BIRATIONAL GEOMETRY OF THE MODULI SPACE OF PURE SHEAVES ON QUADRIC SURFACE

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ABSTRACT. We study birational geometry of the moduli space of stable sheaves on a quadric surface with Hilbert polynomial $5m+1$ and $c_1 = (2,3)$. We describe a birational map between the moduli space and a projective bundle over a Grassmannian as a composition of smooth blow-ups/downs.

RÉSUMÉ. Dans cette note, nous étudions la géométrie birationnelle de l'espace des modules des faisceaux stables sur une quadrique, de polynôme de Hilbert $5m+1$ et de classes de Chern (1,2). Pour cela, nous donnons une application birationnelle entre l'espace des modules et un fibré projectif au dessus d'une Grassmanienne qui est une composition d'éclatements et de contractions lisses.

1. INTRODUCTION

The geometry of the moduli space of sheaves on a projective plane has been studied in various viewpoints, for instance curve counting, the strange duality conjecture, and birational geometry via Bridgeland stability. For a detailed description of the motivation, see [CM15] and references therein. Even further, for small degree cases, it was possible to classify all rational contractions ([CM15, Section 1.3]) and compute the cohomology ring of the moduli space ([CM15, Theorem 1.2]).

It is natural to extend this result to del Pezzo surfaces. In this paper, we consider the next simplest case of a quadric surface. Here we construct a flip between the moduli space of sheaves and a projective bundle, and show that their common blown-up space is the moduli space of stable pairs ([LP93]). We expect that this analysis provides some insight to the study of a general Bridgeland wall-crossing over the moduli space of shaves on a del Pezzo surface. To the authors knowledge, there is no explicit study of wall-crossings in the case of moduli spaces of torsion sheaves on smaller degree del Pezzo surfaces.

Let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric surface in $\mathbb{P}^3$ with a very ample polarization $L := \mathcal{O}_Q(1,1)$. For the convenience of the reader, we start with a list of relevant moduli spaces.

Definition 1.1. (1) Let $\mathbf{M} := \mathbf{M}_L(Q,(2,3),5m+1)$ be the moduli space of stable sheaves F on Q with $c_1(F) = c_1(\mathcal{O}_Q(2,3))$ and $\chi(F(m)) = 5m+1$.

(2) Let $\mathbf{M}^\alpha := \mathbf{M}^\alpha_L(Q,(2,3),5m+1)$ be the moduli space of α-stable pairs (s,F) with $c_1(F) = c_1(\mathcal{O}_Q(2,3))$ and $\chi(F(m)) = 5m+1$ ([LP93] and [He98, Theorem 2.6]). Let $\mathbf{M}^+ := \mathbf{M}^\epsilon$ for $0 < \epsilon \ll 1$.

(3) Let $\mathbf{G} = \mathrm{Gr}(2,4)$ and let $\mathbf{G}_1$ be the blow-up of $\mathbf{G}$ along $\mathbb{P}^1$ that parametrizes projective lines in $Q \subset \mathbb{P}^3$ of type $(1,0)$ (Section 2.1).

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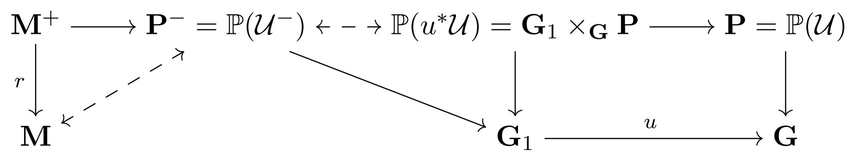
(4) Let $\mathbf{P} := \mathbb{P}(\mathcal{U})$ and $\mathbf{P}^- := \mathbb{P}(\mathcal{U}^-)$, where $\mathcal{U}$ (resp. $\mathcal{U}^-$) is a rank 10 vector bundle over $\mathbf{G}$ (resp. $\mathbf{G}_1$) defined in (3) in Section 2.1 (resp. Section 3.3).

The aim of this paper is to explain and justify the following commutative diagram between moduli spaces.

(1)



We have to explain two flips (dashed arrows) on the diagram.

One of key ingredients is the *elementary modification* of vector bundles ([Mar73]), sheaves ([HL10, Section 2.B]), and pairs ([CC16, Section 2.2]). It has been widely used in the study of sheaves on a smooth projective variety. Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X and $\mathcal{Q}$ be a vector bundle on a smooth divisor $Z \subset X$ with a surjective map $\mathcal{F}|_Z \twoheadrightarrow \mathcal{Q}$. The elementary modification of $\mathcal{F}$ along Z is the kernel of the composition

$$\mathrm{elm}_Z(\mathcal{F}) := \ker(\mathcal{F} \twoheadrightarrow \mathcal{F}|_Z \twoheadrightarrow \mathcal{Q}).$$

A similar definition is valid for sheaves and pairs, too. Note that the category of pairs is abelian ([He98, Theorem 1.3]).

On $\mathbf{G}_1$, let $\mathcal{U}^- := \mathrm{elm}_{Y_{10}}(u^*\mathcal{U})$ be the elementary transformation of $u^*\mathcal{U}$ along a smooth divisor Y_{10} (Section 2.1).

Proposition 1.2. *Let $\mathbf{P}^- = \mathbb{P}(\mathcal{U}^-)$. The flip $\mathbf{P}^- \dashrightarrow \mathbb{P}(u^*\mathcal{U}) = \mathbf{G}_1 \times_{\mathbf{G}} \mathbb{P}(\mathcal{U})$ is a composition of a blow-up and a blow-down. The blow-up center in $\mathbf{P}^-$ (resp. $\mathbb{P}(u^*\mathcal{U})$) is a $\mathbb{P}^1$ (resp. $\mathbb{P}^7$)-bundle over Y_{10}.*

Theorem 1.3. *There is a flip between $\mathbf{M}$ and $\mathbf{P}^-$ which is a blow-up followed by a blow-down, and the master space is $\mathbf{M}^+$, the moduli space of $+$-stable pairs (Definition 1.1 (2)).*

As the referee pointed out, all morphisms in (1) are SL_2-equivariant for the natural SL_2-action on the second ruling of Q. Thus one may expect an SL_2-quotient version of the main result. We didn't pursue this direction because we could not find any new explicit moduli theoretic interpretation.

As applications, we compute the Poincaré polynomial of $\mathbf{M}$ and show the rationality of $\mathbf{M}$ (Corollary 3.8) which were obtained by Maican by different methods ([Mai16]). Since each step of the birational transform is described in terms of blow-ups/downs along explicit subvarieties, in principle the cohomology ring and the Chow ring of $\mathbf{M}$ can be obtained from that of $\mathbf{G}$. Also one may aim for the completion of Mori's program for $\mathbf{M}$. We will carry on these projects in forthcoming papers.

