EXAMPLES OF FUNNY CONES OF CURVES

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1. PRODUCT OF ELLIPTIC CURVES

Let E be a smooth projective complex irreducible genus g curve. $X = E \times E$. Pick $p \in E$. Let $f_1 = \{p\} \times E$, $f_2 = E \times \{p\}$, and δ be the diagonal.

- f_1, f_2, δ are independent in $\mathrm{N}_1(X)$.
- For general E, f_1, f_2, δ span $\mathrm{N}_1(X)$.
- $f_1\delta = f_2\delta = f_1f_2 = 1$, $f_1^2 = f_2^2 = 0$, $\delta^2 = 2 - 2g$.

If $g = 1$, X is an abelian surface.

Lemma 1.1. *(1)* $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)$.
(2) A class $\alpha \in \mathrm{N}^1(X)_{\mathbb{R}}$ *is nef* $\Leftrightarrow \alpha^2 \geq 0$ *and* $\alpha \cdot h \geq 0$ *for some* h *ample.*

Proof. (1) On a surface, $\mathrm{Nef}(X) \subset \overline{\mathrm{NE}}(X)$ and we have equality $\Leftrightarrow C^2 \geq 0$ for all irreducible curves $C \subset X$.

If τ is a general automorphism of X, then $\tau C \equiv_{num} C$ and τC and C intersect properly. $\tau C \cdot C = C^2 \geq 0$.

(2) $\Rightarrow$ is true for all surfaces. For $\Leftarrow$, first let's show that if D is a divisor such that $D^2 > 0$ and $D \cdot H > 0$ then for $n \gg 0$, nD is effective.

For $n \gg 0$, $D \cdot H > 0$ so $h^2(nD) = h^0(K - nD) = 0$ because $(K - nD) \cdot H$ becomes a negative number. So from

$$h^0(nD) - h^1(nD) + h^2(nD) = \frac{1}{2}nD(nD - K) + 1 + p_a,$$

we get

$$h^0(nD) \geq \frac{1}{2}n^2D^2 - \frac{1}{2}nDK + 1 + p_a > 0,$$

because $K \equiv 0$. Therefore nD is effective.

Then such D is nef because $D^2 > 0$.

Since all α is a limit of such D, α is also nef because nef cone is closed. ∎

Let $af_1 + bf_2 + c\delta$ is nef. Then $(af_1 + bf_2 + c\delta)^2 \geq 0$ and $(af_1 + bf_2 + c\delta)(f_1 + f_2 + \delta) \geq 0$. By solving these inequalities, we get $ab + ac + bc \geq 0$ and $a + b + c \geq 0$. So it is a circular cone.

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2. Blow up of $\mathbb{P}^2$

(See [Har77, V. Ex.4.15].) Take $\mathbb{P}^2$ and blow-up at ≥ 10 general points. Then $\mathrm{N}_1(X) = \langle e_i, \ell \rangle$, where ℓ is the line class and e_i are exceptional divisors. Fix $1 \gg \epsilon > 0$ such that $h = \ell - \epsilon \sum e_i$ is ample.

Then there exists a sequence $C_i \subset X$ of smooth rational curves such that $C_i^2 = -1$, $C_i \cdot h \to \infty$. These C_i are extremal rays of $\overline{\mathrm{NE}}(X)$. Thus $\overline{\mathrm{NE}}(X)$ has infinitely many extremal rays.

On the other hand, $C_i \cdot K = -1$ by adjunction formula. So all C_i lie on an affine hyperplane $K_{-1} := \{C \in \overline{\mathrm{NE}}(X) \mid C \cdot K = -1\}$. If we draw a section of $\overline{\mathrm{NE}}(X)$, (See [Laz04, Figure 1.7].) then C_i forms a set of points on the section such that whose accumulation points lie on $K_{\perp} = \{C \in \overline{\mathrm{NE}}(X) \mid C \cdot K = 0\}$.

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. 2

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. 2