AMPLENESS

ADRIAN BRUNYATE, HAN-BOM MOON

Definition 0.1. A line bundle L on a complete scheme X is **very ample** if there exists an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $i^*\mathcal{O}(1) = L$. L is **ample** if there is $m \in \mathbb{N}$ such that L^m is very ample.

Theorem 0.2. *The followings are equivalent.*

1. *L is ample.*
2. *For any coherent sheaf F on X, there exists m_1 such that for all $m \geq m_1$, $H^i(X, L^m \otimes F) = 0$ for $i > 0$.*
3. *There exists m_2 such that for all $m \geq m_2$, $F \otimes L^m$ is globally generated.*

Two Cartier divisors D_1, D_2 are **numerically equivalent** if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve C.

Theorem 0.3 (Nakai-Moishezon). *L is ample if and only if for every irreducible closed subvariety V,*

$$\int_V c_1(L)^{\dim V} > 0.$$

Corollary 0.4. *Ampleness is preserved by numerical equivalence.*

Example 0.5. Let $\mathbb{F}_1 = \mathrm{Bl}_p\mathbb{P}^2$. Then $\mathrm{NS}(\mathbb{F}_1) = \langle e, f\rangle$ where e is the exceptional divisor, f is the proper transform of the line passing through p. Now $e^2 = -1, e \cdot f = 1, f^2 = 0$. So both e and f are not ample (self-intersection is not positive).

My claim is that an effective curve on $\mathbb{F}_1$ is a non-negative linear combination of e and f. First of all, the pull-back of hyperplane on $\mathbb{P}^2$ is $e + f$. It is not hard to see it is a base-point-free. Also, f is another base-point-free divisor because it is a fiber on the ruled surface $\mathbb{F}_1$. So if we have an effective curve C on $\mathbb{F}_1$, then both $C \cdot (e + f)$ and $C \cdot f$ are non-negative. If $C = ae + bf$, then $(ae + bf)(e + f) = b$ and $(ae + bf)f = a$. Therefore both a and b are non-negative and C is an effective linear combination of e and f. Thus the cone of effective curves is generated by e and f.

Now let $ce + df$ be an ample divisor. By Nakai-Moishezon criterion, it is ample if and only if

- $(ce + df)^2 = -c^2 + 2cd > 0$ (the case that V is a point) and

Date: February 2, 2013.

- $(ce + df)(ae + bf) = -ac + ad + bc > 0$ for all non-negative a and b (the case that V is a curve).

Exercise 0.6. Check that we can replace the second condition to a simpler condition that $(ce + df)e = -c + d > 0$ and $(ce + df)f = c > 0$.

Thus we can obtain that $ce + df$ is ample if and only if $c > d > 0$.