$\mathbb{Q}$-DIVISORS AND $\mathbb{R}$-DIVISORS

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By introducing $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors, we can apply some techniques from analysis (like perturbation) to study divisors.

Definition 0.1.

$$\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{Div}_{\mathbb{Z}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

So for $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, $D = \sum c_i A_i$ where $c_i \in \mathbb{Q}$ and $A_i \in \mathrm{Div}_{\mathbb{Z}}(X)$. By finding a common denominator, we can write D as $D = cA$ where $A \in \mathrm{Div}_{\mathbb{Z}}(X)$ and $c \in \mathbb{Q}$.

$D \in \sum c_i A_i$ is **effective** if $c_i > 0$ and A_i are effective divisors.

Definition 0.2. (Support of D) Let $E \subset X$ be a codimension 1 subset. Then E is a **support** of D if $D = \sum c_i A_i$ such that $\cup |A_i| \subset E$.

We can define intersection product on $\mathrm{Div}_{\mathbb{Q}}(X)$: Do intersections on integral divisors and extend linearly. Also we can define the numerical equivalence.

$$N^1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

Finally, two $\mathbb{Q}$-divisors D_1, D_2 are linearly equivalent if

$$rD_1 \equiv_{\mathrm{lin},\mathbb{Z}} rD_2$$

for some $r \in \mathbb{Z}$.

Example 0.3. It can happen for $D_1, D_2 \in \mathrm{Div}_{\mathbb{Z}}$, $D_1 \not\equiv_{\mathrm{lin},\mathbb{Z}} D_2$ but $D_1 \equiv_{\mathrm{lin},\mathbb{Q}} D_2$. Let X be an elliptic curve. Take a 2-torsion point $P \in X$ and the identity $O \in X$. Then $P - O \not\equiv_{\mathrm{lin},\mathbb{Z}} 0$, but $2(P - O) \equiv_{\mathrm{lin},\mathbb{Z}} 0$, so $P - O \equiv_{\mathrm{lin},\mathbb{Q}} 0$.

We can also define pull-back of $\mathbb{Q}$-divisor.

Definition 0.4. A $\mathbb{Q}$-divisor D is **ample** if one of following holds.

(1) $D = \sum c_i A_i$ where $c_i > 0$ and A_i are ample.
(2) $\exists$ an integer $r > 0$ such that rD is integral and ample.
(3) $D^{\dim V} \cdot [V] > 0$ for any irreducible closed subvariety $V \subset X$.

Proposition 0.5. *Let X be a projective variety. Let H be an ample $\mathbb{Q}$-divisor and E be an arbitrary Cartier divisor. Then $H + \epsilon E$ is ample for every $0 < |\epsilon| \ll 1$.*

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Thus the ampleness is an open condition. Therefore $N^1(X)_{\mathbb{Q}}$ is spanned by ample divisors.

Remark 0.6. If X is normal,

$$\mathrm{Div}_{\mathbb{Z}}(X) \hookrightarrow \mathrm{WDiv}_{\mathbb{Z}}(X).$$

So

$$\mathrm{Div}_{\mathbb{Q}}(X) \hookrightarrow \mathrm{Wdiv}_{\mathbb{Q}}(X).$$

$D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$ is $\mathbb{Q}$**-Cartier** if it lies in the image of this inclusion.

Example 0.7. Consider a quadric cone $V(xy - z^2)$ in $\mathbb{P}^3$. A line class $V(y, z)$ passing through the origin is a Weil divisor but not Cartier. But $A = 2D$ is Cartier (See [Har77, II.6.5.2].) Now A is linearly equivalent to $V(x)$ and $V(y)$. Thus

$$A \cdot A = \dim_k k[x, y, z]/(xy - z^2)_{(x,y,z)}/(x, y) = \dim_k k[z]/(z^2) = 2.$$

Hence

$$D \cdot D = \frac{1}{2}A \cdot \frac{1}{2}A = \frac{1}{4}A \cdot A = \frac{1}{2}.$$

Remark 0.8. We can define the notion of $\mathbb{R}$-divisor by replacing $\mathbb{Q}$ by $\mathbb{R}$. We want to define the notion of ample divisor as Definition 0.4. (1) $\Rightarrow$ (3) is clear, but (3) $\Rightarrow$ (1) is proved by Campana and Paternell in 70's. (2) is meaningless because we can't take a common denominator for real numbers.

REFERENCES

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. 2