BIRATIONAL GEOMETRY OF THE MODULI SPACE OF PURE SHEAVES ON QUADRIC SURFACE

KIRYONG CHUNG AND HAN-BOM MOON

ABSTRACT. We study birational geometry of the moduli space of stable sheaves on a quadric surface with Hilbert polynomial $5m + 1$ and $c_1 = (2, 3)$. We describe a birational map between the moduli space and a projective bundle over a Grassmannian as a composition of smooth blow-ups/downs.

1. INTRODUCTION

The geometry of the moduli space of sheaves on a del Pezzo surface has been studied in various viewpoints, for instance curve counting, the strange duality conjecture, and birational geometry via Bridgeland stability. For a detailed description of the motivation, see [CM15] and references therein. In this paper we continue the study of birational geometry of the moduli space of torsion sheaves on a del Pezzo surface, which was initiated in [CM15]. More precisely, here we construct a flip between the moduli space of sheaves and a projective bundle, and show that their common blown-up space is the moduli space of stable pairs ([LP93]), in the case of a quadric surface.

Let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric surface in $\mathbb{P}^3$ with a very ample polarization $L := \mathcal{O}_Q(1, 1)$. For the convenience of the reader, we start with a list of relevant moduli spaces.

Definition 1.1. (1) Let $M := M_L(Q, (2, 3), 5m + 1)$ be the moduli space of stable sheaves $F$ on $Q$ with $c_1(F) = c_1(\mathcal{O}_Q(2, 3))$ and $\chi(F(m)) = 5m + 1$.

(2) Let $M^\alpha := M^\alpha_L(Q, (2, 3), 5m + 1)$ be the moduli space of $\alpha$-stable pairs $(s, F)$ with $c_1(F) = c_1(\mathcal{O}_Q(2, 3))$ and $\chi(F(m)) = 5m + 1$ ([LP93] and [He98, Theorem 2.6]).

(3) Let $G = \text{Gr}(2, 4)$ and let $G_1$ be the blow-up of $G$ along $\mathbb{P}^1$ (Section 2.1).

(4) Let $P := \mathbb{P}(\mathcal{U})$ and $P^- := \mathbb{P}(\mathcal{U}^-)$, where $\mathcal{U}$ (resp. $\mathcal{U}^-$) is a rank 10 vector bundle over $G$ (resp. $G_1$) defined in (2) in Section 2.1 (resp. Section 3.3).

The aim of this paper is to explain and justify the following commutative diagram between moduli spaces.

\[
\begin{array}{ccccccc}
M^+ & \longrightarrow & P^- & \longrightarrow & \mathbb{P}(\mathcal{U}^-) & \longrightarrow & \mathbb{P}(u^*\mathcal{U}) = G_1 \times_G P & \longrightarrow & P = \mathbb{P}(\mathcal{U}) \\
\downarrow r & & \downarrow & & \downarrow & & \downarrow & & \\
M & \longrightarrow & G_1 & \longrightarrow & G
\end{array}
\]

We have to explain two flips (dashed arrows) on the diagram.

One of key ingredients is the elementary modification of vector bundles ([Mar73]), sheaves ([HL10, Section 2.B]), and pairs ([CC16, Section 2.2]). It has been widely used in the study of sheaves on...
a smooth projective variety. Let \( F \) be a vector bundle on a smooth projective variety \( X \) and \( Q \) be a vector bundle on a smooth divisor \( Z \subset X \) with a surjective map \( F|_Z \to Q \). The elementary modification of \( F \) along \( Z \) is the kernel of the composition

\[
\text{elm}_Z(F) := \ker(F \to F|_Z \to Q).
\]

A similar definition is valid for sheaves and pairs, too.

On \( G_1 \), let \( U^- := \text{elm}_{Y_{10}}(u^*U) \) be the elementary transformation of \( u^*U \) along a smooth divisor \( Y_{10} \) (Section 2.1).

**Proposition 1.2.** Let \( P^- = \mathbb{P}(U^-) \). The flip \( P^- \dashrightarrow \mathbb{P}(u^*U) = G_1 \times_G \mathbb{P}(U) \) is a composition of a blow-up and a blow-down. The blow-up center in \( P^- \) (resp. \( \mathbb{P}(u^*U) \)) is a \( \mathbb{P}^1 \) (resp. \( \mathbb{P}^7 \))-bundle over \( Y_{10} \).

**Theorem 1.3.** There is a flip between \( M \) and \( P^- \) which is a blow-up followed by a blow-down, and the master space is \( M^+ \), the moduli space of \(+\)-stable pairs.

As applications, we compute the Poincaré polynomial of \( M \) and show the rationality of \( M \) (Corollary 3.8) which were obtained by Maican by different methods ([Mai16]). Since each step of the birational transform is described in terms of blow-ups/downs along explicit subvarieties, in principle the cohomology ring and the Chow ring of \( M \) can be obtained from that of \( G \). Also one may aim for the completion of Mori’s program for \( M \). We will carry on these projects in forthcoming papers.

## 2. Relevant moduli spaces

In this section we give definitions and basic properties of some relevant moduli spaces.

