*\mathcal{E}' \cong \mathcal{E}xt^2_\pi(\mathcal{E}', \omega_\pi)^*$ by [LP93, Corollary 8.19]. Hence we have $w^*\mathcal{U}^* \cong (w^*\mathcal{U})^* \cong (\pi_*(\mathcal{E}'))^* \cong \mathcal{E}xt^2_\pi(\mathcal{E}', \omega_\pi) \twoheadrightarrow \mathcal{L}'$. Therefore we obtain a morphism $q : \mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P}$.

By the proof of Lemma 3.5, the modified pair does not depend on the choice of a $(2,3)$-curve, so $q : \mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P} \setminus p^{-1}(t(Y_{10}))$ is indeed a contraction of $\mathbf{M}_2^+$ and the image of $\mathbf{M}_2^+$ is Y_{01}. Recall that the exceptional divisor $\mathbf{M}_2^+$ is $|\mathcal{O}_Q(2,3)| \times |\mathcal{O}_Q(0,1)| \cong \mathbb{P}^{11} \times \mathbb{P}^1$. Note that the sheaf F in the pair $(s, F) \in \mathbf{M}_2^+$ is parametrized by $\mathbb{P}^{11} = |\mathcal{O}_Q(2,3)| = \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_Q(-2,-2)[1], \mathcal{O}_Q(0,1))$. It follows also from the fact that each F fits into a triangle $0 \to \mathcal{O}_Q(0,1) \to F \to \mathcal{O}_Q(-2,-2)[1] \to 0$. By analyzing $T_F\mathbf{M} = \mathrm{Ext}^1(F,F)$ (which is similar to [CC16, Lemma 3.4]), one can see that $\mathcal{N}_{\mathbf{M}_2/\mathbf{M}}|_{\mathbb{P}^{11}} \cong \mathrm{Ext}^1(\mathcal{O}_Q(0,1), \mathcal{O}_Q(-2,-2)[1]) \otimes \mathcal{O}_{\mathbb{P}^{11}}(-1) \cong \mathrm{H}^0(\mathcal{O}_Q(0,1))^* \otimes \mathcal{O}_{\mathbb{P}^{11}}(-1)$. Thus $\mathcal{N}_{\mathbf{M}_2^+/\mathbf{M}^+} \cong \mathcal{O}_{\mathbb{P}^{11}\times\mathbb{P}^1}(-1,-1)$ and q is a smooth blow-down by Fujiki-Nakano criterion. □

Thus we have two different contractions of $\mathbf{M}^+$, one is $\mathbf{M}$ obtained by contracting all $\mathbb{P}^1$-fibers on $\mathbf{M}_2^+$, and the other is:

Definition 3.6. Let $\mathbf{M}^-$ be the contraction of $\mathbf{M}^+$ which is obtained by contracting all $\mathbb{P}^{11}$-fibers on $\mathbf{M}_2^+$. We define $\mathbf{M}_i^-$ as the image of $\mathbf{M}_i^+$ for the contraction $\mathbf{M}^+ \to \mathbf{M}^-$.

3.3. **The second elementary modification and $\mathbf{M}^-$.** Recall that $u : \mathbf{G}_1 \to \mathbf{G}$ is the blow-up of $\mathbf{G}$ along the $\mathbb{P}^1$ parameterizing $(1,0)$-lines in Q, and Y_{10} is the exceptional divisor. Let $\mathcal{W}$ be the cokernel of the universal morphism ϕ on $\mathbf{G} \times Q$ in Section 2.1. Let $\mathcal{V} := (u \times \mathrm{id})^*\mathcal{W}$ be the pull-back of $\mathcal{W}$ along the map $u \times \mathrm{id} : \mathbf{G}_1 \times Q \to \mathbf{G} \times Q$. Then for $([\ell], t) \in Y_{10}$, $\mathcal{V}|_{([\ell],t)\times Q}$ fits into a non-split exact sequence $0 \to \mathcal{O}_\ell(1) \to \mathcal{V}|_{([\ell],t)\times Q} \to \mathcal{O}_Q(1,3) \to 0$. By relativizing it over $Y_{10} \times Q$, we obtain $0 \to \mathcal{S} \to \mathcal{V}|_{Y_{10}\times Q} \to \mathcal{Q} \to 0$. Let $\mathcal{V}^-$ be the elementary modification $\mathrm{elm}_{Y_{10}\times Q}(\mathcal{V}, \mathcal{Q}) := \ker(\mathcal{V} \twoheadrightarrow \mathcal{V}|_{Y_{10}\times Q} \twoheadrightarrow \mathcal{Q})$ along $Y_{10} \times Q$. Note that over $([\ell], t) \in \mathbf{G}_1$, $\mathcal{V}^-|_{([\ell],t)\times Q}$ fits into a non-split exact sequence $0 \to \mathcal{O}_Q(1,3) \to \mathcal{V}^-|_{([\ell],t)\times Q} \to \mathcal{O}_\ell(1) \to 0$ because the elementary modification interchanges the sub/quotient sheaves. Let $\pi_1 : \mathbf{G}_1 \times Q \to \mathbf{G}_1$ be the projection into the first factor. Then $\mathcal{U}^- := \pi_{1*}\mathcal{V}^-$ is a rank 10 bundle over $\mathbf{G}_1$. Let $\mathbf{P}^- := \mathbb{P}(\mathcal{U}^-)$.

