Let's count points!

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Three kingdoms in mathematical worlds

$$\begin{array}{ccc} \text{Geometry} & \longleftrightarrow & \text{Combinatorics} \\ & \nwarrow\!\!\searrow \quad\quad \swarrow\!\!\nearrow & \\ & \text{Algebra} & \end{array}$$

- Geometry: study of figures/spaces, curves, surfaces, higher dimensional spaces, their lengths, volumes, curvatures, symmetries, $\ldots$
- Combinatorics: study of finite sets, graphs, counting, $\cdots$
- Algebra: study of calculation of symbols, numbers, polynomials, matrices, equations, $\cdots$

There are many surprising interactions of these three seemingly unrelated worlds.

Part I

Introduction - integral points and integral polytopes

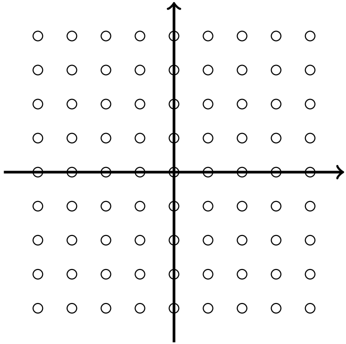
Integral points

An integral point of $\mathbb{R}^n$ is a point $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ such that each coordinate x_i is an integer.

Examples:

- $(2, -1, 5)$: an integral point in $\mathbb{R}^3$
- $(-2, \frac{2}{5})$: not an integral point in $\mathbb{R}^2$

$\mathbb{Z}^n$: set of n-dimensional integral points.

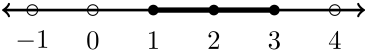


Integral polytope

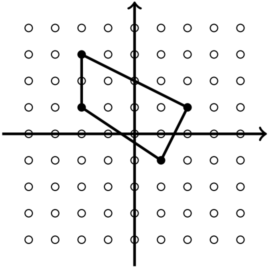
In an n-dimensional space $\mathbb{R}^n$, an integral polytope is a figure (convex set) generated by finitely many integral points.

Examples:

- A one-dimensional integral polytope is a finite interval whose endpoints are integers.



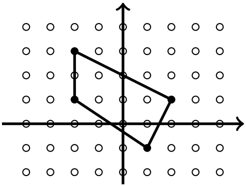
- A two-dimensional integral polytope is a polygon whose vertices are all integral points.



Main question

Question

For an integral polytope P, count the number of integral points on P.



Two related questions:

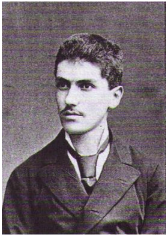
Question (Computational aspect)

Find a fast way counting the number of integral points on an integral polytope P.

Question (Theoretical aspect)

Describe the number of integral points on P in terms of some geometric properties/quantities of P.

Pick's theorem



Georg Alexander Pick, 1859 - 1942

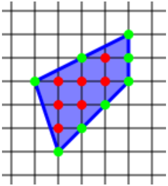
Theorem (Pick, 1899)

Let P be a two-dimensional integral polytope. Let i be the number of integral points in the interior of P and b be the number of integral points on the boundary of P. Then the following equation holds.

$$\mathit{Area}(P) = i + \frac{b}{2} - 1$$

Pick's theorem

$$\text{Area}(P) = i + \frac{b}{2} - 1$$



For instance, in the picture above, $i = 7$, $b = 8$. So

$$\text{Area} = 7 + \frac{8}{2} - 1 = 10.$$

Simple corollary: Every two-dimensional integral polytope has a half-integer area.

Goal

I will present and prove an interesting formula which counts the number of integral points (indeed, describes the set of integral points completely) on a given integral polytope P.

- The result is true for arbitrary dimension, but I will give a proof for two-dimensional case, for notational simplicity.
- The same proof works for every dimension.