### 2.1. Grassmannian as a moduli space of Kronecker quiver representations

The moduli space of representations of a Kronecker quiver parametrizes the isomorphism classes of stable sheaf homomorphisms

\[
O_Q(0, 1) \longrightarrow O_Q(1, 2)^{\oplus 2}
\]

up to the natural action of the automorphism group \( \mathbb{C}^* \times GL_2/\mathbb{C}^* \cong GL_2 \). For two vector spaces \( E \) and \( F \) of dimension 1 and 2 respectively and \( V^* := H^0(Q, L) \), the moduli space is constructed as \( G := \text{Hom}(F, V^* \otimes E)/GL_2 \cong V^* \otimes E \otimes F^*/GL_2 \) with an appropriate linearization ([Kin94]). Note that the GL_2 acts as a row operation on the space of \( 2 \times 4 \) matrices, \( G \cong \text{Gr}(2, 4) \).

Let \( H(n) := \text{Hilb}^n(Q) \), the Hilbert scheme of \( n \) points on \( Q \). \( H(2) \) is birational to \( G \) because a general \( Z \in H(2) \), \( I_Z(2, 3) \) has a resolution of the form (1). For any \( Z \in H(2) \), let \( \ell_Z \) be the unique line in \( \mathbb{P}^3 \supset Q \) containing \( Z \). Then either \( \ell_Z \cap Q = Z \) or \( \ell_Z \subset Q \). In the second case, the class of \( \ell_Z \) is of the type \((1, 0)\) or \((0, 1)\). Let \( Y_{10} \) (resp. \( Y_{01} \)) be the locus of subschemes such that \( \ell_Z \) is a line of the type \((1, 0)\) (resp. \((0, 1)\)). Then \( Y_{10} \) and \( Y_{01} \) are two disjoint subvarieties which are isomorphic to a \( \mathbb{P}^2 \)-bundle over \( \mathbb{P}^1 \).

**Proposition 2.1 ([BC13, Example 6.1]).** There exists a morphism \( t : H(2) \longrightarrow G_1 \xrightarrow{u} G \). The first (resp. the second) map contracts the divisor \( Y_{01} \) (resp. \( Y_{10} \)) to \( \mathbb{P}^1 \). If \( \ell_Z \cap Q = Z \), then \( t(Z) = I_Z(2, 3) \).

If \( Z \in Y_{10} \), then \( t(Z) = E_{10} \in \mathbb{P}(\text{Ext}^1(O_Q(1, 3), O_{\ell_2}(1))) = \{\text{pt}\} \). If \( Z \in Y_{01} \), then \( t(Z) = E_{01} \in \mathbb{P}(\text{Ext}^1(O_Q(2, 2), O_{\ell_2})) = \{\text{pt}\} \).
There is a universal morphism \( \phi : p_1^* \mathcal{F} \otimes p_2^* \mathcal{O}_Q(0,1) \to p_1^* \mathcal{E} \otimes p_2^* \mathcal{O}_Q(1,2) \) where \( p_1 : G \times Q \to G \) and \( p_2 : G \times Q \to Q \) are two projections ([Kin94]). Let \( \mathcal{U} \) be the cokernel of \( p_{1*} \phi \). On the stable locus, \( p_{1*} \phi \) is injective. Thus we have an exact sequence
\[
0 \to \mathcal{F} \otimes H^0(\mathcal{O}_Q(0,1)) \to \mathcal{E} \otimes H^0(\mathcal{O}_Q(1,2)) \to \mathcal{U} \to 0
\]
and \( \mathcal{U} \) is a rank 10 vector bundle. Let \( \mathbb{P} := \mathbb{P}(\mathcal{U}) \).

2.2. Moduli space \( M \) of stable sheaves. Recall that \( M := M_L(Q, (2,3), 5m + 1) \) is the moduli space of stable sheaves \( F \) on \( Q \) with \( c_1(F) = c_1(\mathcal{O}_Q(2,3)) \) and \( \chi(F(m)) = 5m + 1 \). There are four types of points in \( M \) ([Mai16, Theorem 1.1]). Let \( C \in |\mathcal{O}_Q(2,3)| \).

1. \( F = \mathcal{O}_C(p + q) \), where the line \( (p, q) \) is not contained in \( Q \);
2. \( F = \mathcal{O}_C(p + q) \), where the line \( (p, q) \) in \( Q \) is of type \((1,0)\);
3. \( F = \mathcal{O}_C(0,1) \);
4. \( F \) fits into a non-split extension \( 0 \to \mathcal{O}_E \to F \to \mathcal{O}_\ell \to 0 \) where \( E \) is a \((2,2)\)-curve and \( \ell \) is a \((0,1)\)-line.

Let \( M_i \) be the locus of sheaves of the form \((i)\). Then \( M_i \) is a subvariety of codimension \(i\). \( M_1 \) is a \( \mathbb{P}^3 \)-bundle over \( \mathbb{P}^2 \times \mathbb{P}^1 \). \( M_2 \) is isomorphic to \( |\mathcal{O}_Q(2,3)| \). Finally, \( M_3 \) is a \( \mathbb{P}^1 \)-bundle over \( |\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)| \). \( M_1 \cap M_2 = M_1 \cap M_3 = \emptyset \), but \( M_{23} := M_2 \cap M_3 \cong |\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)| \) ([Mai16, Theorem 1.1]). Note that \( \dim H^0(F) = 1 \) in general, but \( M_2 \) parametrizes sheaves that \( \dim H^0(F) = 2 \).