The following proposition completes the proof of Theorem 1.3.

Proposition 3.7. *The projective bundle $\mathbf{P}^-$ is isomorphic to $\mathbf{M}^-$ in Definition 3.6.*

Proof. Since the elementary modification has been done locally around $Y_{10}\times Q$, $\mathbb{P}(u^*\mathcal{U})$ and $\mathbf{P}^-$ are isomorphic over $\mathbf{G}_1 \setminus Y_{10}$. On the other hand, set theoretically, it is straightforward to see that the image of q is $\mathbf{P} \setminus p^{-1}(t(Y_{10}))$, where $p : \mathbf{P} \to \mathbf{G}$ is the structure morphism. So we have a birational morphism $\mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P} \setminus p^{-1}(t(Y_{10})) \cong \mathbb{P}(u^*\mathcal{U}) \setminus p^{-1}(Y_{10}) \cong \mathbf{P}^- \setminus p^{-1}(Y_{10})$ (here we used the same notation p for the projections $\mathbb{P}(u^*\mathcal{U}) \to \mathbf{G}_1$ and $\mathbf{P}^- \to \mathbf{G}_1$). By Proposition 3.2, this map is a blow-down along $\mathbf{M}_2^+$, thus we have an isomorphism $\tau : \mathbf{P}^- \setminus p^{-1}(Y_{10}) \to \mathbf{M}^- \setminus \mathbf{M}_1^-$. So we have a birational map $\tau : \mathbf{P}^- \dashrightarrow \mathbf{M}^-$, where its undefined locus is $p^{-1}(Y_{10})$.

On the other hand, since the flipped locus for $\mathbf{M}^\infty \dashrightarrow \mathbf{M}^+$ is $\mathbf{M}_3^+$, we have an isomorphism $\mathbf{M}^- \setminus (\mathbf{M}_2^- \cup \mathbf{M}_3^-) \cong \mathbf{M}^+ \setminus (\mathbf{M}_2^+ \cup \mathbf{M}_3^+) \cong \mathbf{M}^\infty \setminus (\mathbf{M}_2^\infty \cup \mathbf{M}_3^\infty)$ (Here $\mathbf{M}_i^\infty$ is defined in an obvious way.). Also $\tau^{-1}(\mathbf{M}_2^- \cup \mathbf{M}_3^-) = p^{-1}(Y_{01})$. Hence if we restrict the domain of τ, then we have $\sigma : \mathbf{P}^- \setminus p^{-1}(Y_{01}) \dashrightarrow \mathbf{M}^- \setminus (\mathbf{M}_2^- \cup \mathbf{M}_3^-) \cong \mathbf{M}^\infty \setminus (\mathbf{M}_2^\infty \cup \mathbf{M}_3^\infty)$ whose undefined locus is $p^{-1}(Y_{10})$. Therefore σ can be regarded as a map into a relative Hilbert scheme. Note that $\mathbf{M}_2^\infty \cup \mathbf{M}_3^\infty$ is the locus of $(2,3)$-curves passing through two points lying on a $(0,1)$-line.

We claim that σ is extended to a morphism $\tilde{\sigma} : \mathbf{P}^- \setminus p^{-1}(Y_{01}) \to \mathbf{M}^-$ such that $\tilde{\sigma}(p^{-1}(Y_{10})) = \mathbf{M}_1^- \cong \mathbf{M}_1^\infty$. To show this, it is enough to check that $\mathcal{V}^-$ over Y_{10} provides a flat family of the twisted ideal sheaf of Hilbert scheme of two points lying on $(1,0)$-type lines. Note that $\mathcal{V}^-$ fits into a non-split extension $0 \to \mathcal{O}_Q(1,3) \to \mathcal{V}^-|_{([\ell],t)\times Q} \to \mathcal{O}_\ell(1) \to 0$. By a diagram chasing similar to the second paragraph of the proof of Lemma 3.3, one can check that $\mathcal{V}^-|_{([\ell],t)\times Q} \cong I_{Z,Q}(2,3)$ where $Z \subset \ell$ and ℓ is a $(1,0)$-line.

Now two maps τ and $\tilde{\sigma}$ coincide over the intersection $\mathbf{P}^- \setminus p^{-1}(Y_{10} \cup Y_{01})$ of domains, so we have a birational morphism $\mathbf{P}^- \to \mathbf{M}^-$. Since $\rho(\mathbf{P}^-) = 3 = \rho(\mathbf{M}^-)$ and both of them are smooth, this map is an isomorphism. □

The modification on $\mathbf{G}_1 \times Q$ descends to $\mathbf{G}_1$. Then Proposition 1.2 follows from a general result of Maruyama ([Mar73]).