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2. Relevant moduli spaces

In this section we give definitions and basic properties of some relevant moduli spaces.

2.1. **Grassmannian as a moduli space of Kronecker quiver representations.** The moduli space of representations of a Kronecker quiver parametrizes the isomorphism classes of stable sheaf homomorphisms

$$\mathcal{O}_Q(0,1) \longrightarrow \mathcal{O}_Q(1,2)^{\oplus 2} \tag{2}$$

up to the natural action of the automorphism group $\mathbb{C}^* \times \mathrm{GL}_2/\mathbb{C}^* \cong \mathrm{GL}_2$. For two vector spaces E and F of dimension 1 and 2 respectively and $V^* := \mathrm{H}^0(Q, L)$, the moduli space is constructed as $\mathbf{G} := \mathrm{Hom}(F, V^* \otimes E)//\mathrm{GL}_2 \cong V^* \otimes E \otimes F^*//\mathrm{GL}_2$ with an appropriate linearization ([Kin94]). We regard $\mathbf{G}$ as a moduli space of complexes. But also note that the GL_2 acts as a row operation on the space of 2×4 matrices, thus $\mathbf{G} \cong \mathrm{Gr}(2, V^*) \cong \mathrm{Gr}(2, 4)$.

Let $\mathbf{H}(n)$ be the Hilbert scheme of n points on Q. There is a birational map $\mathbf{H}(2) \dashrightarrow \mathbf{G}$ which maps Z to a resolution of $I_Z(2,3)$ of the type (2). For any $Z \in \mathbf{H}(2)$, let ℓ_Z be the unique line in $\mathbb{P}^3 \supset Q$ containing Z. Then either $\ell_Z \cap Q = Z$ or $\ell_Z \subset Q$. In the second case, the class of ℓ_Z is of the type $(1,0)$ or $(0,1)$. Let Y_{10} (resp. Y_{01}) be the locus of subschemes such that ℓ_Z is a line of the type $(1,0)$ (resp. $(0,1)$). Then Y_{10} and Y_{01} are two disjoint subvarieties which are isomorphic to a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$.

Proposition 2.1 ([BC13, Example 6.1])**.** *There exists a morphism $t : \mathbf{H}(2) \longrightarrow \mathbf{G}_1 \xrightarrow{u} \mathbf{G}$. The first (resp. the second) map contracts the divisor Y_{01} (resp. Y_{10}) to $\mathbb{P}^1$. If $\ell_Z \cap Q = Z$, then $t(Z)$ is (a resolution of) $I_Z(2,3)$. If $Z \in Y_{10}$, then $t(Z)$ is (a resolution of) $E_{10} \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_Q(1,3), \mathcal{O}_{\ell_Z}(1))) = \{\mathrm{pt}\}$. If $Z \in Y_{01}$, then $t(Z)$ is (a resolution of) $E_{01} \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_Q(2,2), \mathcal{O}_{\ell_Z})) = \{\mathrm{pt}\}$.*

The morphism $\wedge^2 V^* \otimes \mathrm{H}^0(\mathcal{O}_Q(0,1)) \to V^* \otimes V^* \otimes \mathrm{H}^0(\mathcal{O}_Q(0,1)) \to V^* \otimes \mathrm{H}^0(\mathcal{O}_Q(1,2))$ induces the *universal morphism* $\phi : p_1^*\mathcal{O}_{\mathbf{G}}(-1) \otimes p_2^*\mathcal{O}_Q(0,1) \to p_1^*\mathcal{S} \otimes p_2^*\mathcal{O}_Q(1,2)$ where $p_1 : \mathbf{G} \times Q \to \mathbf{G}$ and $p_2 : \mathbf{G} \times Q \to Q$ are two projections ([Kin94, Proposition 5.3]) and $\mathcal{S}$ is the universal subbundle of $\mathbf{G}$. Let $\mathcal{U}$ be the cokernel of $p_{1*}\phi$. On the stable locus, $p_{1*}\phi$ is injective. Thus we have an exact sequence

$$0 \to \mathcal{O}_G(-1) \otimes \mathrm{H}^0(\mathcal{O}_Q(0,1)) \stackrel{p_{1*}\phi}{\to} \mathcal{S} \otimes \mathrm{H}^0(\mathcal{O}_Q(1,2)) \to \mathcal{U} \to 0 \tag{3}$$

and $\mathcal{U}$ is a rank 10 vector bundle. Let $\mathbf{P} := \mathbb{P}(\mathcal{U})$.

2.2. **Moduli space M of stable sheaves.** Recall that $\mathbf{M} := \mathbf{M}_L(Q, (2,3), 5m+1)$ is the moduli space of stable sheaves F on Q with $c_1(F) = c_1(\mathcal{O}_Q(2,3))$ and $\chi(F(m)) = 5m+1$. There are four types of points in $\mathbf{M}$ ([Mai16, Theorem 1.1]). Let $C \in |\mathcal{O}_Q(2,3)|$.

(0) $F = \mathcal{O}_C(p+q)$, where the line $\langle p, q\rangle$ is not contained in Q;
(1) $F = \mathcal{O}_C(p+q)$, where the line $\langle p, q\rangle$ in Q is of type $(1,0)$;
(2) $F = \mathcal{O}_C(0,1)$;
(3) F fits into a non-split extension $0 \to \mathcal{O}_E \to F \to \mathcal{O}_\ell \to 0$ where E is a $(2,2)$-curve and ℓ is a $(0,1)$-line.

Let $\mathbf{M}_i$ be the locus of sheaves of the form (i). Each $\mathbf{M}_i$ is a subvariety of codimension i in $\mathbf{M}$ and for $i > 0$, $\mathbf{M}_i$ is closed. $\mathbf{M}_1$ is a $\mathbb{P}^9$-bundle over $\mathbb{P}^2 \times \mathbb{P}^1$. $\mathbf{M}_2$ is isomorphic to $|\mathcal{O}_Q(2,3)| \cong \mathbb{P}^{11}$. $\mathbf{M}_3$ is a singular subvariety which admits a finite birational map from a $\mathbb{P}^1$-bundle over

$|\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)|$. $\mathbf{M}_1 \cap \mathbf{M}_2 = \mathbf{M}_1 \cap \mathbf{M}_3 = \emptyset$ ([Mai16, Theorem 1.1]), but $\mathbf{M}_2$ and $\mathbf{M}_3$ intersect. Note that $\dim \mathrm{H}^0(F) = 1$ generically, but $\mathbf{M}_2$ parametrizes sheaves such that $\dim \mathrm{H}^0(F) = 2$.