2.3. Moduli spaces of stable pairs. A pair \((s, F)\) consists of \( F \in \text{Coh}(Q) \) and a section \( \mathcal{O}_Q \xrightarrow{\delta} F \). Fix \( \alpha \in \mathbb{Q}_{>0} \). A pair \((s, F)\) is called \( \alpha \)-semistable (resp. \( \alpha \)-stable) if \( F \) is pure and for any proper subsheaf \( F' \subset F \), the inequality
\[
\frac{P(F')(m) + \delta \cdot \alpha}{r(F')} \leq \left(\frac{P(F)(m) + \alpha}{r(F)}\right)
\]
holds for \( m \gg 0 \). Here \( \delta = 1 \) if the section \( s \) factors through \( F' \) and \( \delta = 0 \) otherwise. Let \( M^\alpha := M^\alpha_L(Q, (2,3), 5m + 1) \) be the moduli space of \( S \)-equivalence classes of \( \alpha \)-semistable pairs whose support have a class \( c_1(\mathcal{O}_Q(2,3)) \) ([LP93, Theorem 4.12] and [He98, Theorem 2.6]). The extremal case that \( \alpha \) is sufficiently large (resp. small) is denoted by \( \alpha = \infty \) (resp. \( \alpha = + \)). The deformation theory of pairs is studied in [He98, Corollary 1.6 and Corollary 3.6].

**Proposition 2.2.**

1. There exists a natural forgetful map \( r : M^+ \to M \) which maps \((s, F)\) to \( F \).
2. ([He98, Section 4.4]) The moduli space \( M^\infty \) of \( \infty \)-stable pairs is isomorphic to the relative Hilbert scheme of two points on the complete linear system \(|\mathcal{O}_Q(2,3)|\).

The birational map \( M^\infty \dashrightarrow M^+ \) is analyzed in [Mai16, Theorem 5.7]. It turns out that this is a single flip over \( M^4 \) and is a composition of a smooth blow-up and a smooth blow-down. The blow-up center \( M^*_3 \) is isomorphic to a \( \mathbb{P}^2 \)-bundle over \(|\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)| \) where a fiber \( \mathbb{P}^2 \) parameterizes two points lying on a \((0,1)\)-line. After the flip, the flipped locus on \( M^+ \) is \( M^+_3 \).

For the forgetful map \( r : M^+ \to M \), we define \( M^+_i := r^{-1}(M_i) \) if \( i \neq 3 \) and \( M^+_3 \) is the proper transform of \( M_3 \). It contracts \( M^+_3 \), which is a \( \mathbb{P}^1 \)-bundle over \( M_2 \) and \( M^+ \setminus M^+_3 \cong M \setminus M_2 \). Maican proved that \( r \) is a smooth blow-up along the Brill-Noether locus \( M_2 \) ([Mai16, Proposition 5.8]).
3. Decomposition of the birational map between $M$ and $P$

In this section we prove Proposition 1.2 and Theorem 1.3 by describing the birational map between $M$ and $P$.

3.1. Construction of a birational map $M^+ \rightarrow P$.

**Lemma 3.1.** There exists a surjective morphism $w : M^+ \rightarrow G$ which maps $(s, \mathcal{O}_C(p + q)) \in M^+_0$ to $I_{\{p,q\}}(2,3)$, maps $(s, \mathcal{O}_C(p + q)) \in M^+_1$ to the line $(p, q)$ of the type $(1, 0)$, maps $(s, F) \in M^+_2$ to a $(0, 1)$-line determined by a section, and maps $(s, F) \in M^+_3$ to $\ell$ (see Section 2.2 for the notation), a $(0, 1)$-line.

*Proof.* By Proposition 2.2, $M^\infty$ is the relative Hilbert scheme of 2 points on the universal $(2, 3)$-curves, which is a $\mathbb{P}^9$-bundle over $H(2)$ ([CC16, Lemma 2.3]). By composing with $t : H(2) \rightarrow G$ in Proposition 2.1, we have a morphism $M^\infty \rightarrow G$. On the other hand, since the flip $M^\infty \rightarrow M^+$ is the composition of a single blow-up/down, the blown-up space $\widetilde{M}^\infty$ admits two morphisms to $M^\infty$ and $M^+$, and the flipped locus is $M^+_3$. Note that each point in $M^+_3$ can be regarded as a collection of data $(E, \ell, e)$ where $E$ is a $(2, 2)$-curve, $\ell$ is a $(0, 1)$-line, and $e \in \mathbb{P}^1 \mathcal{E}xt^1(\mathcal{O}_E, \mathcal{O}_E)$. The fiber $\widetilde{M}^\infty \rightarrow M^+$ over the point in the blow-up center $M^+_3$ is a $\mathbb{P}^2$ which parameterizes two points on $\ell$. The composition map $\widetilde{M}^\infty \rightarrow M^\infty \rightarrow G$ is constant along the $\mathbb{P}^2$, because $G$ does not remember points on the line $\ell \subset Q$. By the rigidity lemma, $\widetilde{M}^\infty \rightarrow G$ factors through $M^+$ and we obtain a map $w : M^+ \rightarrow G$. \hfill $\Box$

Note that $M^+_1 \cong M_1$ is a $\mathbb{P}^9$-bundle over $\mathbb{P}^2 \times \mathbb{P}^1$ and $M^+_2$ is a $\mathbb{P}^1$-bundle over $|\mathcal{O}_Q(2, 3)| \cong \mathbb{P}^{11}$. They are disjoint divisors on $M^+$.