Proof of Proposition 1.2. Let $\pi_1 : \mathbf{G}_1 \times Q \to \mathbf{G}_1$ be the projection. We claim that $\mathcal{U}^- = \mathrm{elm}_{Y_{10}}(u^*\mathcal{U}, \pi_{1*}\mathcal{Q}) \cong \pi_{1*}\mathrm{elm}_{Y_{10}\times Q}(\mathcal{V}, \mathcal{Q})$. Indeed, from $0 \to \mathcal{V}^- \to \mathcal{V} \to \mathcal{Q} \to 0$, we have $0 \to \pi_{1*}\mathcal{V}^- \to \pi_{1*}\mathcal{V} = u^*\mathcal{U} \to \pi_{1*}\mathcal{Q} \to R^1\pi_{1*}\mathcal{V}^- \to R^1\pi_{1*}\mathcal{V}$. It is sufficient to show that $R^1\pi_{1*}\mathcal{V}^- = 0$. By using the resolution of $\mathcal{V}$ given by the universal morphism ϕ, we have $R^1\pi_{1*}\mathcal{V} = 0$. Over $\mathbf{G}_1 \setminus Y_{10}$, the last two terms are isomorphic. Over Y_{10}, from $\mathrm{H}^1(\mathcal{O}_Q(1,3)) = \mathrm{H}^1(\mathcal{O}_\ell(1)) = 0$ and the description of $\mathcal{V}^-|_{([\ell],t)}$, we obtain $R^1\pi_{1*}\mathcal{V}^- = 0$.

Note that $u^*\mathcal{U}|_{Y_{10}}$ fits into a *vector bundle* sequence $0 \to \pi_{1*}\mathcal{S} \to u^*\mathcal{U}|_{Y_{10}} \to \pi_{1*}\mathcal{Q} \to 0$ and $\mathrm{rank}\ \pi_{1*}\mathcal{S} = 2$ and $\mathrm{rank}\ \pi_{1*}\mathcal{Q} = 8$. The result follows from [Mar73, Theorem 1.3]. □

As a direct application of Theorem 1.3, we compute the Poincaré polynomial of $\mathbf{M}$ which matches with the result in [Mai16, Theorem 1.2].

Corollary 3.8. *(1) The moduli space* $\mathbf{M}$ *is rational;*

(2) The Poincaré polynomial of $\mathbf{M}$ *is*

$$P(\mathbf{M}) = q^{13} + 3q^{12} + 8q^{11} + 10q^{10} + 11q^9 + 11q^8 + 11q^7 + 11q^6 + 11q^5 + 11q^4 + 10q^3 + 8q^2 + 3q + 1.$$

Proof. Now $\mathbf{M}$ is birational to a $\mathbb{P}^9$-bundle over $\mathbf{G}$, so we obtain Item (1). Item (2) is a straightforward calculation using

$$P(\mathbf{M}) = P(\mathbb{P}^{11}) - P(\mathbb{P}^1) + P(\mathbf{M}^-) = P(\mathbb{P}^{11}) - P(\mathbb{P}^1) + P(\mathbb{P}^9)(P(\mathbf{G}) + (P(\mathbb{P}^2) - 1)P(\mathbb{P}^1)).$$

□

References

[BC13] Aaron Bertram and Izzet Coskun. The birational geometry of the Hilbert scheme of points on surfaces. In *Birational geometry, rational curves, and arithmetic*, pages 15–55. Springer, New York, 2013. 2

[CC16] Jinwon Choi and Kiryong Chung. Moduli spaces of -stable pairs and wall-crossing on $\mathbb{P}^2$. *J. Math. Soc. Japan*, 68(2):685–789, 2016. 1, 4, 5, 7

[CM15] Kiryong Chung and Han-Bom Moon. Chow ring of the moduli space of stable sheaves supported on quartic curves. arXiv:1506.00298, To appear in Quarterly Journal of Mathematics, 2015. 1, 4, 5

[He98] Min He. Espaces de modules de systèmes cohérents. *Internat. J. Math.*, 9(5):545–598, 1998. 1, 3, 4, 5, 6

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. 1

[Kin94] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994. 2, 3, 6

[LP93] Joseph Le Potier. Systèmes cohérents et structures de niveau. *Astérisque*, (214):143, 1993. 1, 3, 4, 6

[Mai16] Mario Maican. Moduli of sheaves supported on curves of genus two in a quadric surface. *arXiv:1612.03566*, 2016. 2, 3, 8

[Mar73] M. Maruyama. On a family of algebraic vector bundles. *Number Theory, Algebraic Geometry, and Commutative Algebra*, pages 95–149, 1973. 1, 8

DEPARTMENT OF MATHEMATICS EDUCATION, KYUNGPOOK NATIONAL UNIVERSITY, 80 DAEHAKRO, BUKGU, DAEGU 41566, KOREA

E-mail address: krchung@knu.ac.kr

DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NY 10458

E-mail address: hmoon8@fordham.edu