2.3. **Moduli spaces of stable pairs.** A pair (s, F) consists of $F \in \mathsf{Coh}(Q)$ and a section $\mathcal{O}_Q \stackrel{s}{\to} F$. Fix $\alpha \in \mathbb{Q}_{>0}$. A pair (s, F) is called *α-semistable* (resp. *α-stable*) if F is pure and for any proper subsheaf $F' \subset F$, the inequality

$$\frac{P(F')(m) + \delta \cdot \alpha}{r(F')} \leq (<) \frac{P(F)(m)) + \alpha}{r(F)}$$

holds for $m \gg 0$. Here $\delta = 1$ if the section s factors through F' and $\delta = 0$ otherwise. Let $\mathbf{M}^\alpha := \mathbf{M}^\alpha_L(Q, (2,3), 5m+1)$ be the moduli space of S-equivalence classes of α-semistable pairs (s, F) such that the support of F has a class $c_1(\mathcal{O}_Q(2,3))$ ([LP93, Theorem 4.12] and [He98, Theorem 2.6]). The extremal case that α is sufficiently large (resp. small) is denoted by $\alpha = \infty$ (resp. $\alpha = +$). The deformation theory of pairs is studied in [He98, Corollary 1.6 and Corollary 3.6].

Proposition 2.2. *(1) ([CCM16, Lemma 2.2 (3)]) There exists a natural forgetful map $r : \mathbf{M}^+ \longrightarrow \mathbf{M}$ which maps (s, F) to F.*

(2) ([He98, Section 4.4]) The moduli space $\mathbf{M}^\infty$ of ∞-stable pairs is isomorphic to the relative Hilbert scheme of two points on the complete linear system $|\mathcal{O}_Q(2,3)|$.

The birational map $\mathbf{M}^\infty \dashrightarrow \mathbf{M}^+$ is analyzed in [Mai16, Theorem 5.7]. It turns out that this is a single flip over $\mathbf{M}^4$ and is a composition of a smooth blow-up and a smooth blow-down. By identifying the space $\mathbf{M}^\infty$ as the relative Hilbert scheme (Proposition 2.2 (2)), the blow-up center is isomorphic to a $\mathbb{P}^2$-bundle over $|\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)|$ where a fiber $\mathbb{P}^2$ parameterizes two points lying on a $(0,1)$-line. After the flip, the flipped locus, denoted by $\mathbf{M}_3^+$, on $\mathbf{M}^+$ is a $\mathbb{P}^1$-bundle over $|\mathcal{O}_Q(2,2)| \times |\mathcal{O}_Q(0,1)| \cong \mathbb{P}^8 \times \mathbb{P}^1$. For the forgetful map $r : \mathbf{M}^+ \to \mathbf{M}$, we define $\mathbf{M}_i^+ := r^{-1}(\mathbf{M}_i)$ if $i \neq 3$. Then $r(\mathbf{M}_3^+) = \mathbf{M}_3$, but $r : \mathbf{M}_3^+ \to \mathbf{M}_3$ is a birational finite map (This implies that $\mathbf{M}_3$ is not normal). The map r contracts $\mathbf{M}_2^+$, which is a $\mathbb{P}^1$-bundle over $\mathbf{M}_2$ and $\mathbf{M}^+ \setminus \mathbf{M}_2^+ \cong \mathbf{M} \setminus \mathbf{M}_2$. Maican proved that r is a smooth blow-up along the Brill-Noether locus $\mathbf{M}_2$ ([Mai16, Proposition 5.8]).

3. Decomposition of the birational map between $\mathbf{M}$ and $\mathbf{P}$

In this section we prove Proposition 1.2 and Theorem 1.3 by describing the birational map between $\mathbf{M}$ and $\mathbf{P}$.

3.1. Construction of a birational map $\mathbf{M}^+ \dashrightarrow \mathbf{P}$.

Lemma 3.1. *There exists a surjective morphism $w : \mathbf{M}^+ \longrightarrow \mathbf{G}$ which maps $(s, \mathcal{O}_C(p+q)) \in \mathbf{M}_0^+$ to $I_{\{p,q\}}(2,3)$, maps $(s, \mathcal{O}_C(p+q)) \in \mathbf{M}_1^+$ to the line $\langle p, q\rangle$ of the type $(1,0)$, maps $(s, F) \in \mathbf{M}_2^+$ to a $(0,1)$-line determined by a section, and maps $(s, F) \in \mathbf{M}_3^+$ to ℓ (see Section 2.2 for the notation), a $(0,1)$-line.*

Proof. By Proposition 2.2, $\mathbf{M}^\infty$ is the relative Hilbert scheme of two points on the universal $(2,3)$-curves, which is a $\mathbb{P}^9$-bundle over $\mathbf{H}(2)$ ([CC16, Lemma 2.3]). By composing with $t : \mathbf{H}(2) \to \mathbf{G}$ in Proposition 2.1, we have a morphism $\mathbf{M}^\infty \to \mathbf{G}$. On the other hand, since the flip $\mathbf{M}^\infty \to \mathbf{M}^+$

is the composition of a single blow-up/down, the blown-up space $\widetilde{\mathbf{M}}^\infty$ admits two morphisms to $\mathbf{M}^\infty$ and $\mathbf{M}^+$, and the flipped locus is $\mathbf{M}_3^+$. Note that each point in $\mathbf{M}_3^+$ can be regarded as a collection of data (E, ℓ, e) where E is a $(2,2)$-curve, ℓ is a $(0,1)$-line, and $e \in \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_\ell, \mathcal{O}_E)$. The fiber $\widetilde{\mathbf{M}}^\infty \to \mathbf{M}^+$ over the point in the blow-up center $\mathbf{M}_3^+$ is a $\mathbb{P}^2$ which parameterizes two points on ℓ. The composition map $\widetilde{\mathbf{M}}^\infty \to \mathbf{M}^\infty \to \mathbf{G}$ is constant along the $\mathbb{P}^2$, because $\mathbf{G}$ does not remember points on the line $\ell \subset Q$. By the rigidity lemma ([KM98, Lemma 1.6]), $\widetilde{\mathbf{M}}^\infty \to \mathbf{G}$ factors through $\mathbf{M}^+$ and we obtain a map $w : \mathbf{M}^+ \to \mathbf{G}$. □

Note that $\mathbf{M}_1^+ \cong \mathbf{M}_1$ is a $\mathbb{P}^9$-bundle over $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbf{M}_2^+$ is a $\mathbb{P}^1$-bundle over $|\mathcal{O}_Q(2,3)| \cong \mathbb{P}^{11}$. They are disjoint divisors on $\mathbf{M}^+$.