**Proposition 3.2.** There is a birational morphism $q : M^+ \setminus M^+_1 \rightarrow P = \mathbb{P}(U)$ such that $p \circ q : M^+ \setminus M^+_1 \rightarrow P \rightarrow G$ coincides with $w|_{M^+ \setminus M^+_1}$ in Lemma 3.1. Furthermore, $q$ is the smooth blow-down along $M^+_2$.

The proof consists of several steps. Since $P = \mathbb{P}(U)$ is a projective bundle over $G$, it is sufficient to construct a surjective homomorphism $w^*U^* \rightarrow L \rightarrow 0$ over $M^+ \setminus M^+_1$ for some $L \in \text{Pic}(M^+ \setminus M^+_1)$, or equivalently, a bundle morphism $0 \rightarrow L^* \rightarrow w^*U$.

Recall that a family $(\mathcal{L}, \mathcal{F})$ of pairs on a scheme $S$ is a collection of data $\mathcal{L} \in \text{Pic}(S), \mathcal{F} \in \text{Coh}(S \times Q)$, which is a flat family of pure sheaves, and a surjective morphism $\text{Ext}^2_{\pi}(\mathcal{F}, \omega_{\pi}) \rightarrow \mathcal{L}$ where $\pi : S \times Q \rightarrow S$ is the projection and $\omega_{\pi}$ is the relatively dualizing sheaf (See [LP93, Section 4.3] for the explanation why we take the dual.). Now let $(\mathcal{L}, \mathcal{F})$ be the universal pair ([He98, Theorem 4.8]) on $M^+ \times Q$. By applying $\text{Hom}(-, \mathcal{O})$ to $\text{Ext}^2_{\pi}(\mathcal{F}, \omega_{\pi}) \rightarrow \mathcal{L}$, we obtain $0 \rightarrow L^* \rightarrow \text{Hom}(\text{Ext}^2_{\pi}(\mathcal{F}, \omega_{\pi}), \mathcal{O})$. It can be shown that $\text{Hom}(\text{Ext}^2_{\pi}(\mathcal{F}, \omega_{\pi}), \mathcal{O}) \cong \text{Ext}^1_{\pi}(\text{Ext}^1_{\pi}(\mathcal{F}, \mathcal{O}), \mathcal{O})$ (see [CM15, Section 3.2]). So we have a non-zero element $e \in \text{Ext}^1_{\pi}(\text{Ext}^1_{\pi}(\mathcal{F}, \mathcal{O}), \mathcal{O}) \cong \text{Ext}^1(\mathcal{F}, \mathcal{O}, \pi^*\mathcal{L})$ ([CM15, Section 3.2]), which provides $0 \rightarrow \pi^*\mathcal{L} \rightarrow \mathcal{E} \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{O}) \rightarrow 0$ on $M^+ \times Q$. By taking $\text{Hom}_{\pi}(-, \omega_{\pi})$, we have $\text{Ext}^2_{\pi}(\mathcal{E}, \omega_{\pi}) \rightarrow \text{Ext}^2_{\pi}(\pi^*\mathcal{L}, \omega_{\pi}) \cong L^* \rightarrow 0$ because $\mathcal{L}$ is a line bundle. This implies the existence of a flat family of pairs $(\mathcal{L}^*, \mathcal{E})$ on $M^+ \times Q$. We may explicitly describe this construction fiberwisely in the following way. Let $(s, F) \in M^+$. Let $F^D := \mathcal{E}xt^1(F, \omega_Q)$. For a non-zero section $s \in H^0(F) \cong H^1(F^D)^* \cong \text{Ext}^1(F^D(2, 2), (s^*) \otimes \mathcal{O}_Q)$, we
have a pair \((s^*, G)\) given by
\[
0 \to (s^*) \otimes \mathcal{O}_Q \to G \to F^D(2, 2) \to 0.
\]

**Lemma 3.3.** The map \((s, F) \mapsto (s^*, G)\) defines a dominant rational map \(M^+ \dashrightarrow P = \mathbb{P}(U)\), which is regular on \(M^+ \setminus (M^+_1 \sqcup M^+_2)\).

**Proof.** Since we have a relative construction of pairs, it suffices to describe the extension \((s^*, G)\) set theoretically. If \((s, F) \in M^+_1 \sqcup M^+_2\), then \(F \cong \mathcal{O}_C(p + q) \cong I_{Z,C}(0, -1)\) for some curve \(C\) and \(Z = \{p, q\} \in \mathbb{H}(2)\) such that the line \(\ell_Z\) containing \(Z\) is not in \(Q\) ([He98, Section 4.4]). Then \(F^D(2, 2) \cong I_{Z,C}(2, 3)\). Since \(\text{Ext}^1(F^D(2, 2), \mathcal{O}_Q) \cong H^1(F^D)* \cong H^0(F) \cong \mathbb{C}\), from \(0 \to \mathcal{O}_Q(-2, -3) \cong I_{C,Q} \to I_{Z,Q} \to I_{Z,C} \to 0\), we obtain \(G = I_{Z,Q}(2, 3)\). If \((s, F) \in M^+_1\), then we have an element \((s^*, G)\) in \(P\) because \(G\) has a resolution of the form \(\mathcal{O}_Q(0, 1) \to \mathcal{O}_Q(1, 2)^{\oplus 2}\). However, if \((s, F) \in M^+_1\), then we have \(0 \to I_{\ell_{Z,Q}(2, 3)} \to G = I_{Z,Q}(2, 3) \to I_{Z,C}(2, 3) \to 0\) and \(I_{\ell_{Z,Q}(2, 3)} = \mathcal{O}_Q(1, 3)\).

