NEF LINE BUNDLES AND DIVISORS - II

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Theorem 0.1 (Kleiman)**.** *If D is a $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$-Cartier divisor on a complete scheme. Suppose that D is nef ($C \cdot D \geq 0$ for all irreducible curve C). Then*

$$D^{\dim V} \cdot V \geq 0$$

for all closed subvariety $V \subset X$.

Therefore, a nef divisor is a limit of ample divisors. See Corollary 0.2 below.

Proof. Step 1. We may assume that X is irreducible and reduced.
By some basic properties of nefness we discussed in last week. By using Chow's lemma, we may also assume that X is projective. So let X be a projective variety.

Step 2. We will use induction on dimension.
If $\dim X = 1$, it is obvious. So assume the theorem for $\dim X = n - 1$ i.e, $D^k \cdot V \geq 0$ for any subvariety V with $\dim V = k \leq n - 1$. Thus we need to show $D^n \geq 0$.

Step 3. Suppose that D is a $\mathbb{Q}$-divisor with $D^n < 0$.
Pick an ample $\mathbb{Z}$-divisor H on X. Consider

$$P(t) = (D + tH)^n = D^n + nD^{n-1} \cdot Ht + \binom{n}{2} D^{n-2}H^2t^2 + \cdots + H^n t^n.$$

$P(0) = D^n < 0$. Then $D^{n-k}H^k \geq 0$ for all $1 \leq k \leq n$, because 1) H^k is a $\mathbb{Q}$-effective cycle, and 2) induction hypothesis. Furthermore, H^n is strictly positive by Nakai-Moishezon criterion.

Since the leading coefficient of $P(t)$ is positive, there exists $t_0 > 0$ such that $P(t_0) = 0$. Pick $t_1 \in \mathbb{Q}$ with $t_1 > t_0$. Then $P(t_1) > 0$.

Step 4.A $\mathbb{Q}$-divisor $D + t_1 H$ is ample.
For any irreducible subvariety V with $\dim V = k$, $(D + t_1 H)^k \cdot V > 0$ because all terms are nonnegative and $H^k \cdot V > 0$. Also $(D + t_1 H)^n > 0$ by the construction of t_1. Therefore by Nakai-Moishezon criterion again, it is ample.

Step 5. Make a contradiction.
Now

$$P(t) = (D + tH)^n = (D + tH)(D + tH)^{n-1}$$
$$= D(D + tH)^{n-1} + tH(D + tH)^{n-1} =: R(t) + S(t).$$

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For all $t_1 \in \mathbb{Q}$ with $t_1 > t_0$, $D+t_1H$ is ample, so $(D+t_1H)^{n-1}$ is an effective $\mathbb{Q}$-curve. Thus $D(D+t_1H)^{n-1} \geq 0$ because D is nef. Therefore $R(t_1) \geq 0$ for all $t_1 > t_0$. By continuity, $R(t_0) \geq 0$.

On the other hand, $S(t_0) > 0$ since

$$S(t_0) = t_0H(D+t_0H)^{n-1} = t_0HD^{n-1} + t_0^2H^2D^{n-2} + \cdots t_0^2H^n,$$

where all terms are nonnegative and the last term is strictly positive. $0 = P(t_0) = R(t_0) + S(t_0) > 0$, arise a contradiction.

Step 6.Case of $\mathbb{R}$-divisor.

Now suppose that D is a nef $\mathbb{R}$-divisor. Fix $H_1, \cdots, H_r$ be ample divisor which span $N^1(X)_{\mathbb{R}}$. Consider $D + \epsilon_1H_1 + \cdots + \epsilon_rH_r$ with $0 \leq \epsilon_i \ll 1$. We can choose ϵ_i such that $D + \epsilon_1H_1 + \cdots + \epsilon_rH_r$ is a nef $\mathbb{Q}$-divisor. By the previous case of $\mathbb{Q}$-divisors, $(D + \epsilon_1H_1 + \cdots + \epsilon_rH_r)^{\dim V} \cdot V \geq 0$ for all irreducible closed subvariety $V \subset X$. By continuity again, $D^{\dim V} \cdot V \geq 0$, so it is nef. □

Corollary 0.2. *Let D be a nef divisor and H be an ample divisor. Then $D + \epsilon H$ is ample for all $\epsilon > 0$.*

Proof. (In a case of $\mathbb{Q}$-divisor) For any irreducible subvariety V with $\dim V = k$, $(D+\epsilon H)^k \cdot V > 0$, because $D^{k-i}H^iV \geq 0$ for all $k \geq 1$ and $H^k \cdot V > 0$. □