Part II

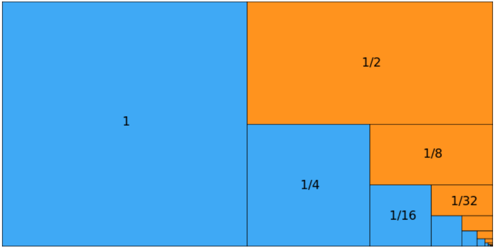
Preliminary - Geometric series

Geometric series

A geometric series is an infinite sum with constant ratio between successive terms.

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \sum_{n=0}^{\infty} ar^n$$

It appears in many different geometric contexts. For instance:



This picture shows us that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = 2.$$

Geometric series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \sum_{n=0}^{\infty} ar^n$$

Indeed, if $|r| < 1$, we can evaluate this sum by using a very simple formula.

First of all, consider the sum of first $n+1$ terms:

$$S := a + ar + ar^2 + \cdots + ar^n$$

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n+1}$$

$$(1-r)S = S - rS = (a + ar + ar^2 + \cdots + ar^n) - (ar + ar^2 + ar^3 + \cdots + ar^{n+1})$$

$$= a - ar^{n+1} = a(1 - r^{n+1})$$

$$\Rightarrow S = \frac{a(1 - r^{n+1})}{1 - r}$$

$$a + ar + ar^2 + ar^3 + \cdots = \lim_{n\to\infty} S = \lim_{n\to\infty} \frac{a(1 - r^{n+1})}{1 - r} = \frac{a}{1 - r}$$

Geometric series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Here are some explicit examples we will consider later:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \ (a = 1, r = x)$$

$$x^3 + x^4 + x^5 + \cdots = \frac{x^3}{1-x} \ (a = x^3, r = x)$$

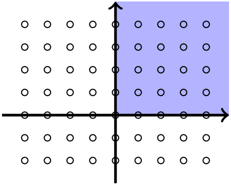
$$1+x^{-1}+x^{-2}+x^{-3}+\cdots = 1+x^{-1}+(x^{-1})^2+(x^{-1})^3+\cdots = \frac{1}{1-x^{-1}} = \frac{x}{x-1}$$

$$\begin{aligned} 1 + xy^{-1} + x^2y^{-2} + x^3y^{-3} + \cdots &= 1 + xy^{-1} + (xy^{-1})^2 + (xy^{-1})^3 + \cdots \\ &= \frac{1}{1-xy^{-1}} = \frac{y}{y-x} \end{aligned}$$

Some variations

Exercise: Compute the sum of all monomials $x^i y^j$ on the first quadrant, that is, $x^i y^j$ with $i, j \geq 0$.

$$1 + x + y + x^2 + xy + y^2 + x^3 + x^2y + xy^2 + y^3 + \cdots = ?$$



If we change the order of terms with respect to the degree of x:

$$\begin{aligned}
& 1 + x + y + x^2 + xy + y^2 + x^3 + x^2y + xy^2 + y^3 + \cdots \\
&= (1 + y + y^2 + \cdots) + (x + xy + xy^2 + \cdots) + (x^2 + x^2y + x^2y^2 + \cdots) + \cdots \\
&= \frac{1}{1-y} + \frac{x}{1-y} + \frac{x^2}{1-y} + \cdots = \frac{1}{1-y} + \frac{1}{1-y} \cdot x + \frac{1}{1-y} \cdot x^2 + \cdots \\
&= \frac{\frac{1}{1-y}}{1-x} = \frac{1}{(1-x)(1-y)}.
\end{aligned}$$

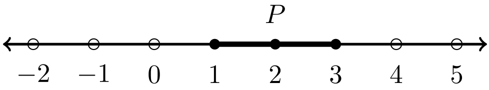
Part III

Main theorem

First example

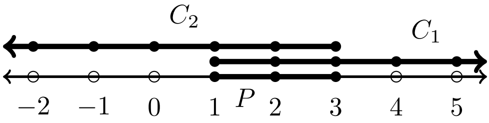
Before stating the main theorem, let's compute some simple examples.

Consider a finite interval $P = [1, 3]$ on the real line. This is a one-dimensional integral polytope.



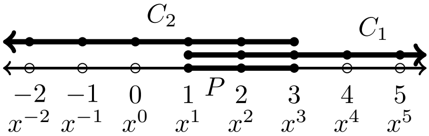
There are three integral points on it, 1, 2, and 3.