In particular, \(\text{Hom}(\mathcal{O}_Q(1, 3), G) \neq 0\) and \(G\) does not admit a resolution \(\mathcal{O}_Q(0, 1) \to \mathcal{O}_Q(1, 2)^{\oplus 2}\). So \(G \not\in G\).

Suppose that \((s, F) \in M^+_2 \setminus M^+_2\). Then \(F\) fits into a non-split extension \(0 \to \mathcal{O}_E \to F \to \mathcal{O}_\ell \to 0\). Apply \(\text{Hom}(-, \omega_Q)\), then we have \(0 \to \mathcal{O}_E(0, 1) \to F^D(2, 2) \to \mathcal{O}_E(2, 2) \to 0\). Since \(\text{Ext}^1(\mathcal{O}_E(2, 2), \mathcal{O}_Q) \cong \text{Ext}^1(F^D(2, 2), \mathcal{O}_Q) \cong \mathbb{C}\), the sheaf \(G\) is given by the pull-back:
\[
\begin{array}{c}
0 \\ \downarrow \\
\mathcal{O}_Q \\
\downarrow \\
0 \\
\end{array} 
\begin{array}{c}
\mathcal{O}_Q(0, 2) \\
\downarrow \\
\mathcal{O}_Q(2, 2) \\
\downarrow \\
\mathcal{O}_Q(3, 2) \\
\downarrow \\
0 \\
\end{array} 
\begin{array}{c}
\mathcal{O}_E(0, 1) \\
\downarrow \\
\mathcal{O}_E(2, 2) \\
\downarrow \\
\mathcal{O}_E(3, 2) \\
\downarrow \\
0 \\
\end{array}
\begin{array}{c}
0 \\
\downarrow \\
G \\
\downarrow \\
F^D(2, 2) \\
\downarrow \\
0 \\
\end{array}
\]

By applying the snake lemma to (4), we conclude that the unique non-split extension \(G\) lies on \(0 \to \mathcal{O}_E(0, 1) \to G \to \mathcal{O}_Q(2, 2) \to 0\). Hence \(G \in G\) (Proposition 2.1) and we have an element \((s^*, G) \in P\).

Now suppose that \((s, F) \in M^+_2\), so \(F = \mathcal{O}_C(0, 1)\). Then \(F^D(2, 2) = \mathcal{O}_C(2, 2)\). So we have \(0 \to (s^*) \otimes \mathcal{O}_Q \to G \to \mathcal{O}_C(2, 2) \to 0\). By the snake lemma (Consult the proof of [CM15, Lemma 3.7].), \(G\) fits into \(0 \to \mathcal{O}_Q(2, 2) \to G \to \mathcal{O}_\ell \to 0\) where \(\ell\) is the line of type \((0, 1)\) determined by the section \(s\). So \(\text{Hom}(\mathcal{O}_Q(2, 2), G) \neq 0\) and this implies \(G\) does not admit a resolution \(\mathcal{O}_Q(0, 1) \to \mathcal{O}_Q(1, 2)^{\oplus 2}\). Thus the correspondence is not well-defined on \(M^+_2\).

**\(\square\)**

**3.2. The first elementary modification and the extension of the domain.** We can extend the morphism in Lemma 3.3 by applying an elementary modification of pairs ([CC16, Section 2.2]) on \(M^+_2\).

**Lemma 3.4.** There exists an exact sequence of pairs \(0 \to (0, K) \to (L^*, m) : E^*_1|_{M^+_2} \times Q \to (L^*, \mathcal{O}_Z) \to 0\) where \(Z\) is the pull-back of the universal family of \((0, 1)\)-lines to \(M^+_2 \times Q\) and \(K_{m|Q} \cong \mathcal{O}_Q(2, 2)\) for \(m = [(s, F)] \in M^+_2\).

**Proof.** The last part of the proof of Lemma 3.3 tells us that there is an exact sequence of sheaves \(0 \to \mathcal{O}_Q(2, 2) \to \mathcal{E}|_{M^+_2 \times Q} \to \mathcal{O}_Z \to 0\). Now it is sufficient to show that for each fiber \(G = \mathcal{E}|_{((s, F)) \times Q},\) the section \(s^*\) of \(G\) does not come from \(H^0(\mathcal{O}_Q(2, 2))\). If it is, we have an injection \(\mathcal{O}_Q \subset \mathcal{O}_Q(2, 2)\) whose cokernel is \(\mathcal{E}_q(2, 2)\) for some elliptic curve \(E\). By the snake lemma once again, we obtain \(0 \to \mathcal{O}_E(2, 2) \to F^D(2, 2) = \mathcal{O}_C(2, 2) \to \mathcal{O}_\ell \to 0\). It violates the stability of \(F^D(2, 2)\).
Let \((\mathcal{L}', \mathcal{E}')\) be the elementary modification of \((\mathcal{L}^*, \mathcal{E})\) along \(M_2^+\), that is,

\[
\text{Ker}(\mathcal{L}^*, \mathcal{E}) \to (\mathcal{L}'|_{M_2^+}, \mathcal{E}'|_{M_2^+ \times Q}) \to (\mathcal{L}', \mathcal{O}_Z))
\]

**Lemma 3.5.** For a point \(m = [(s, F = O_C(0, 1))] \in M_2^+, \) the modified pair \((\mathcal{L}', \mathcal{E}')|_{\{m\} \times Q}\) fits into a non-split exact sequence \(0 \to (s', \mathcal{O}_E) \to (s', \mathcal{E}'|_{\{m\} \times Q}) \to (0, \mathcal{O}_Q(2, 2)) \to 0\) where \(E\) is a \((0, 1)\)-line.

