Cone theorems.

Recall.

Thm. (Kleiman). X: proj. var.

D: $\mathbb{R}$-Cartier divisor.

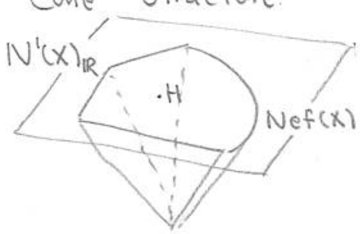
D is nef $\Longleftrightarrow$ $D^{\dim V} \cdot V \geq 0$

$\forall\, V$: irred. closed subvar.

Cor. D: nef. H: ample.

$\Rightarrow D + \varepsilon H$: ample. $\forall\, \varepsilon > 0$.

Cone structure.



Thm. ① $\mathrm{int}(\mathrm{Nef}(X)) = \mathrm{Amp}(X)$

② $\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X)$

Def. ① $\gamma = \sum a_i C_i$ ⋯ $\mathbb{R}$-lin. comb. of irred. curves.

γ_1, γ_2 are numerically equivalent if

$(D\cdot\gamma_1) = (D\cdot\gamma_2)$ $\forall\, D \in N^1(X)_{\mathbb{R}}$.

② $N_1(X)_{\mathbb{R}}$ = v. sp of. num. equi. classes of curves.

$\exists$ pairing. $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$

$D \cdot \gamma \longmapsto (D\cdot\gamma)$

Def. $NE(X) = \{ \sum a_i C_i \mid a_i \geq 0 \}$.

$\overline{NE}(X) = \overline{NE(X)}$.

$\mathrm{Nef}(X) = \overline{NE}(X)^* = \{ D \in N^1(X)_{\mathbb{R}} \mid (D\cdot\gamma) \geq 0.\ \forall \gamma \in \overline{NE}(X) \}$

$\Rightarrow \overline{NE}(X) = \mathrm{Nef}(X)^* = \{ \gamma \in N_1(X)_{\mathbb{R}} \mid (D.\gamma) \geq 0\ \forall D \in \mathrm{Nef}(X) \}$

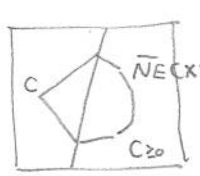
Ex) X: proj. surf. $(N^1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{R}})$

① $\mathrm{Nef}(X) \subset \overline{NE}(X)$.

② $\mathrm{Nef}(X) = \overline{NE}(X) \Longleftrightarrow C^2 \geq 0.$ $\forall$ irred. C

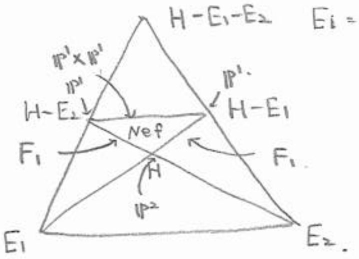
③ $C^2 \leq 0 \Rightarrow \overline{NE}(X) = \mathrm{Span}_{\geq 0}\{ C, C_{\geq 0} \}$

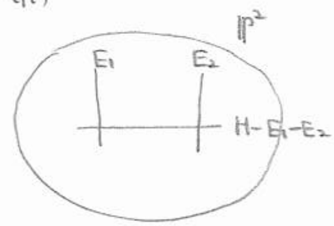
C: irred. $\quad C_{\geq 0} = \{ C' \mid (C'.C) \geq 0 \}$



Ex). $X = Bl_{p_1,p_2}\mathbb{P}^2$. $H = \pi^*\mathcal{O}(H)$. $H^2 = 1$. $E_1^2 = E_2^2 = -1$. $H\cdot E_i = 0$. $E_1\cdot E_2 = 0$.

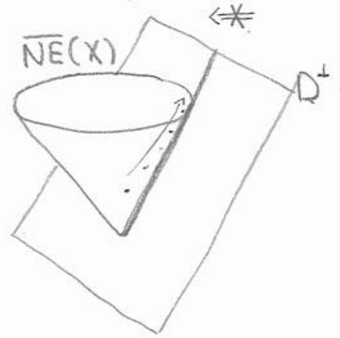
$E_i = \pi^{-1}(p_i)$





Rmk. H: ample $\Longrightarrow$ $H\cdot C > 0$. $\forall$ irred. curve.
$\not\Longleftarrow$

H: ample $\Longrightarrow$ $H\cdot C > \varepsilon$ $\forall$ irred curve.
$\not\Longleftarrow$ $\varepsilon > 0$.



Thm. (Seshadri).

X: proj. var. D: $\mathbb{R}$-Cartier divisor.

D: ample $\Longleftrightarrow$ $\exists\, \varepsilon > 0$.

$$\frac{(D\cdot C)}{\mathrm{mult}_x C} \geq \varepsilon.$$

$\forall x \in X$. $x \in C \subset X$.

| $\mathrm{mult}_x C = \deg \mathbb{P}(C)_x$

$\mathbb{P}(C)_x$: $\underset{\text{proj.}}{\hat{}}$tangent cone of C at x.

mult. 3 

$P_t = \dim_{\mathbb{C}}\left(\sum_{i=1}^{t} m_x^{i-1}/m_x^{i}\right)$. ... poly. of deg 1.

$= at + b$.

$\mathrm{mult}_x C = a$.

proof. ($\Rightarrow$).

Pick m. s.t. mD is very ample.

$\Rightarrow$ $(mD).C) \geq \mathrm{mult}_x C$ because

we can take a div. $E \in |mD|$ s.t

E & C intersect at x generically.

$\Rightarrow$ $(D.C) \geq \frac{1}{m}\cdot \mathrm{mult}_x C$.

($\Leftarrow$). Use induction on dim. X.

ETS $D^n > 0$. (Nakai - Moishezon)

$x \in X$. sm. pt.

$\mu : X' = Bl_x X \longrightarrow X$ $\quad E$: except. div.

Claim. $\mu^* D - \varepsilon E$ is nef.

If then $(\mu^* D)^n - \varepsilon^n = (\mu^* D - \varepsilon E)^n \geq 0$.

$$\underset{D^n - \varepsilon^n}{\overset{(\mu^* D)^n - \varepsilon^n}{\parallel}}$$

$\Rightarrow$ $D^n \geq \varepsilon^n > 0$. Done.

$C' \subset X'$ irred. curve.

① $C' \subset E \Rightarrow (\mu^* D - \varepsilon E) \cdot C' = -\varepsilon E \cdot C' > 0$.

② $C' \not\subset E \Rightarrow \mu(C') =: C$.

$$(\mu^* D - \varepsilon E) \cdot C' = \mu^* D \cdot C' - \varepsilon E \cdot C'$$
$$= D \cdot C - \varepsilon \operatorname{mult}_x C. \geq 0.$$

$\Rightarrow$ $\mu^* D - \varepsilon E$ is nef.