For each endpoint, we can draw a ray.



The overlap of these two rays is precisely P.

First example



For each ray C_i, we can make a geometric series $[C_i]$ by adding all x^i's where i is an integer on the ray.

$$[C_1] = x + x^2 + x^3 + x^4 + \cdots = \frac{x}{1-x}$$

$$[C_2] = x^3 + x^2 + x + 1 + x^{-1} + \cdots = \frac{x^3}{1-x^{-1}} = \frac{x^4}{x-1}$$

Let's add them. Then we have:

$$[C_1] + [C_2] = \frac{x}{1-x} + \frac{x^4}{x-1} = \frac{x}{1-x} - \frac{x^4}{1-x}$$

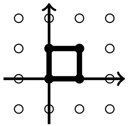
$$= \frac{x-x^4}{1-x} = \frac{x(1-x^3)}{1-x} = \frac{x(1-x)(1+x+x^2)}{1-x} = x(1+x+x^2) = x+x^2+x^3$$

This is precisely the sum of x^i for i on the interval P!

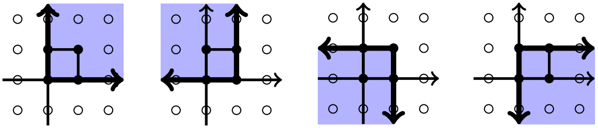
Second example

Now let's look at a two-dimensional example.

Consider a unit square S on the plane:



For each vertex v_k, we can make an internal angle C_k.

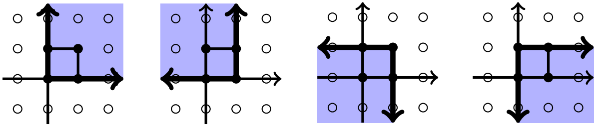


C_1 C_2 C_3 C_4

As we have done before, by adding all $x^i y^j$ on C_k, we can make an infinite sum $[C_k]$. For instance,

$$[C_1] = 1 + x + y + x^2 + xy + y^2 + x^3 + x^2y + xy^2 + y^3 + \cdots = \frac{1}{(1-x)(1-y)}.$$

Second example



C_1 C_2 C_3 C_4

By similar computations, we obtain:

$$[C_1] = \frac{1}{(1-x)(1-y)}$$

$$[C_2] = x+1+xy+x^{-1}+y+xy^2+\cdots = \sum_{i\le 1, j\ge 0} x^i y^j = \frac{x}{(1-x^{-1})(1-y)} = \frac{x^2}{(x-1)(1-y)}$$

$$[C_3] = \sum_{i\le 1, j\le 1} x^i y^j = \frac{xy}{(1-x^{-1})(1-y^{-1})} = \frac{x^2y^2}{(x-1)(y-1)}$$

$$[C_4] = \sum_{i\ge 0, j\le 1} x^i y^j = \frac{y}{(1-x)(1-y^{-1})} = \frac{y^2}{(1-x)(y-1)}$$

Second example

$$[C_1] = \frac{1}{(1-x)(1-y)}, \quad [C_2] = \frac{x^2}{(x-1)(1-y)},$$
$$[C_3] = \frac{x^2y^2}{(x-1)(y-1)}, \quad [C_4] = \frac{y^2}{(1-x)(y-1)}$$

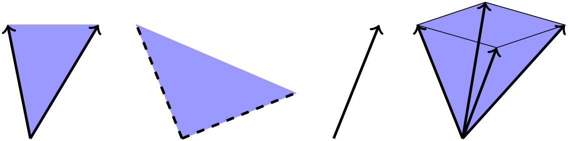
If we add them together, we have:

$$\frac{1}{(1-x)(1-y)} + \frac{x^2}{(x-1)(1-y)} + \frac{x^2y^2}{(x-1)(1-y)} + \frac{y^2}{(1-x)(y-1)}$$
$$= \frac{1-x^2+x^2y^2-y^2}{(1-x)(1-y)} = \frac{(1-x)(1-y)(1+x)(1+y)}{(1-x)(1-y)}$$
$$= (1+x)(1+y) = 1+x+y+xy$$