**Proof.** An elementary modification of pairs interchanges the sub pair with the quotient pair ([He98, Lemma 4.24]). Thus we obtain the sequence. It remains to show that the sequence is non-split. We will show that the normal bundle \(N_{M_2^+/M^+}\) at \(m\) is canonically isomorphic to \(H^0(\mathcal{O}_E)^*\). Then the element \(m\) corresponds to the projective equivalent class of nonzero elements in \(H^0(\mathcal{O}_E)^* \cong \text{Ext}^1((0, \mathcal{O}_Q(2, 2)), (s', \mathcal{O}_E))\), so it is non-split.

The pair \((s, F)\) fits into \(0 \to (0, \mathcal{O}_Q(-2, -2)) \to (s, \mathcal{O}_Q(0, 1)) \to (s, F) \to 0\). Thus we have

\[
0 \to \text{Ext}^0((0, \mathcal{O}_Q(-2, -2)), (s, F)) \to \text{Ext}^1((s, F), (s, F)) \to \text{Ext}^1((s, \mathcal{O}_Q(0, 1)), (s, F)) \to \cdots.
\]

The first term \(\text{Ext}^0((0, \mathcal{O}_Q(-2, -2)), (s, F)) \cong H^0(\mathcal{O}_C(2, 2, 3)) \cong C^{11}\) is the deformation space of curves \(C\) on \(Q\). The second term \(\text{Ext}^1((s, F), (s, F))\) is \(T_m \mathcal{M}^+\) ([He98, Theorem 3.12]). For the third term, by [He98, Theorem 3.12], we have

\[
0 \to \text{Hom}(s, H^0(F)/\langle s \rangle) \to \text{Ext}^1((s, \mathcal{O}_Q(0, 1)), (s, F)) \to \text{Ext}^1(\mathcal{O}_Q(0, 1), F) \overset{\delta}{\to} \text{Hom}(s, H^1(F)).
\]

The first term \(\text{Hom}(s, H^0(F)/\langle s \rangle) = C\) is the deformation space of the line \(E\) in \(Q\) determined by the section \(s\). By Serre duality, \(\phi : H^0(\mathcal{O}_Q(0, 1))^* \to H^0(\mathcal{O}_Q)^*\) and the kernel is \(H^0(\mathcal{O}_E(0, 1))^* \cong H^0(\mathcal{O}_E)^*\). This proves our assertion. \(\square\)

Recall that the modified pair \((\mathcal{L}', \mathcal{E}')\) provides a natural surjection \(\mathcal{E}xt^2_\pi(\mathcal{E}', \omega_\pi) \to \mathcal{L}'\) on \(M^+ \times Q\). It is straightforward to check that \(\mathcal{E}xt^2_\pi(\mathcal{E}', \omega_\pi)\) has rank 10 at each fiber, thus it is locally free.

**Proof of Proposition 3.2.** We claim that there exists a surjection \(w^*U^* \to \mathcal{L}' \to 0\) up to a twisting by a line bundle on \(M^+ \setminus \mathcal{M}^+_1\). Then there is a morphism \(M^+ \setminus \mathcal{M}^+_1 \to \mathcal{P}\).

Consider the following commutative diagram

\[
\begin{array}{ccc}
(M^+ \setminus \mathcal{M}^+_1) \times Q & \xrightarrow{w'} & G \times Q \\
\pi & & \pi \\
M^+ \setminus \mathcal{M}^+_1 & \xrightarrow{w} & G.
\end{array}
\]

Note that \(U = \pi_*\mathcal{W}\) where \(\mathcal{W} = \text{coker}(\phi)\) is the universal quotient on \(G \times Q\) (Section 2.1). One can check that \(\mathcal{W}\) is flat over \(G\). By its construction of \(w, \mathcal{E}'|_{\{m\} \times Q} \cong w'^*\mathcal{W}|_{\{m\} \times Q}\) restricted to each point \(m \in M^+ \setminus \mathcal{M}^+_1\). The universal property of \(G\) (as a quiver representation space [Kin94, Proposition 5.6]) tells us that \(w'^*\mathcal{W} \cong \mathcal{E}'\) up to a twisting by a line bundle on \(M^+ \setminus \mathcal{M}^+_1\). The base change property implies that there exists a natural isomorphism (up to a twisting by a line bundle) \(w^*U = w'^*(\pi_*\mathcal{W}) = \pi_*\mathcal{E}' \cong \mathcal{E}xt^2_\pi(\mathcal{E}', \omega_\pi)^*\) by [LP93, Corollary 8.19]. Hence we have \(w^*U^* \cong (w^*U)^* \cong (\pi_*\mathcal{E}')^* \cong \mathcal{E}xt^2_\pi(\mathcal{E}', \omega_\pi) \to \mathcal{L}'\). Therefore we obtain a morphism \(q : M^+ \setminus \mathcal{M}^+_1 \to \mathcal{P}\).