Proposition 3.2. *There is a birational morphism $q : \mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P} = \mathbb{P}(\mathcal{U})$ such that $p \circ q : \mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P} \to \mathbf{G}$ coincides with $w|_{\mathbf{M}^+ \setminus \mathbf{M}_1^+}$ in Lemma 3.1. Furthermore, q is the smooth blow-down along $\mathbf{M}_2^+$.*

The proof consists of several steps. Since $\mathbf{P} = \mathbb{P}(\mathcal{U})$ is a projective bundle over $\mathbf{G}$, it is sufficient to construct a surjective homomorphism $w^*\mathcal{U}^* \to \mathcal{L} \to 0$ over $\mathbf{M}^+ \setminus \mathbf{M}_1^+$ for some $\mathcal{L} \in \mathrm{Pic}(\mathbf{M}^+ \setminus \mathbf{M}_1^+)$, or equivalently, a *bundle* morphism $0 \to \mathcal{L}^* \to w^*\mathcal{U}$.

Recall that a family $(\mathcal{L}, \mathcal{F})$ of pairs on a scheme S is a collection of data $\mathcal{L} \in \mathrm{Pic}(S)$, $\mathcal{F} \in \mathsf{Coh}(S \times Q)$, which is a flat family of pure sheaves, and a surjective morphism $\mathcal{E}xt^2_\pi(\mathcal{F}, \omega_\pi) \twoheadrightarrow \mathcal{L}$ where $\pi : S \times Q \to S$ is the projection and ω_π is the relatively dualizing sheaf (See [LP93, Section 4.3] for the explanation why we take the dual.). Now let $(\mathcal{L}, \mathcal{F})$ be the universal pair ([He98, Theorem 4.8]) on $\mathbf{M}^+ \times Q$. By applying $\mathcal{H}om(-, \mathcal{O})$ to $\mathcal{E}xt^2_\pi(\mathcal{F}, \omega_\pi) \twoheadrightarrow \mathcal{L}$, we obtain $0 \to \mathcal{L}^* \to \mathcal{H}om(\mathcal{E}xt^2_\pi(\mathcal{F}, \omega_\pi), \mathcal{O})$. It can be shown that $\mathcal{H}om(\mathcal{E}xt^2_\pi(\mathcal{F}, \omega_\pi), \mathcal{O}) \cong \mathcal{E}xt^1_\pi(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}), \mathcal{O})$ (see [CM15, Section 3.2]). So we have a non-zero element $e \in \mathrm{Hom}(\mathcal{L}^*, \mathcal{E}xt^1_\pi(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}), \mathcal{O})) \cong \mathrm{Ext}^1(\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}), \pi^*\mathcal{L})$ ([CM15, Section 3.2]), which provides $0 \to \pi^*\mathcal{L} \to \mathcal{E} \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}) \to 0$ on $\mathbf{M}^+ \times Q$. By taking $\mathcal{H}om_\pi(-, \omega_\pi)$, we have $\mathcal{E}xt^2_\pi(\mathcal{E}, \omega_\pi) \to \mathcal{E}xt^2_\pi(\pi^*\mathcal{L}, \omega_\pi) \cong \mathcal{L}^* \to 0$ because $\mathcal{L}$ is a line bundle. This implies the existence of a flat family of pairs $(\mathcal{L}^*, \mathcal{E})$ on $\mathbf{M}^+ \times Q$. We may explicitly describe this construction fiberwisely in the following way. Let $(s, F) \in \mathbf{M}^+$. Let $F^D := \mathcal{E}xt^1(F, \omega_Q)$. For a non-zero section $s \in \mathrm{H}^0(F) \cong \mathrm{H}^1(F^D)^* \cong \mathrm{Ext}^1(F^D(2,2), \mathcal{O}_Q)$, we have a pair (s^*, G) given by

$$0 \to \mathcal{O}_Q \xrightarrow{s^*} G \to F^D(2,2) \to 0. \tag{4}$$

The first isomorphism comes from [Cho12, Proposition 4.2.8], and the section s^* is the one-dimensional vector space dual to s ([LP93a, Theorem 5.5]).

Lemma 3.3. *The map $(s, F) \mapsto (s^*, G)$ defines a dominant rational map $\mathbf{M}^+ \dashrightarrow \mathbf{P} = \mathbb{P}(\mathcal{U})$, which is regular precisely on $\mathbf{M}^+ \setminus (\mathbf{M}_1^+ \sqcup \mathbf{M}_2^+)$.*

Proof. Since we have a relative construction of pairs, it suffices to describe the extension (s^*, G) set theoretically. If $(s, F) \in \mathbf{M}_0^+ \sqcup \mathbf{M}_1^+$, then $F \cong \mathcal{O}_C(p+q) \cong I^D_{Z,C}(0,-1)$ for some curve C and $Z = \{p, q\} \in \mathbf{H}(2)$ such that the line ℓ_Z containing Z is not in Q ([He98, Section 4.4]). Then $F^D(2,2) \cong I_{Z,C}(2,3)$. Since $\mathrm{Ext}^1(F^D(2,2), \mathcal{O}_Q) \cong \mathrm{H}^1(F^D)^* \cong \mathrm{H}^0(F) \cong \mathbb{C}$, from $0 \to \mathcal{O}_Q(-2,-3) \cong I_{C,Q} \to I_{Z,Q} \to I_{Z,C} \to 0$, we obtain $G = I_{Z,Q}(2,3)$. If $(s, F) \in \mathbf{M}_0^+$, then we have an element $(s^*, G) \in \mathbf{P}$ because G has a resolution of the form $\mathcal{O}_Q(0,1) \to \mathcal{O}_Q(1,2)^{\oplus 2}$. However, if $(s, F) \in \mathbf{M}_1^+$, then we have $0 \to I_{\ell_Z,Q}(2,3) \to G = I_{Z,Q}(2,3) \to I_{Z,\ell_Z}(2,3) \to 0$ and $I_{\ell_Z,Q}(2,3) = \mathcal{O}_Q(1,3)$,

$I_{Z,\ell_Z}(2,3) = \mathcal{O}_{\ell_Z}(1)$. In particular, $\mathrm{Hom}(\mathcal{O}_Q(1,3), G) \neq 0$ and G does not admit a resolution $\mathcal{O}_Q(0,1) \to \mathcal{O}_Q(1,2)^{\oplus 2}$. So $G \notin \mathbf{G}$.