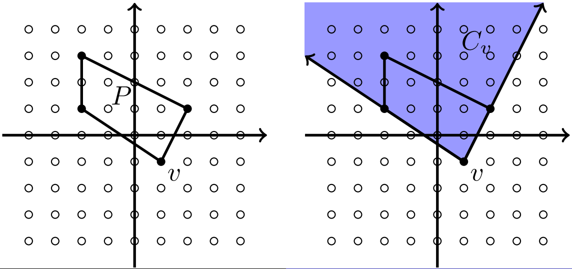
which is exactly, the sum of x^iy^j on the square S.

Some terminologies

A cone with the apex v is a geometric figure that is generated by several rays starting from v.

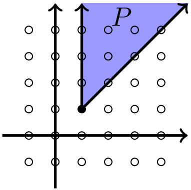


Let P be an integral polytope and v be a vertex. The inner tangent cone C_v to P at v is the internal angle we've described before.



Some teminologies

For any polytope or cone P, let $[[P]]$ be the (possibly infinite) sum of all $x^i y^j$'s where (i,j) is in P.



$$[[P]] = xy + xy^2 + x^2y^2 + xy^3 + x^2y^3 + x^3y^3 + \cdots$$

Let $[P]$ be a representation of $[[P]]$ as a rational function.

$$\begin{aligned} [P] &= (xy + x^2y^2 + x^3y^3 + \cdots) + (xy^2 + x^2y^3 + x^3y^4 + \cdots) + \cdots \\ &= \frac{xy}{1-xy} + \frac{xy^2}{1-xy} + \cdots = \frac{\frac{xy}{1-xy}}{1-y} = \frac{xy}{(1-xy)(1-y)} \end{aligned}$$

If P is an integral polytope, $[[P]] = [P]$ is a finite sum.

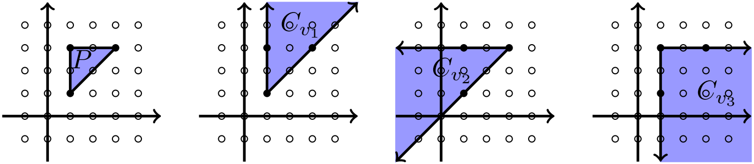
One can generalize them to n-dimensional polytopes.

Main theorem

Theorem (Brion, 1988)

Let P be an integral polytope with vertices $v_1, v_2, \cdots, v_k$. Then

$$[P] = \sum_{i=1}^{k} [C_{v_i}].$$



$$\begin{aligned} [C_{v_1}] + [C_{v_2}] + [C_{v_3}] &= \frac{xy}{(1-y)(1-xy)} + \frac{x^5y^4}{(x-1)(xy-1)} + \frac{xy^4}{(1-x)(y-1)} \\ &= xy + xy^2 + x^2y^2 + xy^3 + x^2y^3 + x^3y^3 = [P] \end{aligned}$$

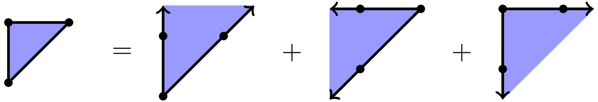
Main theorem

Theorem (Brion, 1988)

Let P be an integral polytope with vertices $v_1, v_2, \cdots, v_k$. Then

$$[P] = \sum_{i=1}^{k} [C_{v_i}].$$

Schematically, we can say:



Part IV

The proof

First reduction - Induction on dimension

Theorem

Let P be an integral polytope with vertices $v_1, v_2, \cdots, v_k$. Then

$$[P] = \sum_{i=1}^{k} [C_{v_i}].$$

The statement of the theorem is saying that "for every n-dimensional integral polytope, a certain equation holds".

It is a nice idea to use mathematical induction on the dimension of integral polytopes.