By the proof of Lemma 3.5, the modified pair does not depend on the choice of a \((2, 3)\)-curve, so \(q : M^+ \setminus \mathcal{M}^+_1 \to \mathcal{P} \setminus p^{-1}(t(Y_{10}))\) is indeed a contraction of \(M^+_2\) and the image of \(M^+_2\) is \(Y_{01}\). Recall
that the exceptional divisor $M^+_2$ is $|\mathcal{O}_Q(2,3)| \times |\mathcal{O}_Q(0,1)| \cong \mathbb{P}^{11} \times \mathbb{P}^1$. Note that the sheaf $F$ in the pair $(s, F) \in M^+_2$ is parametrized by $\mathbb{P}^{11} = |\mathcal{O}_Q(2,3)| = \mathbb{P} \text{Ext}^1(\mathcal{O}_Q(-2,-2)[1], \mathcal{O}_Q(0,1))$. It follows also from the fact that each $F$ fits into a triangle $0 \to \mathcal{O}_Q(0,1) \to F \to \mathcal{O}_Q(-2,-2)[1] \to 0$. By analyzing $T_F M = \text{Ext}^1(F, F)$ (which is similar to [CC16, Lemma 3.4]), one can see that $\mathcal{N}_{M_2/M^+} \cong \mathcal{N}_{M_2/M^+} \cong \text{Ext}^1(\mathcal{O}_Q(0,1), \mathcal{O}_Q(-2,-2)[1]) \otimes \mathcal{O}_{\mathbb{P}^{11}}(-1) \cong H^0(\mathcal{O}_Q(0,1))^* \otimes \mathcal{O}_{\mathbb{P}^{11}}(-1)$. Thus $\mathcal{N}_{M^+_2/M^+} \cong \mathcal{O}_{\mathbb{P}^{11} \times \mathbb{P}^1}(-1, -1)$ and $q$ is a smooth blow-down by Fujiki-Nakano criterion.

Thus we have two different contractions of $M^+$, one is $M$ obtained by contracting all $\mathbb{P}^1$-fibers on $M^+_2$, and the other is:

**Definition 3.6.** Let $M^-$ be the contraction of $M^+$ which is obtained by contracting all $\mathbb{P}^{11}$-fibers on $M^+_2$. We define $M^-_1$ as the image of $M^+_1$ for the contraction $M^+ \to M^-$.  

### 3.3. The second elementary modification and $M^-$. Recall that $u : G_1 \to G$ is the blow-up of $G$ along the $\mathbb{P}^1$ parameterizing $(1, 0)$-lines in $Q$, and $Y_{10}$ is the exceptional divisor. Let $W$ be the cokernel of the universal morphism $\phi$ on $G \times Q$ in Section 2.1. Let $V := (u \times \text{id})^*W$ be the pullback of $W$ along the map $u \times \text{id} : G_1 \times Q \to G \times Q$. Then for $([\ell], t) \in Y_{10}$, $V_{([\ell], t) \times Q}$ fits into a non-split exact sequence $0 \to \mathcal{O}_\ell(1) \to V_{([\ell], t) \times Q} \to \mathcal{O}_Q(1,3) \to 0$. By relativizing it over $Y_{10} \times Q$, we obtain $0 \to S \to V|_{Y_{10} \times Q} \to Q \to 0$. Let $V^-$ be the elementary modification $\text{elem}_{Y_{10} \times Q}(V, Q) := \ker(V \to V|_{Y_{10} \times Q} \to Q)$ along $Y_{10} \times Q$. Note that over $([\ell], t) \in G_1$, $V^-_{([\ell], t) \times Q}$ fits into a non-split exact sequence $0 \to \mathcal{O}_Q(1,3) \to V^-_{([\ell], t) \times Q} \to \mathcal{O}_\ell(1) \to 0$ because the elementary modification interchanges the sub/quotient sheaves. Let $\pi_1 : G_1 \times Q \to G_1$ be the projection into the first factor. Then $U^- := \pi_1, V^-$ is a rank $10$ bundle over $G_1$. Let $P^- := \mathbb{P}(U^-)$.

The following proposition completes the proof of Theorem 1.3.

**Proposition 3.7.** The projective bundle $P^-$ is isomorphic to $M^-$ in Definition 3.6.

**Proof.** Since the elementary modification has been done locally around $Y_{10} \times Q$, $\mathbb{P}(u^*U)$ and $P^-$ are isomorphic over $G_1 \setminus Y_{10}$. On the other hand, set theoretically, it is straightforward to see that the image of $q$ is $P \setminus p^{-1}(t(Y_{10}))$, where $p : P \to G$ is the structure morphism. So we have a birational morphism $M^+ \setminus M^+_1 \to P \setminus p^{-1}(t(Y_{10})) \cong \mathbb{P}(u^*U) \setminus p^{-1}(Y_{10}) \cong P^- \setminus p^{-1}(Y_{10})$ (here we used the same notation $p$ for the projections $\mathbb{P}(u^*U) \to G_1$ and $P^- \to G_1$). By Proposition 3.2, this map is a blow-down along $M^+_2$, thus we have an isomorphism $\tau : P^- \setminus p^{-1}(Y_{10}) \to M^+ \setminus M^+_1$. So we have a birational map $\tau : P^- \dashrightarrow M^-$, where its undefined locus is $p^{-1}(Y_{10})$.