Suppose that $(s, F) \in \mathbf{M}_3^+ \setminus \mathbf{M}_2^+$. Then F fits into a non-split extension $0 \to \mathcal{O}_E \to F \to \mathcal{O}_\ell \to 0$. Apply $\mathcal{H}om(-, \omega_Q)$, then we have $0 \to \mathcal{O}_\ell(0,1) \to F^D(2,2) \to \mathcal{O}_E(2,2) \to 0$. By taking the functor $\mathrm{Ext}^\bullet(-, \mathcal{O}_Q)$ in this short exact sequence, one can see that $\mathrm{Ext}^1(\mathcal{O}_E(2,2), \mathcal{O}_Q) \cong \mathrm{Ext}^1(F^D(2,2), \mathcal{O}_Q) \cong \mathrm{H}^1(F^D) \cong \mathrm{H}^0(F)^* \cong \mathbb{C}$ because of Serre duality and [Cho12, Proposition 4.2.8]. Hence the sheaf G is given by the pull-back:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_Q & \longrightarrow & \mathcal{O}_Q(2,2) & \longrightarrow & \mathcal{O}_E(2,2) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_Q & \longrightarrow & G & \longrightarrow & F^D(2,2) & \longrightarrow & 0 \end{array} \tag{5}$$

By applying the snake lemma to (5), we conclude that the unique non-split extension G lies on $0 \to \mathcal{O}_\ell(0,1) \to G \to \mathcal{O}_Q(2,2) \to 0$. Hence $G \in \mathbf{G}$ (Proposition 2.1) and we have an element $(s^*, G) \in \mathbf{P}$.

Now suppose that $(s, F) \in \mathbf{M}_2^+$, so $F = \mathcal{O}_C(0,1)$. Then $F^D(2,2) = \mathcal{O}_C(2,2)$. So we have $0 \to \mathcal{O}_Q \stackrel{s^*}{\to} G \to \mathcal{O}_C(2,2) \to 0$. By the snake lemma (consult the proof of [CM15, Lemma 3.7]), G fits into

$$0 \to \mathcal{O}_Q(2,2) \to G \to \mathcal{O}_\ell \to 0 \tag{6}$$

where ℓ is the line of type $(0,1)$ determined by the section s. So $\mathrm{Hom}(\mathcal{O}_Q(2,2), G) \neq 0$ and this implies G does not admit a resolution $\mathcal{O}_Q(0,1) \to \mathcal{O}_Q(1,2)^{\oplus 2}$. Thus the correspondence is not well-defined on $\mathbf{M}_2^+$. □

3.2. **The first elementary modification and the extension of the domain.** We can extend the morphism in Lemma 3.3 by applying an elementary modification of pairs ([CC16, Section 2.2]) on $\mathbf{M}_2^+$.

Lemma 3.4. *There exists an exact sequence of pairs* $0 \to (0, K) \to (\mathcal{L}^*|_{\mathbf{M}_2^+}, \mathcal{E}|_{\mathbf{M}_2^+ \times Q}) \to (\mathcal{L}'', \mathcal{O}_{\mathcal{Z}}) \to 0$ *where* $\mathcal{Z}$ *is the pull-back of the universal family of* $(0,1)$*-lines to* $\mathbf{M}_2^+ \times Q$ *and* $K_{\{m\} \times Q} \cong \mathcal{O}_Q(2,2)$ *for* $m = [(s, F)] \in \mathbf{M}_2^+$.

Proof. By relativizing the short exact sequence (6) in the proof of Lemma 3.3, there is an exact sequence of *sheaves* $0 \to K \to \mathcal{E}|_{\mathbf{M}_2^+ \times Q} \to \mathcal{O}_{\mathcal{Z}} \to 0$. To obtain the short exact sequence of pairs in the statement of the lemma, it is sufficient to show that for each fiber $G = \mathcal{E}|_{\{(s,F)\} \times Q}$, the section s^* of G does not come from $\mathrm{H}^0(\mathcal{O}_Q(2,2))$. If it is, we have an injection $\mathcal{O}_Q \subset \mathcal{O}_Q(2,2)$ whose cokernel is $\mathcal{O}_E(2,2)$ for some curve E of arithmetic genus one. By the snake lemma once again, we obtain $0 \to \mathcal{O}_E(2,2) \to F^D(2,2) = \mathcal{O}_C(2,2) \to \mathcal{O}_\ell \to 0$. It violates the stability of $F^D(2,2)$. □

Let $(\mathcal{L}', \mathcal{E}')$ be the elementary modification of $(\mathcal{L}^*, \mathcal{E})$ along $\mathbf{M}_2^+$, that is,

$$\mathrm{Ker}((\mathcal{L}^*, \mathcal{E}) \twoheadrightarrow (\mathcal{L}^*|_{\mathbf{M}_2^+}, \mathcal{E}|_{\mathbf{M}_2^+ \times Q}) \twoheadrightarrow (\mathcal{L}'', \mathcal{O}_{\mathcal{Z}})).$$

Lemma 3.5. *For a point* $m = [(s, F = \mathcal{O}_C(0,1))] \in \mathbf{M}_2^+$*, the modified pair* $(\mathcal{L}', \mathcal{E}')|_{\{m\} \times Q}$ *fits into a non-split exact sequence* $0 \to (s', \mathcal{O}_\ell) \to (s', \mathcal{E}'|_{\{m\} \times Q}) \to (0, \mathcal{O}_Q(2,2)) \to 0$ *where* ℓ *is a* $(0,1)$*-line.*

Proof. An elementary modification of pairs interchanges the sub pair with the quotient pair ([He98, Lemma 4.24]). Thus we obtain the sequence. It remains to show that the sequence is non-split. We will show that the normal bundle $\mathcal{N}_{\mathbf{M}_2^+/\mathbf{M}^+}$ at m is canonically isomorphic to $\mathrm{H}^0(\mathcal{O}_\ell)^*$. Then the element m corresponds to the projective equivalence class of nonzero elements in $\mathrm{H}^0(\mathcal{O}_\ell)^* \cong \mathrm{Ext}^1((0,\mathcal{O}_Q(2,2)),(s',\mathcal{O}_\ell))$, so it is non-split ([CC16, Theorem 3.3]).