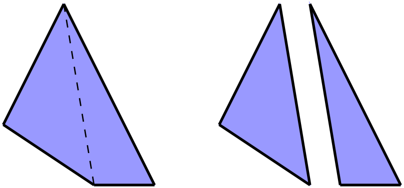
When $n = 1$, the only possible integral polytopes are finite intervals. By the same computation as in the first example, we can check it directly.

So from now, we'll assume that $n = 1$ case is true. I'll give a proof for dimension 2 case, but the same proof works for arbitrary dimensions.

Second reduction - Divide and conquer

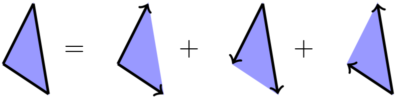
A great idea to solve a mathematical problem is to reduce it into simpler problems.

Every polygon can be decomposed into triangles. For instance, consider the following quadrilateral.



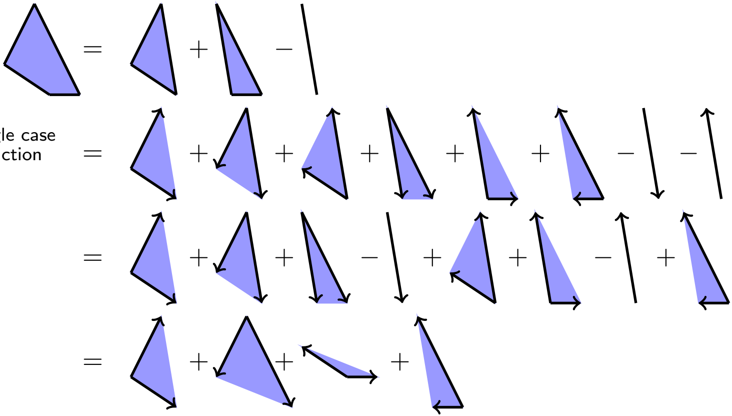
Second reduction - Divide and conquer

Suppose that we can prove the triangle case, that is,



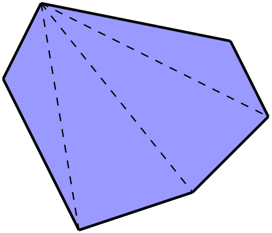
Then it is also true for the quadrilateral case, because:

triangle case induction



Second reduction - Divide and conquer

By applying the same idea several times, we can conclude that it is sufficient to prove the theorem for triangles.



Let's prove the theorem for triangles.

Some algebraic definitions

To complete the proof, we need to define several algebraic notations.

1. $\mathbb{R}[x, y]$: set of polynomials with two variables x and y.
   $x + y^2$, xy, $x + 4x^3$
2. $\mathbb{R}[x^{\pm}, y^{\pm}]$: set of Laurent polynomials.
   $x + y^2$, xy, xy^{-1}, $x^3y^{-2} + 5y^4$
3. $\mathbb{R}[[x, y]]$: set of formal power series.
   $1 + x + y + x^2 + xy + y^2 + \cdots, 1 + x + x^2 + x^3 + \cdots.$
4. $\mathbb{R}[[x^{\pm}, y^{\pm}]]$: set of formal Laurent series.
   $\cdots + x^{-2} + x^{-1} + 1 + x + x^2 + \cdots,$
5. $\mathbb{R}(x, y)$: set of rational functions.
   $\frac{x^2+y}{1+x+y^2}, 1 + x + x^2 + \cdots = \frac{1}{1-x}$

There are some formal Laurent series so that they cannot be written as rational functions.

Infinite sum associated to a cone

Recall: For any polytope or cone C,

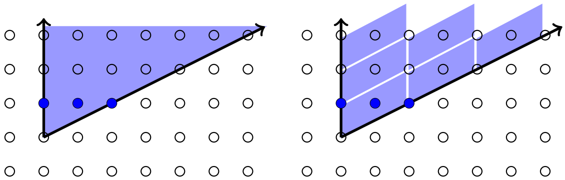
$[[C]]$: infinite sum of all monomials on C. $[[C]] \in \mathbb{R}[[x^{\pm}, y^{\pm}]]$.

$[C]$: its realization as a rational function. $[C] \in \mathbb{R}(x, y)$.

