ASYMPTOTIC THEORY

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Let X be a normal projective variety over $\mathbb{C}$. Recall that if L is an ample line bundle on X, then L^m is very ample for $m \geq m_0$. So we have

$$\begin{aligned} \varphi_{|L^m|} : X &\to \mathbb{P}H^0(X, L^m) \\ x &\mapsto \text{hyperplane of sections } s \in H^0(L^m), s(x) = 0 \end{aligned}$$

If $\{s_0, \cdots, s_N\}$ is a basis of $H^0(X, L^m)$,

$$x \mapsto (s_0(x) : \cdots : s_N(x))$$

Then $\varphi_{|L^m|}(X) \cong X$ since L^m is very ample.

Asymptotic theory studies $\varphi_{|L^m|}$ for arbitrary L.

Definition 0.1.

$$\mathbb{N}(L) := \{n \in \mathbb{N} | H^0(X, L^n) \neq 0\}$$

Note that $\mathbb{N}(L)$ is a semi-group, i.e., $m, n \in \mathbb{N}(L) \Rightarrow n + m \in \mathbb{N}(L)$. There exists e such that for all $m \in \mathbb{N}(L)$ large enough, $e|m$. e is called the **exponent** of L.

Example 0.2. Let N be a semi-group generated by $4, 6$. Then $N = \{4, 6, 8, 10, 12, \cdots\}$. So we can take $e = 2$.

Example 0.3. Let X be an elliptic curve. $L \in \mathrm{Pic}^0(X)$, torsion of order e, i.e., $L^e \cong \mathcal{O}_X$ and $L^m \ncong \mathcal{O}_X$ for $0 < m < e$. Then $\mathbb{N}(L) = \{e, 2e, 3e, \cdots\}$.

For $m \in \mathbb{N}(L)$, we have a rational map

$$\varphi_m = \varphi_{|L^m|} : X \dashrightarrow \mathbb{P}H^0(X, L^m).$$

Let $Y_m = \mathrm{im}\varphi_m \subset \mathbb{P}H^0(X, L^m)$.

Definition 0.4. The **Iitaka dimension** of L is

$$\kappa(X, L) := \max_{m \in \mathbb{N}(L)} \{\dim Y_m\}.$$

If $L = K_X$, then $\kappa(X, K_X)$ is Kodaira dimension of X. If $\mathbb{N}(L) = \{0\}$, set $\kappa(X, L) = -\infty$. By definition, $\kappa(X, L) \leq \dim X$.

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Example 0.5. Let $X = Y \times E$, where E is an elliptic curve and $\eta \in \mathrm{Pic}^0(E)$ is an e-torsion. Let Y be a projective variety and B be a very ample line bundle on Y. Set $L = \pi_1^* B \otimes \pi_2^* \eta = B \boxtimes \eta$. $L^m = \pi_1^* B^m \otimes \pi_2^* \eta^m$. So

$$H^0(L^m) = H^0(Y \times E, \pi_1^* B^m \otimes \pi_2^* \eta^m) = H^0(Y, B^m) \otimes H^0(E, \eta^m).$$

If e does not divide m, $H^0(L^m) = 0$. If $e|m$, $H^0(L^m) \cong H^0(Y, B^m)$. L^m is base point free, and $\kappa(X, L) = \dim Y$.

Remark 0.6. Iitaka dimension is not a deformation invariant (fix X and deform L). For example, if X is any smooth projective variety with nontrivial $\mathrm{Pic}^0(X)$ and take $L \in \mathrm{Pic}^0(X)$. If $L \cong \mathcal{O}_X$ or more generally, if L is torsion, $\kappa(X, L) = 0$. But if L is not torsion, $H^0(X, L^m) = 0$ for all $m > 0$ so $\kappa(X, L) = -\infty$.

Proposition 0.7. *For $m \in \mathbb{N}(L)$ with $m \geq m_0$, $\dim Y_m = \kappa(X, L)$.*

Proof. We may assume that $e(L) = 1$ by replacing L by L^e. Let k_0 be a power realizing Iitaka dimension, i.e., $\dim Y_{k_0} = \kappa(X, L)$. We will show that $\dim Y_{k_0+p} = \dim Y_{k_0} = \kappa(X, L)$. The reason is $\varphi_{|L^{k_0}|}$ factor through $\varphi_{|L^{k_0+p}|}$.

$$\begin{array}{ccc} X & \overset{\varphi_{|L^{k_0+p}|}}{\dashrightarrow} & Y_{k_0+p} \\ & \underset{\varphi_{|L^{k_0}|}}{\searrow} & \downarrow \\ & & Y_{k_0} \end{array}$$

We have a map

$$H^0(L^{k_0}) \overset{\otimes s_p}{\hookrightarrow} H^0(L^{k_0+p})$$

by tensoring a fixed section $s_p \in H^0(L^p)$. So we can regard $H^0(L^{k_0})$ as a sub-linear system of $H^0(L^{k_0+p})$. $\square$