NEF LINE BUNDLES AND DIVISORS

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Recall the Nakai-Moishezon criterion. Let X be a projective variety. Pick $\delta \in N^1(X)_{\mathbb{Q}} = \mathrm{Div}(X)/\mathrm{Num}(X) \otimes \mathbb{Q}$. Then

$$\delta \text{ is ample } \Leftrightarrow \int_V \delta^{\dim V} > 0$$

for all irreducible closed subvariety V with $\dim V > 0$. Thus the limit of ample classes satisfies

$$\int_V \delta^{\dim V} \geq 0.$$

Definition 0.1. Let X be a complete variety. A line bundle L on X is **nef** if

$$\int_C c_1(L) \geq 0$$

for all irreducible complete curve C on X. For $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$ is **nef** if $D \cdot C \geq 0$ for all irreducible complete curve C on X.

Properties of nefness

Let X be a complete variety, L be a line bundle on X.

- If $f : Y \to X$ is proper, L is nef then f^*L is nef.
- If $f : Y \to X$ is proper and surjective, f^*L is nef then L is nef.
- L is nef $\Leftrightarrow$ L_{red} is nef on X_{red}.
- L is nef $\Leftrightarrow$ $L|_V$ is nef for all irreducible component of X.

Proof. Fix C irreducible curve on Y. Then by projection formula,

$$f^*L \cdot [C] = L \cdot [f_*C] = \begin{cases} 0 & f_*C = \mathrm{pt} \\ \geq 0 & f_*C = C'. \end{cases}$$

Fix C irreducible curve on X. Then there is a curve $\widetilde{C}$ on Y such that $f_*[\widetilde{C}] = m[C]$ by taking sufficient hyperplane intersections of $f^{-1}(C)$. Then

$$L \cdot m[C] = L \cdot f_*[\widetilde{C}] \geq 0,$$

so $L \cdot [C] \geq 0$. □

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Let X be a complete variety and L be a globally generated line bundle on X. Then L is nef, because for any curve C and $x \in C$, we can find a section of L does not vanish at x, so $L \cdot C = \deg L|_C \geq 0$.

Proposition 0.2. *Let X be a complete variety, D be an effective divisor. Suppose that $N_{D/X} = \mathcal{O}_D(D)$ is nef. Then D is nef.*

Proof. Fix an irreducible curve C on X. If $C \not\subset D$, then $C \cdot D \geq 0$. If $C \subset D$,

$$D \cdot C = \deg \mathcal{O}_C(D) = \deg \mathcal{O}_D(D)|_C \geq 0$$

by nefness. □

Proposition 0.3. *Let X be a complete variety and G be a connected algebraic group acts on X transitively. Then any effective divisor D on X is nef.*

Typical examples are abelian varieties, $\mathbb{P}^n$, Grassmannians, flag varieties.

Proof. Fix an effective divisor D. Fix an irreducible curve C. If $C \subset D$, then we can find $g \in G$ such that gD does not pass a point $x \in C$. Thus $gD \cdot C = D \cdot C \geq 0$ because $gD \equiv D$ since G is a connected group. □

Theorem 0.4. *(Kleiman's theorem) Let X be a complete variety, D be a nef $\mathbb{R}$-divisor on X. Then $D^k \cdot V \geq 0$ for all irreducible closed subvariety $V \subset X$ of $\dim V = k$.*