ALGEBRAIC FIBER SPACE AND STABLE BASE LOCUS

XIAOYAN HU, HAN-BOM MOON

Definition 0.1. An **algebraic fiber space** is a surjective projective mapping $f : X \to Y$ where X, Y are reduced and irreducible variety such that $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Remark 0.2.
(1) All fibers are connected. In the category of analytic varieties, this is clear. For an open subset U,
$$f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U)).$$
If $f^{-1}\{y\}$ is not connected, then we can take $y \in U$ (in analytic category) such that $f^{-1}(U)$ is not connected. Thus by taking a function $f^{-1}(U) \to \mathbb{C}$ which is not a pull-back, it is easy to show that $\mathcal{O}_Y(U)$ is a proper subgroup of $\mathcal{O}_X(f^{-1}(U))$. In the case of algebraic varieties, we need to use an idea of formal schemes. See [Har77, III.11].
(2) If X is normal, then Y is normal.
(3) If $f : X \to Y$ is a surjective projective morphism, such that all fibers are connected and Y is normal, then f is an algebraic fiber space. For the proof, see [Har77, Corollary III.11.4].

Lemma 0.3. *Let $f : X \to Y$ be a fiber space and L be a line bundle on Y. Then*
$$H^0(X, f^*L) = H^0(Y, L).$$

Proof.
$$H^0(X, f^*L) = H^0(Y, f_*f^*L) = H^0(f_*\mathcal{O}_X \otimes L) = H^0(Y, \mathcal{O}_Y \otimes L) = H^0(Y, L).$$
□

Lemma 0.4. *Let $f : X \to Y$ be a fiber space. Then*
$$f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$
is injective.

Proof. If $f^*L = \mathcal{O}_X$, $\mathcal{O}_Y = f_*\mathcal{O}_X = f_*f^*L = f_*\mathcal{O}_X \otimes L = \mathcal{O}_Y \otimes L = L$. Thus $\ker f^* = \{0\}$. □

Definition 0.5. Let $\mathrm{Bs}(|D|)$ be the base locus of the linear system $|D|$. The **stable base locus** is
$$B(D) = \bigcap_{m \geq 1} \mathrm{Bs}(|mD|).$$

Date: April 12, 2013.

Proposition 0.6. *(1) $B(D)$ is a unique minimal element of $\{\mathrm{Bs}(|mD|)\}_{m \geq 1}$.*

(2) There exists m_0 such that

$$B(D) = \mathrm{Bs}(|km_0 D|)$$

for all $k \geq 1$.

REFERENCES

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. 1