On the other hand, since the flipped locus for $M^\infty \dashrightarrow M^+$ is $M^+_2$, we have an isomorphism $M^- \setminus (M^2 \cup M^+_3) \cong M^+ \setminus (M^2 \cup M^+_3) \cong M^\infty \setminus (M^2 \cup M^+_3)$. (Here $M^\infty_1$ is defined in an obvious way.) Also $\tau^{-1}(M^2 \cup M^+_3) = p^{-1}(Y_{10})$. Hence if we restrict the domain of $\tau$, then we have $\sigma : P^- \setminus p^{-1}(Y_{10}) \dashrightarrow M^- \setminus (M^2 \cup M^+_3) \cong M^\infty \setminus (M^2 \cup M^+_3)$ whose undefined locus is $p^{-1}(Y_{10})$. Therefore $\sigma$ can be regarded as a map into a relative Hilbert scheme. Note that $M^2 \cup M^+_3$ is the locus of $(2, 3)$-curves passing through two points lying on a $(0, 1)$-line.

We claim that $\sigma$ is extended to a morphism $\tilde{\sigma} : P^- \setminus p^{-1}(Y_{10}) \to M^-$ such that $\tilde{\sigma}(p^{-1}(Y_{10})) = M^+_1 \cong M^\infty_1$. To show this, it is enough to check that $V^-$ over $Y_{10}$ provides a flat family of the twisted ideal sheaf of Hilbert scheme of two points lying on $(1, 0)$-type lines. Note that $V^-$ fits into a non-split extension $0 \to \mathcal{O}_Q(1,3) \to V^-_{([\ell], t) \times Q} \to \mathcal{O}_\ell(1) \to 0$. By a diagram chasing similar to
are isomorphic. Over $Y$ of $V$ let

Proof of Proposition 1.2.

of Maruyama ([Mar73]).

obtain $R$ π

rank $\pi$

□

Corollary 3.8.

(1) The moduli space $M$ matches with the result in [Mai16, Theorem 1.2].

Now two maps $\tau$ and $\tilde{\sigma}$ coincide over the intersection $P^\tau \setminus \rho^{-1}(Y_{10} \cup Y_{01})$ of domains, so we have a birational morphism $P^\tau \to M^\tau$. Since $\rho(P^\tau) = 3 = \rho(M^\tau)$ and both of them are smooth, this map is an isomorphism.

The modification on $G_1 \times Q$ descends to $G_1$. Then Proposition 1.2 follows from a general result of Maruyama ([Mar73]).

Proof of Proposition 1.2. Let $\pi_1 : G_1 \times Q \to G_1$ be the projection. We claim that $U^- = \text{elm}_{Y_{10}}(u^*U, \pi_{1*}Q) \cong \pi_{1*}\text{elm}_{Y_{10} \times Q}(V, Q)$. Indeed, from $0 \to V^- \to V \to Q \to 0$, we have $0 \to \pi_{1*}V^- \to \pi_{1*}V = u^*U \to \pi_{1*}Q \to R^1\pi_{1*}V^- \to R^1\pi_{1*}V$. It is sufficient to show that $R^1\pi_{1*}V^- = 0$. By using the resolution of $V$ given by the universal morphism $\phi$, we have $R^1\pi_{1*}V = 0$. Over $G_1 \setminus Y_{10}$, the last two terms are isomorphic. Over $Y_{10}$, from $H^1(\mathcal{O}_Q(1, 3)) = H^1(\mathcal{O}_Y(1)) = 0$ and the description of $V^-|_{(\ell, t)}$, we obtain $R^1\pi_{1*}V^- = 0$.

Note that $u^*U|_{Y_{10}}$ fits into a vector bundle sequence $0 \to \pi_{1*}S \to u^*U|_{Y_{10}} \to \pi_{1*}Q \to 0$ and rank $\pi_{1*}S = 2$ and rank $\pi_{1*}Q = 8$. The result follows from [Mar73, Theorem 1.3].

As a direct application of Theorem 1.3, we compute the Poincaré polynomial of $M$ which matches with the result in [Mai16, Theorem 1.2].

Corollary 3.8. (1) The moduli space $M$ is rational;
(2) The Poincaré polynomial of $M$ is

$$P(M) = q^{12} + 3q^{12} + 8q^{11} + 10q^{10} + 11q^9 + 11q^8 + 11q^7 + 11q^6 + 11q^5 + 11q^4 + 10q^3 + 8q^2 + 3q + 1.$$ 

Proof. Now $M$ is birational to a $\mathbb{P}^9$-bundle over $G$, so we obtain Item (1). Item (2) is a straightforward calculation using

$$P(M) = P(\mathbb{P}^1) - P(\mathbb{P}^1) + P(M^-) = P(\mathbb{P}^1) - P(\mathbb{P}^1) + P(\mathbb{P}^9)(P(G) + (P(\mathbb{P}^2) - 1)P(\mathbb{P}^1)).$$

□

REFERENCES


DEPARTMENT OF MATHEMATICS EDUCATION, KYUNGPOOK NATIONAL UNIVERSITY, 80 DAEHAKRO, BUKGU, DAEGU 41566, KOREA

*E-mail address*: krchung@knu.ac.kr

DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NY 10458

*E-mail address*: hmoon8@fordham.edu