Wait! It is not clear that we can always find a rational function $[C]$.

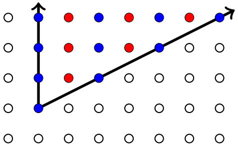
Lemma (Gordan's lemma)

Let C be an integral cone with apex $\mathbf{0}$. Suppose that C does not contain the origin. Then there are finitely many integral points $v_1, v_2, \cdots, v_k$ on C such that any other integral points on C is described as a nonnegative sum of $v_1, v_2, \cdots, v_k$.



Infinite sum associated to a cone

A consequence is that for any integral cone C, we can always find a realization of $[[C]]$ as a rational function $[C]$.



sum of blue terms $= 1+y+x^2y+y^2+x^2y^2+x^4y^2+\cdots = \dfrac{1}{(1-x^2y)(1-y)}$

sum of red terms $= x+xy+x^3y+xy^2+x^3y^2+x^5y^2+\cdots = \dfrac{x}{(1-x^2y)(1-y)}$

$$\text{total sum} = \frac{1+x}{(1-x^2y)(1-y)}$$

Indeed, the above description provides an explicit algorithm finding $[C]$.

Infinite sum associated to a cone

Let PL be the subspace of $\mathbb{R}[[x^{\pm}, y^{\pm}]]$ generated by $[[C]]$ for some cone C (space of polyhedral Laurent series).

Then there is a (unique) map

$$\begin{array}{rcl} p : PL & \to & \mathbb{R}(x, y) \\ [[C]] & \mapsto & [C]. \end{array}$$

The map p preserves the addition, subtraction and polynomial multiplication. In particular, for any $h \in \mathbb{R}[x^{\pm}, y^{\pm}]$, then

$$p(h \cdot [[C]]) = h \cdot [C] = h \cdot p([[C]]).$$

With a formal language, we say that p is an $\mathbb{R}[x^{\pm}, y^{\pm}]$-module homomorphism.

Key observation

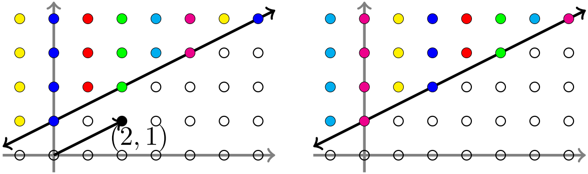
Note that an half-plane H is a union of two cones. So $[[H]] \in PL$. We would like to compute $[H] := p([[H]])$.

Lemma

For a half-plane H, $[H] = 0$.

Proof.

Let (a, b) be any integral vector parallel to the boundary of H.
Then $x^a y^b [[H]] = [[H]]$.
Apply p. Then $x^a y^b [H] = [H]$.
We have $(1 - x^a y^b)[H] = 0$.
By dividing both side by $1 - x^a y^b$, we obtain $[H] = 0$. $\square$



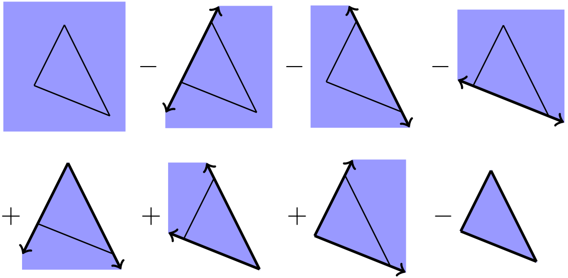
Final step of the proof

For any triangle T, let C_1, C_2, C_3 be three inner tangent cones, H_1, H_2, H_3 be three half-planes generated by three edges, and $\mathbb{R}^2$ be the whole plane.

Consider the following alternating sum:

$$[[\mathbb{R}^2]] - [[H_1]] - [[H_2]] - [[H_3]] + [[C_1]] + [[C_2]] + [[C_3]] - [[T]]$$

Schematically,



This sum is 0. (Why?)

Final step of the proof

$$[[\mathbb{R}^2]] - [[H_1]] - [[H_2]] - [[H_3]] + [[C_1]] + [[C_2]] + [[C_3]] - [[T]] = 0$$

Apply the map p here. Then $[H_1] = [H_2] = [H_3] = 0$.

