VOLUME OF BIG DIVISORS

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Definition 0.1. Let X be a projective variety. A line bundle L is **big** if $\kappa(X, L) = \dim X$.

Definition 0.2. Let X be n dimensional irreducible variety and L be a big line bundle on X.

$$\mathrm{Vol}_X(L) = \limsup_{m \to \infty} \frac{h^0(X, mL)}{m^n/n!}$$

Remark 0.3. By asymptotic Riemann-Roch theorem ([Laz04, Corollary 1.4.41]), if L is nef, $\mathrm{Vol}_X(L) = L^n$.

We will show that vol_X gives a map $\mathbb{N}^1_{\mathbb{R}}(X) \to \mathbb{R}$ and moreover that it is birational in the sense that if $\nu : X' \to X$ is birational map, then

$$\mathrm{Vol}_{X'}(\nu^* L) = \mathrm{Vol}_X(L).$$

Example 0.4. If $X = \mathbb{P}^n$ and $L = \mathcal{O}(k)$, then $h^0(X, mL) = \binom{n+mk}{n}$.

$$\binom{n+mk}{n} \sim \frac{(mk)^n}{n!}$$

So $\mathrm{Vol}_{\mathbb{P}^n}(\mathcal{O}(k)) = k^n$.

Remark 0.5. (1) $\mathrm{Vol}_X(aL) = a^n \mathrm{Vol}_X(L)$.

We need to show

$$\limsup_{k \to \infty} \frac{h^0(X, kL)}{k^n} = \limsup_{k \to \infty} \frac{h^0(X, akL)}{(ak)^n}.$$

Since L is big, there is r_0 such that if $r \in \mathbb{N}(X, L)$ and $r \geq r_0$ then $H^0(X, rL) \neq 0$ ([Laz04, Lemma 2.2.3]). Similarly, we can find $q \gg 0$ such that $H^0(X, (qa - r)L) \neq 0$. Then there are injective morphisms

$$H^0(X, kaL) \hookrightarrow H^0(X, (ka + r)L) \hookrightarrow H^0(X, (q + k)aL).$$

Thus

$$\limsup_{k \to \infty} \frac{h^0(X, kaL)}{(ka)^n} \leq \limsup_{k \to \infty} \frac{h^0(X, (ka + r)L)}{(ka)^n} = \limsup_{k \to \infty} \frac{h^0(X, (ka + r)L)}{(ka + r)^n}.$$

Similarly,

$$\limsup_{k \to \infty} \frac{h^0(X, (ka + r)L)}{(ka + r)^n} \leq \limsup_{k \to \infty} \frac{h^0(X, (q + k)aL)}{(ka + r)^n}$$

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$$= \limsup_{k\to\infty} \frac{h^0(X,(q+k)aL)}{((q+k)a)^n} = \limsup_{k\to\infty} \frac{h^0(X,kaL)}{(ka)^n}.$$

So if we define

$$V_i = \limsup_{k\to\infty} \frac{h^0(X,(ka+i)L)}{(ka+i)^n} = \limsup_{k\to\infty} \frac{h^0(X,(ka+i)L)}{(ka)^n},$$

then $V_i \le V_0$ for all i such that $ka+i \in \mathbb{N}(X,L)$. Since $\mathrm{Vol}_X(L) = \max\{V_i | ka+i \in \mathbb{N}(X,L)\}$, all of them are same to $\mathrm{Vol}_X(L)$.

(2) $\lim_{p\to\infty} \frac{1}{p^n}\mathrm{Vol}(pL-N) = \mathrm{Vol}(L)$ for any N.

If $N = A-B$ for two effective divisors A and B, since L is big, there exists $r>0$ such that $rL-B$ is effective. Then

$$\limsup_{k\to\infty} \frac{h^0(kL-N)}{k^n} = \limsup_{k\to\infty} \frac{h^0((k-r)L-N)}{(k-r)^n}$$
$$= \limsup_{k\to\infty} \frac{h^0((k-r)L-N)}{k^n} = \limsup_{k\to\infty} \frac{h^0(kL-A-(rL-B))}{k^n}$$

So we may assume that N is effective.

Let N' be a very ample line bundle such that $N'+N$ is very ample. Then

$$\mathrm{Vol}_X(pL-(N+N')) \le \mathrm{Vol}_X(pL-N) \le \mathrm{Vol}_X(pL),$$

so

$$\lim_{p\to\infty} \frac{1}{p^n}\mathrm{Vol}_X(pL-(N+N')) \le \lim_{p\to\infty} \frac{1}{p^n}\mathrm{Vol}_X(pL-N)$$
$$\le \lim_{p\to\infty} \frac{1}{p^n}\mathrm{Vol}_X(pL) = \mathrm{Vol}_X(L)$$

Thus it is enough to check a very ample N.

Pick k general elements $E_i \in |N|$. From the exact sequence

$$0 \to H^0(X,k(pL-N)) \to H^0(X,kpL) \to \bigoplus_k H^0(E_i, kpL|_{E_i}),$$

If we take general elements E_i, then $h^0(E_i, mpL|_{E_i})$ is independent from i. Thus

$$h^0(X,m(pL-N)) \ge h^0(X,mpL) - mh^0(X,E_i,mpL|_{E_i})$$

and

$$\mathrm{Vol}_X(pL-N) \ge \mathrm{Vol}_X(pL) - O(p^{n-1}).$$

Therefore, $\lim_{p\to\infty} \frac{1}{p^n}\mathrm{Vol}_X(pL-N) \ge \mathrm{Vol}_X(L)$.

Since $\mathrm{Vol}_X(pL-N) \le \mathrm{Vol}_X(pL) = p^n\mathrm{Vol}_X(L)$, the opposite inequality is clear.

Theorem 0.6. *(Fujita vanishing theorem) For an ample divisor B and a coherent sheaf F, then there exists $n_0>0$ which depends only on B and F, such that*

$$H^i(X, F\otimes \mathcal{O}(nB+A)) = 0$$

for all $n \ge n_0$, $i>0$ and all nef divisor A.

Lemma 0.7. *There exists N such that $N+P$ is effective for all numerically trivial P.*

Proof. Choose an ample divisor B. By Fujita vanishing theorem, there is k such that $H^i(X, kB + A) = 0$ for all nef A. Note that $P + (n - i)B$ is nef for $i < n$. So $H^i(X, kB + P + (n - i)B) = 0$ for $i > 0$. In particular, $kB + P + nB$ is 0-regular with respect to B (See [Laz04, Definition 1.8.4 and Theorem 1.8.5].). So $kB + P + nB$ is globally generated for all P. Thus we can take $N = kB + nB$. □

Proposition 0.8. $\mathrm{Vol}_X(L)$ *is a numerical invariant.*

Proof. Enough to show that $\mathrm{Vol}_X(D + P) \leq \mathrm{Vol}_X(D)$ for a numerically trivial P. By the lemma above, choose N such that $H^0(X, N - pP) \neq 0$ and chose a section for each p. Then we get

$$H^0(X, mp(D + P) - N) \hookrightarrow H^0(X, mpD)$$

So by Remark 0.5 (2) above, we get the result. □

References

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. 1, 3