The $+$-stable pair (s,F) fits into $0 \to (0,\mathcal{O}_Q(-2,-2)) \to (s,\mathcal{O}_Q(0,1)) \to (s,F) \to 0$. Since

$$\mathrm{Ext}^0((s,F),(s,F)) \cong \mathrm{Ext}^0((s,\mathcal{O}_Q(0,1)),(s,F)) \cong \mathrm{Ext}^0(\mathcal{O}_Q(0,1),F) \cong \mathrm{H}^0(\mathcal{O}_C) = \mathbb{C}$$

([He98, Corollary 1.6]), we have

$$0 \to \mathrm{Ext}^0((0,\mathcal{O}_Q(-2,-2)),(s,F)) \to \mathrm{Ext}^1((s,F),(s,F)) \to \mathrm{Ext}^1((s,\mathcal{O}_Q(0,1)),(s,F)) \to \cdots.$$

The first term $\mathrm{Ext}^0((0,\mathcal{O}_Q(-2,-2)),(s,F)) \cong \mathrm{H}^0(\mathcal{O}_C(2,3)) \cong \mathbb{C}^{11}$ is the deformation space of the curve C on Q. The second term $\mathrm{Ext}^1((s,F),(s,F))$ is $\mathcal{T}_m\mathbf{M}^+$ ([He98, Theorem 3.12]). For the third term, by [He98, Theorem 3.12] again, we have

$$0 \to \mathrm{Hom}(s,\mathrm{H}^0(F)/\langle s\rangle) \to \mathrm{Ext}^1((s,\mathcal{O}_Q(0,1)),(s,F)) \to \mathrm{Ext}^1(\mathcal{O}_Q(0,1),F) \xrightarrow{\phi} \mathrm{Hom}(s,\mathrm{H}^1(F)).$$

The first term $\mathrm{Hom}(s,\mathrm{H}^0(F)/\langle s\rangle) = \mathbb{C}$ is the deformation space of the line ℓ in Q determined by the section s. By Serre duality, $\phi : \mathrm{H}^0(\mathcal{O}_Q(0,1))^* \to \mathrm{H}^0(\mathcal{O}_Q)^*$ and the kernel is $\mathrm{H}^0(\mathcal{O}_\ell(0,1))^* \cong \mathrm{H}^0(\mathcal{O}_\ell)^*$. This proves our assertion. □

Recall that the modified pair $(\mathcal{L}',\mathcal{E}')$ provides a natural surjection $\mathcal{E}xt^2_\pi(\mathcal{E}',\omega_\pi) \twoheadrightarrow \mathcal{L}'$ on $\mathbf{M}^+\times Q$. By Lemmas 3.3 and 3.5, it is straightforward to check that $\mathcal{E}xt^2_\pi(\mathcal{E}',\omega_\pi)$ has rank 10 at each fiber, thus it is locally free.

Proof of Proposition 3.2. We claim that there exists a surjection $w^*\mathcal{U}^* \to \mathcal{L}' \to 0$ up to a twisting by a line bundle on $\mathbf{M}^+ \setminus \mathbf{M}_1^+$. Then there is a morphism $\mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P}$.

Consider the following commutative diagram

$$\begin{array}{ccc} (\mathbf{M}^+ \setminus \mathbf{M}_1^+) \times Q & \xrightarrow[w' := w \times \mathrm{id}]{} & \mathbf{G} \times Q \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbf{M}^+ \setminus \mathbf{M}_1^+ & \xrightarrow{w} & \mathbf{G}. \end{array}$$

Note that $\mathcal{U} = \pi_*(\mathcal{W})$ where $\mathcal{W} = \mathrm{coker}(\phi)$ is the universal quotient on $\mathbf{G}\times Q$ (Section 2.1). One can check that $\mathcal{W}$ is flat over $\mathbf{G}$. By construction of w, $\mathcal{E}'|_{\{m\}\times Q} \cong w'^*\mathcal{W}|_{\{m\}\times Q}$ restricted to each point $m \in \mathbf{M}^+ \setminus \mathbf{M}_1^+$. The universal property of $\mathbf{G}$ (as a quiver representation space [Kin94, Proposition 5.6]) tells us that $w'^*\mathcal{W} \cong \mathcal{E}'$ up to a twisting by a line bundle on $\mathbf{M}^+ \setminus \mathbf{M}_1^+$. The base change property implies that there exists a natural isomorphism (up to a twisting by a line bundle) $w^*\mathcal{U} = w^*(\pi_*\mathcal{W}) \cong \pi_*(w'^*\mathcal{W}) = \pi_*\mathcal{E}' \cong \mathcal{E}xt^2_\pi(\mathcal{E}',\omega_\pi)^*$ by [LP93, Corollary 8.19]. Hence we have $w^*\mathcal{U}^* \cong (w^*\mathcal{U})^* \cong (\pi_*(\mathcal{E}'))^* \cong \mathcal{E}xt^2_\pi(\mathcal{E}',\omega_\pi) \twoheadrightarrow \mathcal{L}'$. Therefore we obtain a morphism $q : \mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P}$.

By the proof of Lemma 3.5, the modified pair does not depend on the choice of a $(2,3)$-curve, so $q : \mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P} \setminus p^{-1}(t(Y_{10}))$ is indeed a contraction of $\mathbf{M}_2^+$ and the image of $\mathbf{M}_2^+$ is isomorphic to a $\mathbb{P}^1$. Recall that the exceptional divisor $\mathbf{M}_2^+$ is $|\mathcal{O}_Q(2,3)| \times |\mathcal{O}_Q(0,1)| \cong \mathbb{P}^{11} \times$

$\mathbb{P}^1$. Note that the sheaf F in the pair $(s,F) \in \mathbf{M}_2^+$ is parametrized by $\mathbb{P}^{11} = |\mathcal{O}_Q(2,3)| = \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_Q(-2,-2)[1], \mathcal{O}_Q(0,1))$. By analyzing $T_F\mathbf{M} = \mathrm{Ext}^1(F,F)$ (which is similar to [CC16, Lemma 3.4]), one can see that $\mathcal{N}_{\mathbf{M}_2/\mathbf{M}}|_{\mathbb{P}^{11}} \cong \mathrm{Ext}^1(\mathcal{O}_Q(0,1), \mathcal{O}_Q(-2,-2)[1]) \otimes \mathcal{O}_{\mathbb{P}^{11}}(-1) \cong \mathrm{H}^0(\mathcal{O}_Q(0,1))^* \otimes \mathcal{O}_{\mathbb{P}^{11}}(-1)$. Thus $\mathcal{N}_{\mathbf{M}_2^+/\mathbf{M}^+} \cong \mathcal{O}_{\mathbb{P}^{11}\times\mathbb{P}^1}(-1,-1)$ and q is a smooth blow-down by Fujiki-Nakano criterion [FN72]. □

Thus we have two different contractions of $\mathbf{M}^+$, one is $\mathbf{M}$ obtained by contracting all $\mathbb{P}^1$-fibers on $\mathbf{M}_2^+$, and the other is:

Definition 3.6. Let $\mathbf{M}^-$ be the contraction of $\mathbf{M}^+$ which is obtained by contracting all $\mathbb{P}^{11}$-fibers on $\mathbf{M}_2^+$. We define $\mathbf{M}_i^-$ as the image of $\mathbf{M}_i^+$ for the contraction $\mathbf{M}^+ \to \mathbf{M}^-$.

