NEF LINE BUNDLES AND DIVISORS - II

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A **ruled surface** $\pi : X \to C$ corresponds to a rank 2 vector bundle U over a curve C is defined by $X = \mathbb{P}(U)$. Assume that $\deg U$ is even. So by twisting, we can assume that $\deg U = 0$.

$N^1(X)_{\mathbb{R}} = \langle E = c_1(\mathcal{O}_{\mathbb{P}(U)}(1)), F\rangle$ where $E^2 = 0, EF = 1, F^2 = 0$. So the nef cone is a nonnegative linear combination of E and F.

There are two cases.

(1) (U unstable) There is a quotient line bundle A of U with negative degree a so $\mathbb{P}(A) \hookrightarrow X$. Note that as a numerical class, $[\mathbb{P}(A)] = xE + yF$. $[\mathbb{P}(A)] \cdot F = 1$ because $[\mathbb{P}(A)]$ is a section. So $x = 1$. Moreover, there is a tautological sequence

$$0 \to S \to \pi^*U \to \mathcal{O}_X(1) \to 0.$$

By definition of the tautological quotient $\mathcal{O}_X(1)$, at a point $x \in C$ and its fiber $\mathbb{P}(A)|_x$, $\mathcal{O}_X(1)|_{\mathbb{P}(A)} = A_x$. Therefore $\mathcal{O}_X(1)|_{\mathbb{P}(A)} = A$ and $\mathbb{P}(A) \cdot E = \deg E|_{\mathbb{P}(A)} = \deg \mathcal{O}_X(1)|_{\mathbb{P}(A)} = \deg A = a$. So $y = a$. Thus $[\mathbb{P}(A)]^2 = (E + aF)^2 = 2a < 0$. Therefore $[\mathbb{P}(A)]$ is an extremal ray of $\overline{NE}(X)$. (Another extremal ray is F.)

By taking the dual cone, we conclude that the nef cone is generated by $E - aF$ and F.

(2) (U is semistable) U has no negative degree quotient line bundles. If U is semistable, then so is S^mU. Then $H^0(C, S^mU \otimes A) = \mathrm{Hom}(A^*, S^mU) \neq 0$ only if $\deg A \geq 0$ by semistability of S^mU. Note that any effective curve is a section of $\mathcal{O}_{\mathbb{P}(U)}(m) \otimes \pi^*(A)$ with nonnegative m, because one extremal ray of $\overline{NE}(X)$ is F, the pull-back of $[p] \in C$. And $H^0(X, \mathcal{O}_{\mathbb{P}(U)}(m) \otimes \pi^*A) = H^0(C, S^mU \otimes A)$ by Leray spectral sequence. So we see $\deg A \geq 0$. Thus $\mathrm{Nef}(X) = \overline{NE}(X)$.

On the other hand, if $g(C) > 2$ and C and U are sufficiently general, there a $\deg 0$ line bundle A such that $H^0(C, S^mU \otimes A) = 0$ for all $m \geq 1$ so $E \notin \mathrm{NE}(X)$. In summary, $\mathrm{NE}(X)$ is not closed.

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