Semiample line bundles & Iitaka fibration

Def. X : proj. var. L : line bdl.

L is semiample if

L^m is base point free for some $m>0$.

Rmk. ample $\Rightarrow$ semi-ample $\Rightarrow$ nef

$$\begin{aligned} N(X.L) &= \{ m \mid H^0(X.L^m) \neq 0 \} \\ &\cup \\ M(X.L) &= \{ m \mid |L^m| \text{ is bpf} \} \\ &\Psi \\ & f \cdots \text{ exponent.} \quad e \gg 0 \in M(X.L) \Rightarrow f \mid e \end{aligned}$$

$m \in M(X.L) \Rightarrow \phi_m : X \xrightarrow{|L^m|} Y_m \subset \mathbb{P}H^0(X.L^m).$

Q. what happens if we change m?

X : normal proj. var.

$$\begin{tikzcd} X \arrow[r, "\phi_{km}", "|L^{km}|"'] \arrow[d, "\phi_m"', "|L^m|"] \arrow[dr, "S^k|L^m|"] & Y_{km} \arrow[d, "\nu_k"] \\ Y_m \arrow[r, "\cong"] & Y_m \end{tikzcd}$$

ν_k : projection. $(S^k|L^m| \subset |L^{km}|)$

Rmk. $\nu_k : Y_{km} \longrightarrow Y_m$ is finite.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y_{km} \arrow[d] & \subset \mathbb{P}(V) \arrow[d] \\ & Y_m & \subset \mathbb{P}(W) \end{tikzcd}$$

$\leftarrow$ proper & affine $\Rightarrow$ finite.

Prop. If $k \gg 0$. $X \xrightarrow{\phi_{km}} Y_{km} \xrightarrow{\nu_k} Y_m$

is the Stein factorization of ϕ_m.

i.e. ◦ ϕ_{km} is an alg. fiber sp

◦ ν_k is finite.

Cor. ① ϕ_{km}. Y_{km} are independent of k for $k \gg 0$.

② Y_{km} is normal for $k \gg 0$.

proof.
$$\begin{tikzcd} X \arrow[rr, "\phi_m"] \arrow[dr, "\psi"'] & & Y_m \\ & Y \arrow[ur, "\mu"'] & \end{tikzcd}$$
... stein factorization.

$\mu \leftarrow$ finite

$Y \leftarrow$ normal.

A_m : v.ample on Y_m. s.t. $\phi_m^* A_m = L^m$.

$B := \mu^* A_m$: ample $\Rightarrow B^k$: v. ample

$\psi^* B^k = \psi^* \mu^* A_m^k = \phi_m^* A_m^k = L^{km}$

$H^0(X.L^{km}) = H^0(X.\psi^* B^k) = H^0(Y.B^k)$

ψ: alg fib sp.

$\Rightarrow$ Y is image of $\phi_{km}: X \xrightarrow{|L^{km}|} Y$

$\Rightarrow$ $Y = Y_{km}$.

Thm. ① X: normal proj. var. L: semi ample

$\Rightarrow$ $\exists$ alg. fib. space $\phi: X \longrightarrow Y$

s.t. $Y = Y_m$. $\phi = \phi_m$ for $m \gg 0$.

Y is normal. $L^m = \phi^* A$.

② Indeed $L^f = \phi^* A$ where f is the exponent of $M(X, L)$.

Cor. ① X: normal variety.

L: base point free.

$\Rightarrow$ $\exists m$. s.t. $\forall a, b \geq m$.

$$\begin{array}{cccccc}
 & H^0(X, L^a) & \otimes & H^0(X, L^b) & \twoheadrightarrow & H^0(X, L^{a+b}) \\
 & \cong & & \cong & & \cong \\
X \xrightarrow{\phi} Y & H^0(X, \phi^* A^a) & \otimes & H^0(X, \phi^* A^b) & & H^0(X, \phi^* A^{a+b}) \\
\text{alg. fib sp} & \cong & & \cong & & \cong \\
 & H^0(Y, A^a) & \otimes & H^0(Y, A^b) & \twoheadrightarrow & H^0(Y, A^{a+b})
\end{array}$$

② $R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^m)$ is f.g.

Cor. X: normal proj. var

L: semi ample line bdl.

$\Rightarrow$ $R(X, L)$ is f.g

Proof. L^m is bpf

$\Rightarrow$ $R(X, L)^{(m)} = \bigoplus_{k \geq 0} H^0(X, L^{km})$ is f.g.

$R(X, L)$ is int. ext. of $R(X, L)^{(m)}$

$\Rightarrow$ $R(X, L)$ is f.g.

Rmk. ① $\dim Y = \kappa(X, L)$

② For a fiber F of $\phi: X \longrightarrow Y$,

$\kappa(F, L|_F) = 0$.

$(L^f = \phi^* A \Rightarrow L^f|_F = \mathcal{O}_F$.

$\Rightarrow \kappa(F, L|_F) = 0)$

Q. What can we say for a general L?

For $k \in N(X, L)$,

$\phi_k : X \overset{|L^k|}{\dashrightarrow} Y_k \subset \mathbb{P} H^0(X, L^k)$.

Thm. (Iitaka fibration)

X : normal proj. var.

L : line bdl with $\kappa(X, L) > 0$.

$\exists$ alg. fiber space $\phi_\infty : X_\infty \to Y_\infty$

of normal var. s.t.

① $\forall k \gg 0 \in N(X, L)$.

$$
\begin{tikzcd}
X \arrow[d, dashed, "\phi_k"', "|L^k|"] & X_\infty \arrow[l, "u_\infty"'] \arrow[d, "\phi_\infty"] \arrow[dl, phantom, "\circlearrowright"] \\
Y_k & Y_\infty \arrow[l, dashed, "\nu_k"]
\end{tikzcd}
$$

u_∞, ν_k birational.

② $L_\infty := u_\infty^* L$.

For a general fiber F,

$\kappa(F, L_\infty|_F) = 0$