$[\mathbb{R}^2] = 0$ because $\mathbb{R}^2$ is the union of two half-planes.

So we have $[C_1] + [C_2] + [C_3] - [T] = 0$, or equivalently,

$$[T] = [C_1] + [C_2] + [C_3].$$

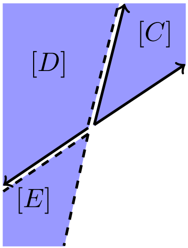
Part V

Final remarks

Ehrhart-MacDonald reciprocity

Recall that for any half-plane H, $[H] = 0$.

We can make an interesting consequence. Consider the following figure:



1. $[C] + [D] = 0$
2. $[D] + [E] = 0$
3. By adding 1 and 2, we have $[C] = [E]$. In other words, any closed cone is equal to the opposite open cone.

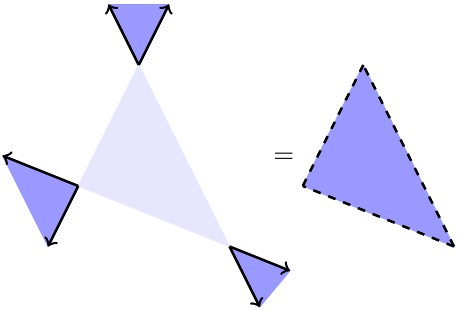
Ehrhart-MacDonald reciprocity

Let T be any triangle. Let E_1, E_2, E_3 be three outer tangent cones, and let C_1, C_2, C_3 be three open inner tangent cones.

Then

$$[E_1] + [E_2] + [E_3] = [C_1] + [C_2] + [C_3] = [T]_{int}$$

where $[T]_{int}$ be the sum of all monomials on the interior of T.



Ehrhart-MacDonald reciprocity

Eugène Ehrhart, 1906 - 2000

Of course, we can generalize it to arbitrary n-dimensional integral polytopes.

Theorem (Ehrhart-MacDonald reciprocity)

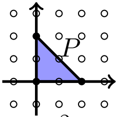
Let P be an integral polytope with vertices $v_1, v_2, \cdots, v_k$. Let E_i be the outer tangent cone to P at v_i. Then

$$[P]_{int} = (-1)^n \sum_{i=1}^{k} [E_i].$$

So... how can we count points?

$$[P] = \sum_{i=1}^{k} [C_{v_i}]$$

The left hand side is the sum of monomials on P.



$$[P] = 1 + x + x^2 + y + xy + y^2$$

The number of integral points on P is $[P](1,1) = 6$.

On the other hand, the right hand side

$$[C_{v_1}]+[C_{v_2}]+[C_{v_3}] = \frac{1}{(1-x)(1-y)} + \frac{y^4}{(y-1)(y-x)} + \frac{x^4}{(x-1)(x-y)}$$

is not defined at $(x,y) = (1,1)$. But we can compute its limit

$$\lim_{(x,y)\to(1,1)} \frac{1}{(1-x)(1-y)} + \frac{y^4}{(y-1)(y-x)} + \frac{x^4}{(x-1)(x-y)} = 6.$$

Original proof

Brion's original proof uses the well-known correspondence

$$\text{integral polytope } P \Longleftrightarrow \text{projective polarized toric variety } (X, L)$$

So we may interpret the result in a purely geometric context.

1. vector space generated by monomials on P: $\mathrm{H}^0(X, L) = \chi(X, L)$.
2. $[P] =$ class of $\chi(X, L)$ in T-equivariant Grothendieck group $K_T^0(X)$.
3. Localization: $\chi(X, L) = \chi(X^T, i^*L) \cdot \gamma_X$. Here $\gamma_X = (\sum_{i \ge 0}(-1)^i[\wedge^i N])^{-1}$, N is the normal bundle to X^T.
4. X^T is a set of torus-invariant points, and for each such point p, $(\chi(X^T, i^*L) \cdot \gamma_X)|_p = [C_p]$.

Thank you!