3.3. **The second elementary modification and $\mathbf{M}^-$.** Recall that $u : \mathbf{G}_1 \to \mathbf{G}$ is the blow-up of $\mathbf{G}$ along the $\mathbb{P}^1$ parameterizing $(1,0)$-lines in Q, and Y_{10} is the exceptional divisor. Let $\mathcal{W}$ be the cokernel of the universal morphism ϕ on $\mathbf{G} \times Q$ in Section 2.1. Let $\mathcal{V} := (u \times \mathrm{id})^*\mathcal{W}$ be the pull-back of $\mathcal{W}$ along the map $u \times \mathrm{id} : \mathbf{G}_1 \times Q \to \mathbf{G} \times Q$. Then for $([\ell], t) \in Y_{10}$, $\mathcal{V}|_{([\ell],t)\times Q}$ fits into a non-split exact sequence $0 \to \mathcal{O}_\ell(1) \to \mathcal{V}|_{([\ell],t)\times Q} \to \mathcal{O}_Q(1,3) \to 0$. By relativizing it over $Y_{10} \times Q$, we obtain $0 \to \mathcal{S} \to \mathcal{V}|_{Y_{10}\times Q} \to \mathcal{Q} \to 0$. Let $\mathcal{V}^-$ be the elementary modification $\mathrm{elm}_{Y_{10}\times Q}(\mathcal{V}, \mathcal{Q}) := \ker(\mathcal{V} \twoheadrightarrow \mathcal{V}|_{Y_{10}\times Q} \twoheadrightarrow \mathcal{Q})$ along $Y_{10} \times Q$. Note that over $([\ell], t) \in \mathbf{G}_1$, $\mathcal{V}^-|_{([\ell],t)\times Q}$ fits into a non-split exact sequence $0 \to \mathcal{O}_Q(1,3) \to \mathcal{V}^-|_{([\ell],t)\times Q} \to \mathcal{O}_\ell(1) \to 0$ because the elementary modification interchanges the sub/quotient sheaves. Let $\pi_1 : \mathbf{G}_1 \times Q \to \mathbf{G}_1$ be the projection into the first factor. Then $\mathcal{U}^- := \pi_{1*}\mathcal{V}^-$ is a rank 10 bundle over $\mathbf{G}_1$. Let $\mathbf{P}^- := \mathbb{P}(\mathcal{U}^-)$.

The following proposition completes the proof of Theorem 1.3.

Proposition 3.7. *The projective bundle $\mathbf{P}^-$ is isomorphic to $\mathbf{M}^-$ in Definition 3.6.*

Proof. Since the elementary modification has been done locally around $Y_{10} \times Q$, $\mathbb{P}(u^*\mathcal{U})$ and $\mathbf{P}^-$ are isomorphic over $\mathbf{G}_1 \setminus Y_{10}$. On the other hand, set theoretically, it is straightforward to see that the image of q is $\mathbf{P} \setminus p^{-1}(t(Y_{10}))$, where $p : \mathbf{P} \to \mathbf{G}$ is the structure morphism. So we have a birational morphism $\mathbf{M}^+ \setminus \mathbf{M}_1^+ \to \mathbf{P} \setminus p^{-1}(t(Y_{10})) \cong \mathbb{P}(u^*\mathcal{U}) \setminus p^{-1}(Y_{10}) \cong \mathbf{P}^- \setminus p^{-1}(Y_{10})$ (here we used the same notation p for the projections $\mathbb{P}(u^*\mathcal{U}) \to \mathbf{G}_1$ and $\mathbf{P}^- \to \mathbf{G}_1$). By Proposition 3.2, this map is a blow-down along $\mathbf{M}_2^+$, thus we have an isomorphism $\tau : \mathbf{P}^- \setminus p^{-1}(Y_{10}) \to \mathbf{M}^- \setminus \mathbf{M}_1^-$. So we have a birational map $\tau : \mathbf{P}^- \dashrightarrow \mathbf{M}^-$, where its undefined locus is $p^{-1}(Y_{10})$.

On the other hand, since the flipped locus for $\mathbf{M}^\infty \dashrightarrow \mathbf{M}^+$ is $\mathbf{M}_3^+$, we have an isomorphism $\mathbf{M}^- \setminus (\mathbf{M}_2^- \cup \mathbf{M}_3^-) \cong \mathbf{M}^+ \setminus (\mathbf{M}_2^+ \cup \mathbf{M}_3^+) \cong \mathbf{M}^\infty \setminus (\mathbf{M}_2^\infty \cup \mathbf{M}_3^\infty)$ (Here $\mathbf{M}_i^\infty$ is defined in an obvious way.). Also $\tau^{-1}(\mathbf{M}_2^- \cup \mathbf{M}_3^-) = p^{-1}(Y_{01})$. Hence if we restrict the domain of τ, then we have $\sigma : \mathbf{P}^- \setminus p^{-1}(Y_{01}) \dashrightarrow \mathbf{M}^- \setminus (\mathbf{M}_2^- \cup \mathbf{M}_3^-) \cong \mathbf{M}^\infty \setminus (\mathbf{M}_2^\infty \cup \mathbf{M}_3^\infty)$ whose undefined locus is $p^{-1}(Y_{10})$. Therefore σ can be regarded as a map into a relative Hilbert scheme. Note that $\mathbf{M}_2^\infty \cup \mathbf{M}_3^\infty$ is the locus of $(2,3)$-curves passing through two points lying on a $(0,1)$-line.

We claim that σ is extended to a morphism $\tilde{\sigma} : \mathbf{P}^- \setminus p^{-1}(Y_{01}) \to \mathbf{M}^-$ such that $\tilde{\sigma}(p^{-1}(Y_{10})) = \mathbf{M}_1^- \cong \mathbf{M}_1^\infty$. To show this, it is enough to check that $\mathcal{V}^-$ over Y_{10} provides a flat family of the twisted ideal sheaf of Hilbert scheme of two points lying on $(1,0)$-type lines. Note that $\mathcal{V}^-$ fits into a non-split extension

$$0 \to \mathcal{O}_Q(1,3) \to \mathcal{V}^-|_{([\ell],t)\times Q} \to \mathcal{O}_\ell(1) \to 0. \tag{7}$$

By a diagram chasing similar to the second paragraph of the proof of Lemma 3.3, one can check that $\mathcal{V}^-|_{([\ell],t)\times Q} \cong I_{Z,Q}(2,3)$ where $Z \subset \ell$ and ℓ is a $(1,0)$-line.

Now two maps τ and $\tilde{\sigma}$ coincide over the intersection $\mathbf{P}^- \setminus p^{-1}(Y_{10} \cup Y_{01})$ of domains, so we have a birational morphism $\mathbf{P}^- \to \mathbf{M}^-$. Since $\rho(\mathbf{P}^-) = 3 = \rho(\mathbf{M}^-)$ and both of them are smooth, this map is an isomorphism. □

The modification on $\mathbf{G}_1 \times Q$ descends to $\mathbf{G}_1$. Proposition 1.2 follows from a general result of Maruyama ([Mar73]).

Proof of Proposition 1.2. Let $\pi_1 : \mathbf{G}_1 \times Q \to \mathbf{G}_1$ be the projection. We claim that $\mathcal{U}^- = \mathrm{elm}_{Y_{10}}(u^*\mathcal{U}, \pi_{1*}\mathcal{Q}) \cong \pi_{1*}\mathrm{elm}_{Y_{10}\times Q}(\mathcal{V}, \mathcal{Q})$. Indeed, from $0 \to \mathcal{V}^- \to \mathcal{V} \to \mathcal{Q} \to 0$, we have $0 \to \pi_{1*}\mathcal{V}^- \to \pi_{1*}\mathcal{V} = u^*\mathcal{U} \to \pi_{1*}\mathcal{Q} \to R^1\pi_{1*}\mathcal{V}^- \to R^1\pi_{1*}\mathcal{V}$. It is sufficient to show that $R^1\pi_{1*}\mathcal{V}^- = 0$. By using the resolution of $\mathcal{W}$ given by the universal morphism ϕ, we have $R^1\pi_{1*}\mathcal{W} = 0$ and this implies $R^1\pi_{1*}\mathcal{V} = 0$. Over $\mathbf{G}_1 \setminus Y_{10}$, $R^1\pi_{1*}\mathcal{V}^-$ and $R^1\pi_{1*}\mathcal{V}$ are isomorphic. For each point $([\ell], t) \in Y_{10}$, $\mathrm{H}^1(\mathcal{V}^-|_{([\ell],t)\times Q}) = 0$ by the exact sequence (7). Therefore we obtain $R^1\pi_{1*}\mathcal{V}^- = 0$.

Note that $u^*\mathcal{U}|_{Y_{10}}$ fits into a *vector bundle* sequence $0 \to \pi_{1*}\mathcal{S} \to u^*\mathcal{U}|_{Y_{10}} \to \pi_{1*}\mathcal{Q} \to 0$ and $\mathrm{rank}\ \pi_{1*}\mathcal{S} = 2$ and $\mathrm{rank}\ \pi_{1*}\mathcal{Q} = 8$. The result follows from [Mar73, Theorem 1.3]. □

As a direct application of Theorem 1.3, we compute the Poincaré polynomial of $\mathbf{M}$ which matches with the result in [Mai16, Theorem 1.2]. We denote the Poincaré polynomial of a smooth projective variety X by $P(X) = \sum_i b_i(X) q^{i/2}$ where $b_i(X)$ is the i-th Betti number of X.

Corollary 3.8. *(1) The moduli space* $\mathbf{M}$ *is rational;*
(2) The Poincaré polynomial of $\mathbf{M}$ *is*

$$P(\mathbf{M}) = q^{13} + 3q^{12} + 8q^{11} + 10q^{10} + 11q^9 + 11q^8 + 11q^7 + 11q^6 + 11q^5 + 11q^4 + 10q^3 + 8q^2 + 3q + 1.$$

Proof. Now $\mathbf{M}$ is birational to a $\mathbb{P}^9$-bundle over $\mathbf{G}$, so we obtain Item (1). Item (2) is a straightforward calculation using

$$P(\mathbf{M}) = P(\mathbb{P}^{11}) - P(\mathbb{P}^1) + P(\mathbf{M}^-) = P(\mathbb{P}^{11}) - P(\mathbb{P}^1) + P(\mathbb{P}^9)(P(\mathbf{G}) + (P(\mathbb{P}^2) - 1)P(\mathbb{P}^1)).$$

□

References

[BC13] Aaron Bertram and Izzet Coskun. The birational geometry of the Hilbert scheme of points on surfaces. In *Birational geometry, rational curves, and arithmetic*, pages 15–55. Springer, New York, 2013. 3

[Cho12] Jinwon Choi. Enumerative invariants for local Calabi-Yau threefolds. Ph.D. Thesis, Univer- sity of Illinois, 2012. 5, 6

[CC16] Jinwon Choi and Kiryong Chung. Moduli spaces of α-stable pairs and wall-crossing on $\mathbb{P}^2$. *J. Math. Soc. Japan*, 68(2):685–789, 2016. 2, 4, 6, 7, 8

[CCM16] Jinwon Choi, Kiryong Chung and Mario Maican. Moduli of sheaves supported on quartic space curves. *Michigan Math. J.*, 65(3):637–671, 2016. 4

[CM15] Kiryong Chung and Han-Bom Moon. Chow ring of the moduli space of stable sheaves supported on quartic curves. arXiv:1506.00298, To appear in Quarterly Journal of Mathematics, 2015. 1, 5, 6

[FN72] Akira Fujiki and Shigeo Nakano. Supplement to "On the inverse of monoidal transformation". *Publ. Res. Inst. Math. Sci.*, 7:637–644, 1971/72. 8

[He98] Min He. Espaces de modules de systèmes cohérents. *Internat. J. Math.*, 9(5):545–598, 1998. 1, 2, 4, 5, 7

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. 2

[Kin94] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994. 3, 7

[KM98] János Kollár and Shigefumi Mori. Birational geometry of algebraic varieties. Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. 5

[LP93a] J. Le Potier. Faisceaux semi-stables de dimension 1 sur le plan projectif. *Rev. Roumaine Math. Pures Appl.*, 38(7-8):635–678, 1993. 5

[LP93] Joseph Le Potier. Systèmes cohérents et structures de niveau. *Astérisque*, (214):143, 1993. 1, 4, 5, 7

[Mai16] Mario Maican. Moduli of sheaves supported on curves of genus two in a quadric surface. *arXiv:1612.03566*, 2016. 2, 3, 4, 9

[Mar73] M. Maruyama. On a family of algebraic vector bundles. *Number Theory, Algebraic Geometry, and Commutative Algebra*, pages 95–149, 1973. 2, 9

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